International Journal of Difference Equations
ISSN 0973-6069, Volume 15, Number 2, pp. 453–460 (2020)
http://campus.mst.edu/ijde

Cauchy Functions for a Higher-Order Quasi-Delta Differential Equation

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Dedicated to Johnny Henderson on the occasion of his 70th birthday.

Abstract

In this paper, we define the Cauchy function, $C(t, s)$, for an nth-order quasi-linear dynamic equation, and show how it can be calculated given a fundamental set of solutions of the quasi-linear dynamic equation. We also show that the quasi-delta derivatives of $C(t, s)$ are Cauchy functions for related lower order quasi-linear dynamic equations.

AMS Subject Classifications: 34A12, 34N05.
Keywords: Time scales, quasi-delta derivatives, contraction mapping principle.

1 Introduction

We define the Cauchy function for a quasi-delta differential equation on a time scale as well as study some of its properties. Partial motivation for this manuscript comes from the work by Akin [1], in which the author studied Cauchy functions for the dynamic equation on a time scale $Px(t) \equiv \sum_{i=1}^{n} p_i(t)x(\sigma^i(t)) = 0$, $t \in \mathbb{T}$. Other motivation stems from the paper by Kaufmann [7], in which the author considered derivatives of Cauchy functions for quasi-differential and quasi-difference equations. Still other motivation comes from the papers by Bohner and Eloe [2], Eloe [5], Erbe, Mathsen, and Peterson [6], and Peterson and Schneider [9]. For more information on Cauchy functions, see [3, 4, 8]. In order for this paper to be self-contained, we present below some results about time scales, most of which can be found in [3].

Received September 28, 2020; Accepted October 24, 2020
Communicated by Douglas R. Anderson

Let $\mathbb{T}$ be a nonempty closed subset of $\mathbb{R}$, and let $\mathbb{T}$ have the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. Then $\mathbb{T}$ is called a *time scale*. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, we define the *forward jump operator*, σ, and the *backward jump operator* ρ, respectively, by

$$\sigma(t) := \inf\{\tau \in \mathbb{T} \mid \tau > t\},$$

$$\rho(r) := \sup\{\tau \in \mathbb{T} \mid \tau < r\},$$

for all $t, r \in \mathbb{T}$. If $\sigma(t) > t$, then t is said to be *right scattered*, and if $\sigma(t) = t$, then t is said to be *right dense*. If $\rho(r) < r$, then r is said to be *left scattered*, and if $\rho(r) = r$, then r is said to be *left dense*. The *graininess* function, $\mu : \mathbb{T} \to \infty$, is defined as $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum, then we define $\mathbb{T}^\kappa$ to be $\mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. We define the *delta derivative* of $f(t)$, $f^\Delta(t)$, to be the number (provided it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, U, of t such that

$$\Big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\Big| \leq \varepsilon\Big|\sigma(t) - s\Big|,$$

for all $s \in U$. Higher order delta derivatives are through the recursive formula, $f^{\Delta^n}(t) = \big(f^{\Delta^{n-1}}\big)^\Delta(t)$.

If $f : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$, $t < \sup \mathbb{T}$, and t is right scattered, then

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

In particular, if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$, whereas, if t is right dense, then $f^\Delta(t) = f'(t)$. If f is differentiable at $t \in \mathbb{T}$, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Also, if f and g are differentiable functions, then

$$(fg)^\Delta = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t).$$

We say that $f : \mathbb{T} \to \mathbb{R}$ is *right-dense continuous* (rd-continuous) provided f is continuous at each right-dense point $t \in \mathbb{T}$, and whenever $t \in \mathbb{T}$ is left-dense, $\lim\limits_{s \to t^-} f(s)$ exists as a finite number. A function $F : \mathbb{T}^\kappa \to \mathbb{R}$ is called a *delta-antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. The integral of f is defined by

$$\int_a^t f(s)\,\Delta s = F(t) - F(a)$$

for $t \in \mathbb{T}$. We will need the Leibniz rule for integration [3, Theorem 1.117, pg 46]. That is, if $f : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ is continuous at (t,t), $f^\Delta(t,\cdot)$ is rd-continuous, and if $g(t) = \displaystyle\int_{t_0}^t f(t,\tau)\,\Delta\tau$, then

$$g^\Delta(t) = \int_{t_0}^t f^\Delta(t,\tau)\,\Delta\tau + f\big(\sigma(t),t\big).$$

The polynomial functions on a time scale $\mathbb{T}$ are defined recursively. Let $h_1(t,s) = t-s$ and define $h_k(t,s), k \geq 2$, by

$$h_k(t,s) = \int_s^t h_{k-1}(\tau,s)\,\Delta\tau.$$

Note that $h_k^\Delta(t,\cdot) = h_k(t,\cdot)$ for all $t \in \mathbb{T}^\kappa, k \in \mathbb{N}$.

The remainder of this paper is organized as follows. We define the Cauchy function for the equation $L_n y = 0$ in Section 2, as well as show how it can be calculated given a fundamental set of solutions for the quasi-delta derivative equation. In Section 3, we prove that if $C(t,s)$ is the Cauchy function for L_n, then $\mathcal{K}(t,s) = \big(1/\varphi_{j+1}(t)\big)L_j C(t,s)$ is the Cauchy function for $\mathcal{D}_{n-j}$, where $\mathcal{D}_0 y(t) = \big(\varphi_{j+1}y\big)(t)$, and $\mathcal{D}_k y(t) = \varphi_{k+j+1}(t)\big(D_{k-1}y\big)^\Delta(t), 1 \leq k \leq n-j$.

2 The Cauchy Function

Let $\varphi_k \in C^{n+1-k}(\mathbb{T},\mathbb{R}), 1 \leq k \leq n+1$, be such that φ_k does not vanish on $\mathbb{T}^{\kappa^{k-1}}$. Define the quasi-delta differential operators, L_k, by

$$\begin{aligned} L_0 y(t) &= \varphi_1(t)y(t), \\ L_k y(t) &= \varphi_{k+1}(t)\big(L_{k-1}y\big)^\Delta(t), \quad 1 \leq k \leq n. \end{aligned}$$

The Cauchy function for the equation

$$L_n y(t) = 0 \tag{2.1}$$

is the unique function $C(t,s)$ defined on $\mathbb{T} \times \mathbb{T}^{\kappa^n}$ that satisfies, for each $s \in \mathbb{T}^{\kappa^n}$,

$$L_n C(t,s) = 0, \tag{2.2}$$

$$L_k C(\sigma(s),s) = 0, \ 0 \leq k \leq n-2, \tag{2.3}$$

$$L_{n-1} C(\sigma(s),s) = \frac{1}{\varphi_{n+1}(s)}. \tag{2.4}$$

Our first theorem shows how to use Wronskians to construct $C(t,s)$, given a fundamental set of solutions of (2.1).

Theorem 2.1. *Let $y_1, y_2, \ldots, y_n$, be n linearly independent solutions of equation* (2.1). *Then the Cauchy function for* (2.1) *is given by*

$$C(t,s) = \frac{W\big(\sigma(s),t\big)}{\varphi_{n+1}(s)W\big(\sigma(s)\big)},$$

where

$$W(s,t) = \begin{vmatrix} L_0y_1(s) & L_0y_2(s) & \cdots & L_0y_n(s) \\ L_1y_1(s) & L_1y_2(s) & \cdots & L_1y_n(s) \\ \vdots & \vdots & & \vdots \\ L_{n-2}y_1(s) & L_{n-2}y_2(s) & \cdots & L_{n-2}y_n(s) \\ y_1(t) & y_2(t) & \cdots & y_n(t) \end{vmatrix}$$

and

$$W\big(\sigma(s)\big) = \begin{vmatrix} L_0y_1(\sigma(s)\big) & L_0y_2(\sigma(s)\big) & \cdots & L_0y_n(\sigma(s)\big) \\ L_1y_1(\sigma(s)\big) & L_1y_2(\sigma(s)\big) & \cdots & L_1y_n(\sigma(s)\big) \\ \vdots & \vdots & & \vdots \\ L_{n-1}y_1(\sigma(s)\big) & L_{n-1}y_2(\sigma(s)\big) & \cdots & L_{n-1}y_n(\sigma(s)\big) \end{vmatrix}.$$

Proof. By expanding the determinant along the last row, we see that $W(s,t)$ is a linear combination of $y_1, y_2, \ldots, y_n$, and hence satisfies (2.1). Furthermore, since

$$L_kW\big(\sigma(s),t\big) = \begin{vmatrix} L_0y_1\big(\sigma(s)\big) & L_0y_2\big(\sigma(s)\big) & \cdots & L_0y_n\big(\sigma(s)\big) \\ L_1y_1\big(\sigma(s)\big) & L_1y_2\big(\sigma(s)\big) & \cdots & L_1y_n\big(\sigma(s)\big) \\ \vdots & \vdots & & \vdots \\ L_{n-2}y_1\big(\sigma(s)\big) & L_{n-2}y_2\big(\sigma(s)\big) & \cdots & L_{n-2}y_n\big(\sigma(s)\big) \\ L_ky_1(t) & L_ky_2(t) & \cdots & L_ky_n(t) \end{vmatrix},$$

for $0 \le k \le n-1$, then $L_kW\big(\sigma(s),\sigma(s)\big) = 0$, $0 \le k \le n-2$. Furthermore, we have $L_{n-1}W\big(\sigma(s),\sigma(s)\big) = W\big(\sigma(s)\big)$. Hence $L_{n-1}C\big(\sigma(s),s\big) = 1/\varphi_{n+1}(s)$, and so $C(t,s)$ satisfies (2.2)–(2.4). □

The Cauchy function is fundamental to the variation of constants formula for solutions of initial value problems.

Theorem 2.2 (Variation of Constants)**.** *Suppose that f is a rd-continuous function and $t_0 \in \mathbb{T}$. Then, the solution of the initial value problem*

$$\begin{aligned} L_ny(t) &= f(t), \quad t \in \mathbb{T}, \\ L_ky(t_0) &= 0,\ 0 \le k \le n-1, \end{aligned}$$

is

$$y(t) = \int_{t_0}^{t} C(t,s)f(s)\,\Delta s,$$

where $C(t,s)$ is the Cauchy function for (2.1).

Proof. Let $y(t) = \int_{t_0}^{t} C(t,s) f(s)\, \Delta s$. Then,

$$L_0 y(t_0) = \varphi_1(t_0) \int_{t_0}^{t_0} C\big(t_0, s\big) f(s)\, \Delta s = 0.$$

For $1 \le k \le n-1$,

$$\begin{aligned} L_k y(t) &= \int_{t_0}^{t} \varphi_{k+1}(t) L_k C(t,s) f(s)\, \Delta s + \varphi_{k+1}(t) L_{k-1} C\big(\sigma(t), t\big) f(t) \\ &= \int_{t_0}^{t} \varphi_{k+1}(t) L_k C(t,s) f(s)\, \Delta s. \end{aligned}$$

Hence, $L_k y(t_0) = 0$, $0 \le k \le n-1$.

Since $L_{n-1} C(t,s) = \dfrac{1}{\varphi_{n+1}(s)}$, we have

$$\begin{aligned} L_n y(t) &= \int_{t_0}^{t} L_n C(t,s) f(s)\, \Delta s + \varphi_{n+1}(t) L_{n-1} C\big(\sigma(t), t\big) f(t) \\ &= 0 + f(t) = f(t), \end{aligned}$$

and the proof is complete. □

3 Derivatives and Integrals of Cauchy Functions

It is well-known that the Cauchy function for $y^{\Delta^n}(t) = 0$ is $h_{n-1}(t,s)$, and the Cauchy function for $y^{\Delta^{n-1}}(t) = 0$ is $h_{n-2}(t,s)$. We know that $h_{n-2}(t,\cdot) = \big(h_{n-1}(t,\cdot)\big)^{\Delta}$. In this section we show that quasi-delta derivatives of the Cauchy function for $L_n y = 0$ are Cauchy functions for related quasi-linear dynamic equations. In particular, we define $D_0 y(t) = (\varphi_2 y)(t)$ and, for $1 \le k \le n-1$, let $D_k y(t) = \varphi_{k+2}(t)\big(D_{k-1} y\big)^{\Delta}(t)$. Our next theorem states the relation between the Cauchy functions for $L_n y = 0$ and $D_{n-1} y(t) = 0$.

Theorem 3.1. *Let $C(t,s)$ be the Cauchy function for $L_n y(t) = 0$. Then, $K(t,s) = \big(1/\varphi_2(t)\big) L_1 C(t,s)$ is the Cauchy function for $D_{n-1} y(t) = 0$.*

Proof. For $2 \le k \le n$, the quasi-delta differential operator $L_k y(t) = \varphi_{k+1}\big(L_{k-1} y\big)^{\Delta}(t)$ can be written as

$$L_k y(t) = \varphi_{k+1}(t) \left(\varphi_k(t) \left(\cdots \left(\varphi_3(t) \big(\varphi_2(t) (\varphi_1(t) y(t))^{\Delta} \big)^{\Delta} \right)^{\Delta} \cdots \right)^{\Delta} \right)^{\Delta}.$$

Since $\varphi_3(t)\big(\varphi_2(t)(\varphi_1(t)y)^\Delta\big)^\Delta = \varphi_3(t)\left(\varphi_2(t)\big(\frac{1}{\varphi_2(t)}(L_1y)(t)\big)\right)^\Delta$, then we see that $L_ky(t) = D_{k-1}\big(1/\varphi_2(t)(L_1y)(t)\big)$. Hence, if $K(t,s) = \big(1/\varphi_2(t)\big)L_1C(t,s)$, then

$$\begin{aligned}
&D_{n-1}K(t,s) = L_nC(t,s) = 0,\\
&D_{k-1}K(\sigma(s),s) = L_kC(\sigma(s),s) = 0,\ 1 \le k \le n-3, \quad \text{and}\\
&D_{n-2}K(\sigma(s),s) = L_{n-1}C(\sigma(s),s) = \frac{1}{\varphi(s)}.
\end{aligned}$$

That is, $K(t,s)$ is the Cauchy function for $D_{n-1}y = 0$ and the proof is complete. □

We can generalize Theorem 3.1. Let $j \in \{1, 2, \ldots, n\}$ be given. Define $\mathcal{D}_0y(t) = (\varphi_{j+1}y)(t)$, and let $\mathcal{D}_ky(t) = \varphi_{k+j+1}(t)\big(\mathcal{D}_{k-1}y\big)^\Delta(t)$ for $1 \le k \le n-j$. The proof of the following corollary is essentially the same as in Theorem 3.1, with the notable exception that $L_ny(t) = \mathcal{D}_{n-j}\big(1/\varphi_{j+1}(t)\big)L_jy(t)$.

Corollary 3.2. *Let $C(t,s)$ be the Cauchy function for $L_ny = 0$. Then, $\mathcal{K}(t,s) = \big(1/\varphi_{j+1}(t)\big)L_jC(t,s)$ is the Cauchy function for $\mathcal{D}_{n-j}y(t) = 0$.*

Again, let $j \in \{1, 2, \ldots, n\}$ be given. We can write $L_ny(t)$ as

$$L_ny(t) = \mathcal{D}_{n-j}\big(1/\varphi_{j+1}(t)\big)L_jy(t).$$

We denote $C_j(t,s)$ to be the Cauchy function associated with L_j, $\mathcal{K}_{n-j}(t,s)$ to be the Cauchy function associated with $\mathcal{D}_{n-j}$, and $C_n(t,s)$ to be the Cauchy function associated with L_n.

Suppose f is rd-continuous, and consider the initial value problem

$$L_ny(t) = \mathcal{D}_{n-j}\big(1/\varphi_{j+1}(t)\big)L_jy(t) = f(t) \tag{3.1}$$
$$L_ky(t_0) = 0,\ 0 \le k \le n-1. \tag{3.2}$$

Let y be the solution of (3.1), (3.2), and let $u(t) = \big(1/\varphi_{j+1}(t)\big)L_jy(t)$. Then, $u(t)$ is the solution of the initial value problem

$$\begin{aligned}
&\mathcal{D}_{n-j}u(t) = 0,\\
&L_ku(t_0) = 0,\ j \le k \le n-1.
\end{aligned}$$

By Theorem 2.2,

$$u(t) = \int_{t_0}^t \mathcal{K}_{n-j}(t,s)f(s)\,\Delta s.$$

Since $L_jy(t) = \varphi_{j+1}(t)u(t)$, then y is the solution of

$$\begin{aligned}
&L_jy(t) = \varphi_{j+1}(t)u(t),\\
&L_ky(t_0) = 0,\ 0 \le k \le j-1.
\end{aligned}$$

Thus,

$$y(t) = \int_{t_0}^{t} C_j(t,\tau)\,\varphi_{j+1}(\tau)\,u(\tau)\,\Delta\tau.$$

We see that

$$\begin{aligned} y(t) &= \int_{t_0}^{t} C_j(t,\tau)\,\varphi_{j+1}(\tau)\int_{t_0}^{\tau} \mathcal{K}_{n-j}(\tau,s)f(s)\,\Delta s\,\Delta\tau \\ &= \int_{t_0}^{t}\int_{t_0}^{\tau} \varphi_{j+1}(\tau)\,C_j(t,\tau)\mathcal{K}_{n-j}(\tau,s)f(s)\,\Delta s\,\Delta\tau. \end{aligned}$$

Using [1, Theorem 10],

$$\int_a^b\int_a^{\tau} F(\tau,s)\,\Delta s\Delta\tau = \int_a^b\int_{\sigma(s)}^{b} F(\tau,s)\,\Delta\tau\Delta s,$$

we have

$$y(t) = \int_{t_0}^{t}\int_{\sigma(s)}^{t} \varphi_{j+1}(\tau)\,C_j(t,\tau)\mathcal{K}_{n-j}(\tau,s)\,\Delta\tau\, f(s)\,\Delta s.$$

Appealing to Theorem 2.2, again we have

$$y(t) = \int_{t_0}^{t} C_n(t,s)f(s)\,\Delta s.$$

Thus,

$$C_n(t,s) = \int_{\sigma(s)}^{t} \varphi_{j+1}(\tau)\,C_j(t,\tau)\mathcal{K}_{n-j}(\tau,s)\,\Delta\tau.$$

We have the following theorem.

Theorem 3.3. *Fix $j \in \{1,2,\dots,n\}$, and suppose that $L_n y(t)$ can be factored as*

$$L_n y(t) = \mathcal{D}_{n-j}\big(1/\varphi_{j+1}(t)\big)L_j y(t).$$

Suppose that $C_j(t,s)$ is the Cauchy function associated with L_j, that $\mathcal{K}_{n-j}(t,s)$ is the Cauchy function associated with $\mathcal{D}_{n-j}$, and that $C_n(t,s)$ is the Cauchy function associated with L_n. Then,

$$C_n(t,s) = \int_{\sigma(s)}^{t} \varphi_{j+1}(\tau)\,C_j(t,\tau)\mathcal{K}_{n-j}(\tau,s)\,\Delta\tau.$$

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