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Time Scales Delta Iyengar-Type Inequalities

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Abstract

Here we give the necessary background on delta time scales approach. Then we present general related time scales delta Iyengar type inequalities for all basic norms. We finish with applications to specific time scales like $\mathbb{R}$, $\mathbb{Z}$ and $q^{\overline{\mathbb{Z}}}$, $q > 1$.

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1 Introduction

We are motivated by the following famous Iyengar inequality (1938), [11].

Theorem 1.1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \quad (1.1)$$

We present generalized analogs of (1.1) to time scales. Motivation comes also from [2–4, 6, 7].

2 Background

Here basics on time scales come from [3, 4, 6–8]. We need the following definition.

Definition 2.1. A time scale is an arbitrary nonempty closed subset of the real numbers, e.g. $\mathbb{R}$, $\mathbb{Z}$, $q^{\mathbb{N}_0} = \{q^k | k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, q > 1\}$.

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Definition 2.2. If $\mathbb{T}$ is a time scale, then we define the forward jump operator $\sigma : \mathbb{T} \longmapsto \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}$, $\forall\, t \in \mathbb{T}$; the backward jump operator $\rho : \mathbb{T} \longmapsto \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} | s < t\}$, $\forall\, t \in \mathbb{T}$; and the graininess function $\mu : \mathbb{T} \to \mathbb{R}_+ = [0, \infty)$, by $\mu(t) = \sigma(t) - t$, $\forall\, t \in \mathbb{T}$. Furthermore for a function $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\sigma}(t) = f(\sigma(t))$, $\forall\, t \in \mathbb{T}$; and $f^{\rho}(t) = f(\rho(t))$, $\forall\, t \in \mathbb{T}$.

In this definition we use $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if t is the maximum of $\mathbb{T}$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if t is the minimum of $\mathbb{T}$).

We call $t \in \mathbb{T}$ right-scattered if $t < \sigma(t)$, $t \in \mathbb{T}$ right-dense if $t = \sigma(t)$, $t \in \mathbb{T}$ left-scattered if $\rho(r) < t$, $t \in \mathbb{T}$ left-dense if $\rho(t) = t$, $t \in \mathbb{T}$ isolated if $\rho(t) < t < \sigma(t)$, $t \in \mathbb{T}$ dense if $\rho(t) = t = \sigma(t)$.

We notice that ρ is an increasing function, so is $\rho^2(t) = \rho(\rho(t)), \ldots,$ so that $\rho^n(t) = \rho\left(\rho^{n-1}(t)\right)$ is increasing in t for $n \in \mathbb{N}$. Since $\mathbb{T}$ is closed subset of $\mathbb{R}$ we have that $\sigma(t), \rho(t) \in \mathbb{T}$, for $t \in \mathbb{T}$.

Definition 2.3 (See [8]). A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous (denoted by C_{rd}) if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits are finite at left-dense points of $\mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is rd-continuous iff f is continuous. Also, if $\mathbb{T} = \mathbb{Z}$, then any function defined on $\mathbb{Z}$ is rd-continuous ([9]).

Definition 2.4 (See [8]). If $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is left-scattered, we let $\mathbb{T}^k := \mathbb{T} - \{\sup \mathbb{T}\}$, otherwise we let $\mathbb{T}^k := \mathbb{T}$ the time scale.

In summary, $\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$

Definition 2.5 (See [8]). Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|, \quad \forall\, s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger [10]) derivative of f at t. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta} = f'$, whereas if $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, the usual forward difference operator.

Theorem 2.6 (Existence of Antiderivatives, see [8]). *Let f be rd-continuous. Then f has an antiderivative F satisfying $F^{\Delta} = f$.*

Definition 2.7 (See [8]). If f is rd-continuous and $t_0 \in \mathbb{T}$, then we define the integral

$$F(t) = \int_{t_0}^{t} f(\tau)\,\Delta\tau \quad \text{for } t \in \mathbb{T}.$$

Therefore for $f \in C_{rd}(\mathbb{T})$ we have by definition

$$\int_a^b f(\tau)\,\Delta\tau = F(b) - F(a),$$

where $F^{\Delta} = f$.

If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

where the integral on the right hand side is the Riemann integral ([9]).

If every point in $\mathbb{T}$ is isolated and $a < b$ are in $\mathbb{T}$, then ([9])

$$\int_a^b f(t)\,\Delta t = \sum_{t=a}^{\rho(b)} f(t)\,\mu(t).$$

Theorem 2.8 (See [8]). *Let f, g be rd-continuous on $\mathbb{T}$, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(1) $\displaystyle\int_a^b (\alpha f(t) + \beta g(t))\,\Delta t = \alpha \int_a^b f(t)\,\Delta t + \beta \int_a^b g(t)\,\Delta t,$

(2) $\displaystyle\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t,$

(3) $\displaystyle\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t,$

(4) $\displaystyle\int_a^b f(t)\,g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\,g(\sigma(t))\,\Delta t,$

(5) $\displaystyle\int_a^a f(t)\,\Delta t = 0,$

(6) $\displaystyle\int_a^b 1\Delta t = b - a.$

Theorem 2.9 (Hölder's Inequality, see [2]). *Let $a, b \in \mathbb{T}$, $a \le b$, and $f, g : \mathbb{T} \to \mathbb{R}$ be rd-continuous. Then*

$$\int_a^b |f(t)|\,|g(t)|\,\Delta t \le \left(\int_a^b |f(t)|^p\,\Delta t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,\Delta t\right)^{\frac{1}{q}}, \tag{2.1}$$

where $p, q > 1 : \dfrac{1}{p} + \dfrac{1}{q} = 1.$

Theorem 2.10 (See [8]). *Let $f, g \in C_{rd}(\mathbb{T})$, $a, b \in \mathbb{T}$, $a \le b$. Then*

1) if $|f(t)| \le g(t)$ on $[a, b) \cap \mathbb{T}$, then $\displaystyle\left|\int_a^b f(t)\,\Delta t\right| \le \int_a^b g(t)\,\Delta t,$

2) if $f(t) \ge 0$, for all $a \le t < b$ and $t \in \mathbb{T}$, then $\displaystyle\int_a^b f(t)\,\Delta t \ge 0.$

Corollary 2.11 (See [3]). *Let $f \in C_{rd}(\mathbb{T})$; $a, b, c \in \mathbb{T}$, with $c \in [a, b]$; $f(t) \geq 0$, $\forall$ $t \in [a, b]$. Then*

$$\int_a^c f(t)\,\Delta t \leq \int_a^b f(t)\,\Delta t.$$

Definition 2.12 (See [3]). For a function $f : \mathbb{T} \to \mathbb{R}$ we consider the second derivative $f^{\Delta\Delta}$ provided f^{Δ} is differentiable on $\mathbb{T}^{k^2} = \left(\mathbb{T}^k\right)^k$ with derivative $f^{\Delta\Delta} = \left(f^{\Delta}\right)^{\Delta} : \mathbb{T}^{k^2} \to \mathbb{R}$. Similarly we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{k^n} \to \mathbb{R}$.

Similarly we define $\sigma^2(t) = \sigma(\sigma(t)), \ldots, \sigma^n(t) = \sigma\left(\sigma^{n-1}(t)\right)$, $n \in \mathbb{N}$. For convenience we put $\rho^0(t) = \sigma^0(t) = t$, $f^{\Delta^0} = f$, $\mathbb{T}^{k^0} = \mathbb{T}$.

Notice $\mathbb{T}^{k^n} \subset \mathbb{T}^{k^l}$, $l \in \{0, 1, ..., n\}$.

Denote by $C_{rd}^n(\mathbb{T})$ the space of all functions $f \in C_{rd}(\mathbb{T})$ such that $f^{\Delta^i} \in C_{rd}(\mathbb{T})$ for $i = 1, \ldots, n \in \mathbb{N}$. In this last case $\mathbb{T}^k = \mathbb{T}$ is needed.

We need the following Taylor formula.

Theorem 2.13 (Taylor's Formula, see [5,9]). *Assume $\mathbb{T}^k = \mathbb{T}$ and $f \in C_{rd}^n(\mathbb{T})$, $n \in \mathbb{N}$ and $s, t \in \mathbb{T}$. Here $h_0(t, s) = 1$, $\forall$ $s, t \in \mathbb{T}$; $k \in \mathbb{N}_0$, and*

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\,\Delta\tau, \quad \forall\, s, t \in \mathbb{T}.$$

(then $h_k^{\Delta}(t, s) = h_{k-1}(t, s)$, for $k \in \mathbb{N}$, $\forall$ $t \in \mathbb{T}$, for each $s \in \mathbb{T}$ fixed). Then

$$f(t) = \sum_{k=0}^{n-1} f^{\Delta^k}(s)\,h_k(t, s) + \int_s^t h_{n-1}(t, \sigma(\tau))\,f^{\Delta^n}(\tau)\,\Delta\tau. \tag{2.2}$$

Remark 2.14 (to Theorem 2.13). By [9], we have $h_1(t, s) = t - s$, $\forall$ $s, t \in \mathbb{T}$.

So if $t \geq s$ then $h_1(t, s) \geq 0$, $h_2(t, s) \geq 0, ..., h_{n-1}(t, s) \geq 0$. However for n odd number $h_{n-1}(t, \sigma(\tau)) \geq 0$ for all $s \leq \tau \leq t$ (see [4], p. 635).

Also it holds ([1])

$$h_k(t, s) \leq \frac{(t-s)^k}{k!}, \quad \forall\, t \geq s, k \in \mathbb{N}_0.$$

Corollary 2.15 (to Theorem 2.13, see [3]). *Assume $f \in C_{rd}^n(\mathbb{T})$ and $s, t \in \mathbb{T}$. Let $m \in \mathbb{N}$ with $m < n$ Then*

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} f^{\Delta^{k+m}}(s)\,h_k(t, s) + \int_s^t h_{n-m-1}(t, \sigma(\tau))\,f^{\Delta^n}(\tau)\,\Delta\tau. \tag{2.3}$$

Proof. Use Theorem 2.13 with n and f substituted by $n-m$ and f^{Δ^m}, respectively. □

Corollary 2.16 (See [3]). *Let $f \in C_{rd}(\mathbb{T})$; $a, b \in \mathbb{T}$, such that $f(t) > 0$, $\forall\, t \in [a,b] \cap \mathbb{T}$, then $\int_a^b f(t)\,\Delta t > 0$.*

We mention also the following.

Lemma 2.17 (See [4, p. 631]). *Let the time scale $\mathbb{T}$ be such that $\mathbb{T}^k = \mathbb{T}$. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, such that $h_0(t,s) \equiv 1$, $\forall\, s,t \in \mathbb{T}$, and $h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\,\Delta\tau$, $\forall\, s,t \in \mathbb{T}$, for all $k \in \mathbb{N}_0$.*

Then $h_k(t,s)$ is continuous in $s \in \mathbb{T}$, for each fixed $t \in \mathbb{T}$; and continuous in $t \in \mathbb{T}$, for each fixed $s \in \mathbb{T}$. Also it holds that $h_k(t,\sigma(s))$ is rd-continuous in $s \in \mathbb{T}$, for each fixed $t \in \mathbb{T}$; for all $k \in \mathbb{N}_0$.

3 Main Results

In this article, we assume that $\mathbb{T}^k = \mathbb{T}$. Next, we present Iyengar type inequality on time scales for all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

Theorem 3.1. *Let $f \in C_{rd}^n(\mathbb{T})$, n is an odd number, $a, b \in \mathbb{T}$; $a \leq b$. Here σ is continuous and $h_{n-1}(t,s)$ jointly continuous. Then*

1)

$$\left| \int_a^b f(t)\,\Delta t - \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b) \right) \right| \leq \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}}$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right) \right], \quad (3.1)$$

$\forall\, x \in [a,b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$, $k = 0,1,...,n-1$, we get from (3.1) *that*

$$\left| \int_a^b f(t)\,\Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}}$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right) \right], \quad (3.2)$$

$\forall\, x \in [a,b] \cap \mathbb{T}$,

2_1) when $x = a$ we get from (3.2) *that*

$$\left| \int_a^b f(t)\,\Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_a^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right), \quad (3.3)$$

2_2) *when* $x = b$ *we get from* (3.2) *that*

$$\left|\int_a^b f(t)\,\Delta t\right| \leq \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right), \tag{3.4}$$

2_3) *by* (3.3) *and* (3.4) *we get*

$$\left|\int_a^b f(t)\,\Delta t\right| \leq \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}$$

$$\min\left\{\left(\int_a^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right), \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right)\right\}, \tag{3.5}$$

and

3) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$, $k = 1, ..., n-1$, *by* (3.1) *we have*

$$\left|\int_a^b f(t)\,\Delta t - \left[f(a)(x-a) + f(b)(b-x)\right]\right| \leq \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right)\right], \tag{3.6}$$

$\forall\, x \in [a,b] \cap \mathbb{T}$.

Proof. By [8, p. 23], we have that $\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} < \infty$. By Theorem 2.13, see (2.2), we have

$$f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a) = \int_a^t h_{n-1}(t,\sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau, \tag{3.7}$$

and

$$f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b) = \int_b^t h_{n-1}(t,\sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau, \tag{3.8}$$

$\forall\, t \in [a,b] \cap \mathbb{T}$.

Then we get

$$\left|f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a)\right| \overset{(3.7)}{\leq} \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau, \tag{3.9}$$

and

$$\left|f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b)\right| \overset{(3.8)}{=} \left|\int_t^b h_{n-1}(t,\sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau\right|$$

$$\leq \left(\int_{t}^{b} h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right) \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}. \tag{3.10}$$

Therefore it holds by (3.9), (3.10)

$$-\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \int_{a}^{t} h_{n-1}(t,\sigma(\tau))\,\Delta\tau \leq f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a)\,h_k(t,a)$$

$$\leq \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \int_{a}^{t} h_{n-1}(t,\sigma(\tau))\,\Delta\tau$$

and

$$-\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_{t}^{b} h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right) \leq f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b)\,h_k(t,b)$$

$$\leq \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_{t}^{b} h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right),$$

$\forall\, t \in [a,b]\cap\mathbb{T}$.

Consequently, we have

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\,h_k(t,a) - \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \int_{a}^{t} h_{n-1}(t,\sigma(\tau))\,\Delta\tau \leq f(t) \tag{3.11}$$

$$\leq \sum_{k=0}^{n-1} f^{\Delta^k}(a)\,h_k(t,a) + \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \int_{a}^{t} h_{n-1}(t,\sigma(\tau))\,\Delta\tau$$

and

$$\sum_{k=0}^{n-1} f^{\Delta^k}(b)\,h_k(t,b) - \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_{t}^{b} h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right) \leq f(t) \tag{3.12}$$

$$\leq \sum_{k=0}^{n-1} f^{\Delta^k}(b)\,h_k(t,b) + \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_{t}^{b} h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right),$$

$\forall\, t \in [a,b]\cap\mathbb{T}$.

Let any $x \in [a,b]\cap\mathbb{T}$, then integrating (3.11), (3.12) we obtain:

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\,h_{k+1}(x,a) - \left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_{a}^{x}\left(\int_{a}^{t} h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right)$$

$$\leq \int_{a}^{x} f(t)\,\Delta t \leq \tag{3.13}$$

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_{k+1}(x,a) + \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right),$$

and

$$-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_{k+1}(x,b) - \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right)$$

$$\leq \int_x^b f(t)\,\Delta t \leq \tag{3.14}$$

$$-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_{k+1}(x,b) + \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}} \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right).$$

Adding (3.13) and (3.14) we derive

$$\sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b) \right) - \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}} \cdot$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right) \right]$$

$$\leq \int_a^b f(t)\,\Delta t \leq \tag{3.15}$$

$$\sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b) \right) + \left\| f^{\Delta^n} \right\|_{\infty,[a,b]\cap\mathbb{T}} \cdot$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))\,\Delta\tau \right) \Delta t \right) \right],$$

$\forall\, x \in [a,b] \cap \mathbb{T}$.

The proof is now complete. □

We continue with the next result.

Theorem 3.2. *Let $f \in C_{rd}^n(\mathbb{T})$, $n \in \mathbb{N}$ is odd, $a, b \in \mathbb{T}$; $a \leq b$. Then*

1)

$$\left| \int_a^b f(t)\,\Delta t - \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b) \right) \right| \leq$$

$$\left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \left\{ \int_a^x (t-\sigma(a))^{n-1}\,\Delta t + \int_x^b (\sigma(b)-t)^{n-1}\,\Delta t \right\}, \tag{3.16}$$

$\forall\, x \in [a,b] \cap \mathbb{T}$, *2) assuming* $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$*,* $k = 0, 1, ..., n-1$*, from* (3.16) *we obtain*

$$\left| \int_a^b f(t)\,\Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \cdot$$

$$\left\{ \int_a^x (t - \sigma(a))^{n-1}\,\Delta t + \int_x^b (\sigma(b) - t)^{n-1}\,\Delta t \right\}, \tag{3.17}$$

$\forall\, x \in [a,b] \cap \mathbb{T}$,

2_1) *when* $x = a$ *by* (3.17) *we get*

$$\left| \int_a^b f(t)\,\Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \left(\int_a^b (\sigma(b) - t)^{n-1}\,\Delta t \right), \tag{3.18}$$

2_2) *when* $x = b$ *by* (3.17) *we get*

$$\left| \int_a^b f(t)\,\Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \left(\int_a^x (t - \sigma(a))^{n-1}\,\Delta t \right), \tag{3.19}$$

2_3) *by* (3.18), (3.19) *we have*

$$\left| \int_a^b f(t)\,\Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \cdot$$

$$\min\left\{ \left(\int_a^b (\sigma(b) - t)^{n-1}\,\Delta t \right), \left(\int_a^b (t - \sigma(a))^{n-1}\,\Delta t \right) \right\}, \tag{3.20}$$

3) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$*,* $k = 1, ..., n-1$*, by* (3.16) *we derive*

$$\left| \int_a^b f(t)\,\Delta t - [f(a)(x-a) + f(b)(b-x)] \right| \leq$$

$$\left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \left\{ \int_a^x (t - \sigma(a))^{n-1}\,\Delta t + \int_x^b (\sigma(b) - t)^{n-1}\,\Delta t \right\}, \tag{3.21}$$

$\forall\, x \in [a,b] \cap \mathbb{T}$.

Proof. Clearly, here it holds $\left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} < \infty$.

Set $h_0(t,s) = 1$, $\forall\, s,t \in \mathbb{T}$; $k \in \mathbb{N}_0$, and $h_{k+1}(t,s) = \displaystyle\int_s^t h_k(\tau,s)\,\Delta\tau$, $\forall\, s,t \in \mathbb{T}$. Easily, it holds $|h_n(t,s)| \leq |t-s|^n$, $\forall\, n \in \mathbb{N}$, $\forall\, s,t \in \mathbb{T}$.

By Theorem 2.13 (3) we have

$$f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a) = \int_a^t h_{n-1}(t, \sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau,$$

and

$$f(t)-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b)=\int_b^t h_{n-1}(t,\sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau,$$

$\forall\, t\in[a,b]\cap\mathbb{T}$.

Then

$$\left|f(t)-\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a)\right|=\left|\int_a^t h_{n-1}(t,\sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau\right|\leq$$

$$\int_a^t |h_{n-1}(t,\sigma(\tau))|\left|f^{\Delta^n}(\tau)\right|\Delta\tau\leq\int_a^t |t-\sigma(\tau)|^{n-1}\left|f^{\Delta^n}(\tau)\right|\Delta\tau\leq$$

$$(t-\sigma(a))^{n-1}\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}.$$

Furthermore we have

$$\left|f(t)-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b)\right|=\left|\int_t^b h_{n-1}(t,\sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau\right|\leq$$

$$\int_t^b |h_{n-1}(t,\sigma(\tau))|\left|f^{\Delta^n}(\tau)\right|\Delta\tau\leq\int_t^b |t-\sigma(\tau)|^{n-1}\left|f^{\Delta^n}(\tau)\right|\Delta\tau\leq$$

$$(\sigma(b)-t)^{n-1}\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}.$$

Therefore it holds

$$-(t-\sigma(a))^{n-1}\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}\leq f(t)-\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a)$$

$$\leq(t-\sigma(a))^{n-1}\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}$$

and

$$-(\sigma(b)-t)^{n-1}\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}\leq f(t)-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b)$$

$$\leq(\sigma(b)-t)^{n-1}\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})},$$

$\forall\, t\in[a,b]\cap\mathbb{T}$.

Consequently it holds

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a)-(t-\sigma(a))^{n-1}\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}\leq f(t)$$

$$\leq\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a)+(t-\sigma(a))^{n-1}\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}$$

and

$$\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b) - (\sigma(b)-t)^{n-1} \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \le f(t)$$

$$\le \sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b) + (\sigma(b)-t)^{n-1} \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})},$$

$\forall\, t \in [a,b]\cap\mathbb{T}$.

Let any $x \in [a,b]\cap\mathbb{T}$, then integrating by integration we have

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_{k+1}(x,a) - \left(\int_a^x (t-\sigma(a))^{n-1} \Delta t \right) \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \tag{3.22}$$

$$\le \int_a^x f(t)\, \Delta t \le$$

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_{k+1}(x,a) + \left(\int_a^x (t-\sigma(a))^{n-1} \Delta t \right) \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})},$$

and

$$-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_{k+1}(x,b) - \left(\int_x^b (\sigma(b)-t)^{n-1} \Delta t \right) \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})}$$

$$\le \int_x^b f(t)\, \Delta t \le$$

$$-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_{k+1}(x,b) + \left(\int_x^b (\sigma(b)-t)^{n-1} \Delta t \right) \left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})}, \tag{3.23}$$

$\forall\, x \in [a,b]\cap\mathbb{T}$.

Adding (3.22) and (3.23), we obtain

$$\sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b) \right) -$$

$$\left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \left\{ \left(\int_a^x (t-\sigma(a))^{n-1} \Delta t \right) + \left(\int_x^b (\sigma(b)-t)^{n-1} \Delta t \right) \right\}$$

$$\le \int_a^b f(t)\, \Delta t \le$$

$$\sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b) \right) +$$

$$\left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \left\{ \left(\int_a^x (t-\sigma(a))^{n-1} \Delta t \right) + \left(\int_x^b (\sigma(b)-t)^{n-1} \Delta t \right) \right\}, \quad (3.24)$$

$\forall\, x \in [a,b] \cap \mathbb{T}$.

The proof is now complete. □

We continue with the next result.

Theorem 3.3. *Let* $f \in C_{rd}^n(\mathbb{T})$, n *is an odd number,* $a, b \in \mathbb{T}$*;* $a \le b$*;* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*;* σ *is continuous and* $h_{n-1}(t,s)$ *is jointly continuous. Then*

1)

$$\left| \int_a^b f(t)\,\Delta t - \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b) \right) \right| \le \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \cdot$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.25)$$

$\forall\, x \in [a,b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$, $k = 0, 1, ..., n-1$, *by* (3.25) *we have that*

$$\left| \int_a^b f(t)\,\Delta t \right| \le \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \cdot$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.26)$$

$\forall\, x \in [a,b] \cap \mathbb{T}$,

2_1) *when* $x = a$ *by* (3.26) *we get*

$$\left| \int_a^b f(t)\,\Delta t \right| \le \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_a^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right), \quad (3.27)$$

2_2) *when* $x = b$ *by* (3.26) *we get*

$$\left| \int_a^b f(t)\,\Delta t \right| \le \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right), \quad (3.28)$$

2_3) *by* (3.27), (3.28) *we derive that*

$$\left| \int_a^b f(t)\,\Delta t \right| \le \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \cdot \quad (3.29)$$

$$\min\left\{\left(\int_a^b\left(\int_t^b h_{n-1}(t,\sigma(\tau))^p\,\Delta\tau\right)^{\frac{1}{p}}\Delta t\right),\left(\int_a^b\left(\int_a^t h_{n-1}(t,\sigma(\tau))^p\,\Delta\tau\right)^{\frac{1}{p}}\Delta t\right)\right\},$$

3) assuming $f^{\Delta^k}(a)=f^{\Delta^k}(b)=0$, $k=1,...,n-1$, by (3.25) *we obtain*

$$\left|\int_a^b f(t)\,\Delta t-[f(a)(x-a)+f(b)(b-x)]\right|\leq\left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})}$$

$$\left[\left(\int_a^x\left(\int_a^t h_{n-1}(t,\sigma(\tau))^p\,\Delta\tau\right)^{\frac{1}{p}}\Delta t\right)+\left(\int_x^b\left(\int_t^b h_{n-1}(t,\sigma(\tau))^p\,\Delta\tau\right)^{\frac{1}{p}}\Delta t\right)\right],\tag{3.30}$$

$\forall\, x\in[a,b]\cap\mathbb{T}$.

Proof. As before, we have

$$K(t,a):=f(t)-\sum_{k=0}^{n-1}f^{\Delta^k}(a)\,h_k(t,a)=\int_a^t h_{n-1}(t,\sigma(\tau))\,f^{\Delta^n}(\tau)\,\Delta\tau,$$

and

$$K(t,b):=f(t)-\sum_{k=0}^{n-1}f^{\Delta^k}(b)\,h_k(t,b)=\int_b^t h_{n-1}(t,\sigma(\tau))\,f^{\Delta^n}(\tau)\,\Delta\tau,$$

$\forall\, t\in[a,b]\cap\mathbb{T}$.

We have that (by use of (2.1))

$$|K(t,a)|\leq\left(\int_a^t h_{n-1}(t,\sigma(\tau))^p\,\Delta\tau\right)^{\frac{1}{p}}\left(\int_a^t\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\right)^{\frac{1}{q}}$$

$$\leq\left(\int_a^t h_{n-1}(t,\sigma(\tau))^p\,\Delta\tau\right)^{\frac{1}{p}}\left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})},$$

and

$$|K(t,b)|=\left|\int_t^b h_{n-1}(t,\sigma(\tau))\,f^{\Delta^n}(\tau)\,\Delta\tau\right|\leq$$

$$\left(\int_t^b h_{n-1}(t,\sigma(\tau))^p\,\Delta\tau\right)^{\frac{1}{p}}\left(\int_t^b\left|f^{\Delta^n}(\tau)\right|^q\Delta\tau\right)^{\frac{1}{q}}$$

$$\leq\left(\int_t^b h_{n-1}(t,\sigma(\tau))^p\,\Delta\tau\right)^{\frac{1}{p}}\left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})},$$

$\forall\, t\in[a,b]\cap\mathbb{T}$.

Hence it holds

$$-\left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \le K(t,a)$$

$$\le \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})}$$

and

$$-\left(\int_t^b h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \le K(t,b)$$

$$\le \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})},$$

$\forall\, t \in [a,b]\cap\mathbb{T}$.

That is,

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a) - \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \le f(t)$$

$$\le \sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_k(t,a) + \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})}$$

and

$$\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b) - \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \le f(t)$$

$$\le \sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_k(t,b) + \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})},$$

$\forall\, t \in [a,b]\cap\mathbb{T}$.

Let any $x \in [a,b]\cap\mathbb{T}$, then by integration we get

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_{k+1}(x,a) - \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \Delta t\right)$$

$$\le \int_a^x f(t)\,\Delta t \le$$

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a)\, h_{k+1}(x,a) + \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \Delta t\right), \tag{3.31}$$

and

$$-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_{k+1}(x,b) - \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p\, \Delta\tau\right)^{\frac{1}{p}} \Delta t\right)$$

$$\leq \int_x^b f(t)\,\Delta t \leq$$

$$-\sum_{k=0}^{n-1} f^{\Delta^k}(b)\, h_{k+1}(x,b) + \left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p\, \Delta\tau\right)^{\frac{1}{p}} \Delta t\right). \tag{3.32}$$

Adding (3.31) and (3.32) we obtain

$$\sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b)\right) -$$

$$\left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \left\{\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p\, \Delta\tau\right)^{\frac{1}{p}} \Delta t\right) +\right.$$

$$\left.\left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p\, \Delta\tau\right)^{\frac{1}{p}} \Delta t\right)\right\}$$

$$\leq \int_a^b f(t)\,\Delta t \leq$$

$$\sum_{k=0}^{n-1} \left(f^{\Delta^k}(a)\, h_{k+1}(x,a) - f^{\Delta^k}(b)\, h_{k+1}(x,b)\right) +$$

$$\left\|f^{\Delta^n}\right\|_{L_q([a,b]\cap\mathbb{T})} \left\{\left(\int_a^x \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p\, \Delta\tau\right)^{\frac{1}{p}} \Delta t\right) +\right.$$

$$\left.\left(\int_x^b \left(\int_t^b h_{n-1}(t,\sigma(\tau))^p\, \Delta\tau\right)^{\frac{1}{p}} \Delta t\right)\right\}, \tag{3.33}$$

$\forall\, x \in [a,b] \cap \mathbb{T}$.

The proof is now complete. □

We continue with the next result.

Theorem 3.4. *Let $f \in C_{rd}^n(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b \in \mathbb{T}$; $a \leq b$. Here σ is continuous and $h_{n-m-1}(t,s)$ is jointly continuous. Then*

1)

$$\left|\int_a^b f^{\Delta^m}(t)\,\Delta t-\left(\sum_{k=0}^{n-m-1}\left(f^{\Delta^{k+m}}(a)\,h_{k+1}(x,a)-f^{\Delta^{k+m}}(b)\,h_{k+1}(x,b)\right)\right)\right|\le$$

$$\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}\left[\left(\int_a^x\left(\int_a^t h_{n-m-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right)+\right.$$

$$\left.\left(\int_x^b\left(\int_t^b h_{n-m-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right)\right], \tag{3.34}$$

$\forall\, x\in[a,b]\cap\mathbb{T}$,

2) assuming $f^{\Delta^{k+m}}(a)=f^{\Delta^{k+m}}(b)=0$, $k=0,1,...,n-m-1$, *we get from* (3.34) *that*

$$\left|\int_a^b f^{\Delta^m}(t)\,\Delta t\right|\le\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}\cdot$$

$$\left[\left(\int_a^x\left(\int_a^t h_{n-m-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right)+\left(\int_x^b\left(\int_t^b h_{n-m-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right)\right], \tag{3.35}$$

$\forall\, x\in[a,b]\cap\mathbb{T}$,

2_1) *when* $x=a$ *we get from* (3.35) *that*

$$\left|\int_a^b f^{\Delta^m}(t)\,\Delta t\right|\le\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}\left(\int_a^b\left(\int_t^b h_{n-m-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right), \tag{3.36}$$

2_2) *when* $x=b$ *we get from* (3.35) *that*

$$\left|\int_a^b f^{\Delta^m}(t)\,\Delta t\right|\le\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}\left(\int_a^b\left(\int_a^t h_{n-m-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right), \tag{3.37}$$

2_3) *by* (3.36), (3.37) *we get*

$$\left|\int_a^b f^{\Delta^m}(t)\,\Delta t\right|\le\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}\cdot$$

$$\min\left\{\left(\int_a^b\left(\int_t^b h_{n-m-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right),\left(\int_a^b\left(\int_a^t h_{n-m-1}(t,\sigma(\tau))\,\Delta\tau\right)\Delta t\right)\right\}, \tag{3.38}$$

and

3) assuming $f^{\Delta^{k+m}}(a)=f^{\Delta^{k+m}}(b)=0$, $k=1,...,n-m-1$, *from* (3.34) *we obtain*

$$\left|\int_a^b f^{\Delta^m}(t)\,\Delta t-\left[f^{\Delta^m}(a)(x-a)+f^{\Delta^m}(b)(b-x)\right]\right|\le\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}\cdot$$

$$\left[\left(\int_a^x\left(\int_a^t h_{n-m-1}\left(t,\sigma\left(\tau\right)\right)\Delta\tau\right)\Delta t\right)+\left(\int_x^b\left(\int_t^b h_{n-m-1}\left(t,\sigma\left(\tau\right)\right)\Delta\tau\right)\Delta t\right)\right], \tag{3.39}$$

$\forall\, x\in[a,b]\cap\mathbb{T}$.

Proof. As in the proof of Theorem 3.1, now using Corollary 2.15 (4). □

We give the following theorem.

Theorem 3.5. *Let* $f\in C_{rd}^{n}\left(\mathbb{T}\right)$*,* $m,n\in\mathbb{N}$*,* $m<n$*,* $n-m$ *is odd,* $a,b\in\mathbb{T}$*;* $a\le b$*. Then*
1)

$$\left|\int_a^b f^{\Delta^m}\left(t\right)\Delta t-\sum_{k=0}^{n-m-1}\left(f^{\Delta^{k+m}}\left(a\right)h_{k+1}\left(x,a\right)-f^{\Delta^{k+m}}\left(b\right)h_{k+1}\left(x,b\right)\right)\right|\le$$

$$\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}\left\{\int_a^x\left(t-\sigma\left(a\right)\right)^{n-m-1}\Delta t+\int_x^b\left(\sigma\left(b\right)-t\right)^{n-m-1}\Delta t\right\}, \tag{3.40}$$

$\forall\, x\in[a,b]\cap\mathbb{T}$,

2) assuming $f^{\Delta^{k+m}}\left(a\right)=f^{\Delta^{k+m}}\left(b\right)=0$*,* $k=0,1,...,n-m-1$*, we get from* (3.40) *that*

$$\left|\int_a^b f^{\Delta^m}\left(t\right)\Delta t\right|\le\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}\cdot$$

$$\left\{\int_a^x\left(t-\sigma\left(a\right)\right)^{n-m-1}\Delta t+\int_x^b\left(\sigma\left(b\right)-t\right)^{n-m-1}\Delta t\right\}, \tag{3.41}$$

$\forall\, x\in[a,b]\cap\mathbb{T}$,

$2_1)$ *when* $x=a$ *by* (3.41) *we get*

$$\left|\int_a^b f^{\Delta^m}\left(t\right)\Delta t\right|\le\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}\left(\int_a^b\left(\sigma\left(b\right)-t\right)^{n-m-1}\Delta t\right), \tag{3.42}$$

$2_2)$ *when* $x=b$ *by* (3.41) *we get*

$$\left|\int_a^b f^{\Delta^m}\left(t\right)\Delta t\right|\le\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}\left(\int_a^x\left(t-\sigma\left(a\right)\right)^{n-m-1}\Delta t\right), \tag{3.43}$$

$2_3)$ *by* (3.42), (3.43) *we have*

$$\left|\int_a^b f^{\Delta^m}\left(t\right)\Delta t\right|\le\left\|f^{\Delta^n}\right\|_{L_1([a,b]\cap\mathbb{T})}\cdot$$

$$\min\left\{\left(\int_a^b\left(\sigma\left(b\right)-t\right)^{n-m-1}\Delta t\right),\left(\int_a^b\left(t-\sigma\left(a\right)\right)^{n-m-1}\Delta t\right)\right\}, \tag{3.44}$$

and

3) assuming $f^{\Delta^{k+m}}(a) = f^{\Delta^{k+m}}(b) = 0$*,* $k = 1, ..., n-m-1$*, from* (3.40) *we obtain*

$$\left| \int_a^b f^{\Delta^m}(t)\,\Delta t - \left[f^{\Delta^m}(a)(x-a) + f^{\Delta^m}(b)(b-x) \right] \right| \leq$$

$$\left\| f^{\Delta^n} \right\|_{L_1([a,b]\cap\mathbb{T})} \left\{ \int_a^x (t-\sigma(a))^{n-m-1}\,\Delta t + \int_x^b (\sigma(b)-t)^{n-m-1}\,\Delta t \right\}, \quad (3.45)$$

$\forall\, x \in [a,b] \cap \mathbb{T}$.

Proof. As in Theorem 3.2, now using Corollary 2.15 (4). □

We also give the next result.

Theorem 3.6. *Let* $f \in C_{rd}^n(\mathbb{T})$*,* $m, n \in \mathbb{N}$*,* $m < n$*,* $n-m$ *is odd,* $a, b \in \mathbb{T}$*;* $a \leq b$*. Here* σ *is continuous and* $h_{n-m-1}(t,s)$ *is jointly continuous. Let also* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Then*

1)

$$\left| \int_a^b f^{\Delta^m}(t)\,\Delta t - \sum_{k=0}^{n-m-1} \left(f^{\Delta^{k+m}}(a)\, h_{k+1}(x,a) - f^{\Delta^{k+m}}(b)\, h_{k+1}(x,b) \right) \right| \leq$$

$$\left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \left[\left(\int_a^x \left(\int_a^t h_{n-m-1}(t,\sigma(\tau))^p\,\Delta\tau \right)^{\frac{1}{p}} \Delta t \right) + \right.$$

$$\left. \left(\int_x^b \left(\int_t^b h_{n-m-1}(t,\sigma(\tau))^p\,\Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.46)$$

$\forall\, x \in [a,b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^{k+m}}(a) = f^{\Delta^{k+m}}(b) = 0$*,* $k = 0, 1, ..., n-m-1$*, we get from* (3.46) *that*

$$\left| \int_a^b f^{\Delta^m}(t)\,\Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \left[\left(\int_a^x \left(\int_a^t h_{n-m-1}(t,\sigma(\tau))^p\,\Delta\tau \right)^{\frac{1}{p}} \Delta t \right) + \right.$$

$$\left. \left(\int_x^b \left(\int_t^b h_{n-m-1}(t,\sigma(\tau))^p\,\Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.47)$$

$\forall\, x \in [a,b] \cap \mathbb{T}$,

$2_1)$ *when* $x = a$ *we get from* (3.47) *that*

$$\left| \int_a^b f^{\Delta^m}(t)\,\Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_a^b \left(\int_t^b h_{n-m-1}(t,\sigma(\tau))^p\,\Delta\tau \right)^{\frac{1}{p}} \Delta t \right), \quad (3.48)$$

$2_2)$ *when* $x = b$ *we get from* (3.47) *that*

$$\left| \int_a^b f^{\Delta^m} (t)\, \Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_a^b \left(\int_a^t h_{n-m-1} (t, \sigma(\tau))^p\, \Delta\tau \right)^{\frac{1}{p}} \Delta t \right), \tag{3.49}$$

$2_3)$ *by* (3.48), (3.49) *we get*

$$\left| \int_a^b f^{\Delta^m} (t)\, \Delta t \right| \leq \left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \cdot$$

$$\min \left\{ \left(\int_a^b \left(\int_t^b h_{n-m-1} (t, \sigma(\tau))^p\, \Delta\tau \right)^{\frac{1}{p}} \Delta t \right), \right.$$

$$\left. \left(\int_a^b \left(\int_a^t h_{n-m-1} (t, \sigma(\tau))^p\, \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right\}, \tag{3.50}$$

and

3) assuming $f^{\Delta^{k+m}} (a) = f^{\Delta^{k+m}} (b) = 0$, $k = 1, ..., n-m-1$, *we get from* (3.46) *that*

$$\left| \int_a^b f^{\Delta^m} (t)\, \Delta t - \left[f^{\Delta^m} (a) (x-a) + f^{\Delta^m} (b) (b-x) \right] \right| \leq$$

$$\left\| f^{\Delta^n} \right\|_{L_q([a,b]\cap\mathbb{T})} \left[\left(\int_a^x \left(\int_a^t h_{n-m-1} (t, \sigma(\tau))^p\, \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) + \right.$$

$$\left. \left(\int_x^b \left(\int_t^b h_{n-m-1} (t, \sigma(\tau))^p\, \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right], \tag{3.51}$$

$\forall\, x \in [a, b] \cap \mathbb{T}$.

Proof. As in Theorem 3.3, by using Corollary 2.15 (4). □

4 Applications

We need the following.

Remark 4.1 (See [8]). i) When $\mathbb{T} = \mathbb{R}$, then $h_k (t, s) = \frac{(t-s)^k}{k!}$, $\forall\, k \in \mathbb{N}_0$, $\forall\, t, s \in \mathbb{R}$, $\sigma(t) = t$, $\int_a^b f(t)\, \Delta t = \int_a^b f(t)\, dt$, $f^{\Delta}(t) = f'(t)$, $f^{\Delta^k} = f^{(k)}$; rd-continuous corresponds to f continuous.

ii) When $\mathbb{T} = \mathbb{Z}$, $h_k(t,s) = \frac{(t-s)^{(k)}}{k!}$, $\forall\, k \in \mathbb{N}_0$, $\forall\, t,s \in \mathbb{Z}$, where $t^{(0)} = 1$, $t^{(k)} = \prod_{i=0}^{k-1}(t-i)$ for $k \in \mathbb{N}$, $\sigma(t) = t+1$,

$$\int_a^b f(t)\,\Delta t = \sum_{t=a}^{b-1} f(t), \quad a < b,$$

$$f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t),$$

$$f^{\Delta^k}(t) = \Delta^k f(t) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} f(t+l),$$

rd-continuous f corresponds to any f.

We also need the following.

Remark 4.2 (See [1, 8]). Consider $q > 1$, $q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\}$, and the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$, which is very important in q-difference equations.

It holds that

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - q^{\nu}s}{\sum_{\mu=0}^{\nu} q^{\mu}}, \quad \forall\, s,t \in \mathbb{T};$$

$$\sigma(t) = qt, \ \ \rho(t) = \frac{t}{q}, \ \ \forall\, t \in \mathbb{T},$$

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \ \forall\, t \in \mathbb{T} - \{0\},$$

$$f^{\Delta}(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s}.$$

Next we give applications of our initial main results.

Theorem 4.3. *Let* $f \in C^n([a,b])$, $n \in \mathbb{N}$ *is odd and* $[a,b] \subset \mathbb{R}$. *Then*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(f^{(k)}(a)(x-a)^{k+1} + (-1)^k f^{(k)}(b)(b-x)^{k+1} \right) \right|$$

$$\leq \frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{(n+1)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right], \tag{4.1}$$

$\forall\, x \in [a,b]$.

Proof. By Theorem 3.1, (3.1). □

We continue with the following.

Theorem 4.4. *Let $f \in C^n([a,b])$, $n \in \mathbb{N}$ is odd, $[a,b] \subset \mathbb{R}$. Then*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(f^{(k)}(a)(x-a)^{k+1} + (-1)^k f^{(k)}(b)(b-x)^{k+1} \right) \right|$$

$$\leq \frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n} \left[(x-a)^n + (b-x)^n \right], \tag{4.2}$$

$\forall\, x \in [a,b]$.

Proof. By Theorem 3.2, (3.16). □

We also give the following.

Theorem 4.5. *Let $f \in C^n([a,b])$, $n \in \mathbb{N}$ is odd and $[a,b] \subset \mathbb{R}$. Let also $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(f^{(k)}(a)(x-a)^{k+1} + (-1)^k f^{(k)}(b)(b-x)^{k+1} \right) \right|$$

$$\leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}} \left(n + \frac{1}{p} \right)} \left[(x-a)^{n+\frac{1}{p}} + (b-x)^{n+\frac{1}{p}} \right], \tag{4.3}$$

$\forall\, x \in [a,b]$.

Proof. By Theorem 3.3, (3.25). □

We continue with the following.

Theorem 4.6. *Let $f : \mathbb{Z} \to \mathbb{R}$, n is an odd number, $a, b \in \mathbb{Z}$; $a \leq b$. Then*

$$\left| \sum_{t=a}^{b-1} f(t) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\Delta^k f(a)(x-a)^{(k+1)} - \Delta^k f(b)(x-b)^{(k+1)} \right) \right| \leq$$

$$\frac{\| \Delta^n f \|_{\infty,[a,b]\cap\mathbb{Z}}}{(n-1)!} \left[\left(\sum_{t=a}^{x-1} \left(\sum_{\tau=a}^{t-1} (t-\tau-1)^{(n-1)} \right) \right) + \left(\sum_{t=x}^{b-1} \left(\sum_{\tau=t}^{b-1} (t-\tau-1)^{(n-1)} \right) \right) \right], \tag{4.4}$$

$\forall\, x \in [a,b] \cap \mathbb{Z}$.

Proof. By Theorem 3.1, (3.1), see also Remark 4.1 (ii). □

We give the next result.

Theorem 4.7. *Let* $f : \mathbb{Z} \to \mathbb{R}$, $n \in \mathbb{N}$ *is odd*, $a, b \in \mathbb{Z}$; $a \leq b$. *Then*

$$\left| \sum_{t=a}^{b-1} f(t) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\Delta^k f(a)(x-a)^{(k+1)} - \Delta^k f(b)(x-b)^{(k+1)} \right) \right| \leq$$

$$\left(\sum_{t=a}^{b-1} |\Delta^n f(t)| \right) \left\{ \sum_{t=a}^{x-1} (t-a-1)^{n-1} + \sum_{t=x}^{b-1} (b+1-t)^{n-1} \right\}, \tag{4.5}$$

$\forall\, x \in [a,b] \cap \mathbb{Z}$.

Proof. By Theorem 3.2, (3.16) and Remark 4.1 (ii). □

We give the following result.

Theorem 4.8. *Let* $f : \mathbb{Z} \to \mathbb{R}$, n *is an odd number,* $a, b \in \mathbb{Z}$; $a \leq b$, *let also* $p, q > 1 :$ $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left| \sum_{t=a}^{b-1} f(t) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\Delta^k f(a)(x-a)^{(k+1)} - \Delta^k f(b)(x-b)^{(k+1)} \right) \right| \leq$$

$$\frac{\left(\sum_{t=a}^{b-1} |\Delta^n f(t)|^q \right)^{\frac{1}{q}}}{(n-1)!} \left[\left(\sum_{t=a}^{x-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-1)} \right)^p \right)^{\frac{1}{p}} \right) + \right.$$

$$\left. \left(\sum_{t=x}^{b-1} \left(\sum_{\tau=t}^{b-1} \left((t-\tau-1)^{(n-1)} \right)^p \right)^{\frac{1}{p}} \right) \right], \tag{4.6}$$

$\forall\, x \in [a,b] \cap \mathbb{Z}$.

Proof. By Theorem 3.3, (3.25) and Remark 4.1 (ii). □

We continue with the following theorem.

Theorem 4.9. *Let* $f \in C_{rd}^n\left(q^{\overline{\mathbb{Z}}}\right)$, $n \in \mathbb{N}$ *is odd,* $a, b \in q^{\overline{\mathbb{Z}}}$; $a \leq b$. *Then*

$$\left| \int_a^b f(t)\,\Delta t - \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) \prod_{\nu=0}^{k} \frac{x - q^\nu a}{\sum_{\mu=0}^{\nu} q^\mu} - f^{\Delta^k}(b) \prod_{\nu=0}^{k} \frac{x - q^\nu b}{\sum_{\mu=0}^{\nu} q^\mu} \right) \right| \leq$$

$$\left\| f^{\Delta^n} \right\|_{L_1\left([a,b] \cap q^{\overline{\mathbb{Z}}}\right)} \left\{ \int_a^x (t-qa)^{n-1}\,\Delta t + \int_x^b (qb-t)^{n-1}\,\Delta t \right\}, \tag{4.7}$$

$\forall\, x \in [a,b] \cap q^{\overline{\mathbb{Z}}}$.

Proof. By Theorem 3.2, (3.16), and Remark 4.2. □

We finish with the next result.

Theorem 4.10. *Let* $f \in C_{rd}^{n}\left(q^{\overline{\mathbb{Z}}}\right)$, $m, n \in \mathbb{N}$*;* $m < n$, $n - m$ *is odd,* $a, b \in q^{\overline{\mathbb{Z}}}$*;* $a \leq b$*. Then*

$$\left| \int_a^b f^{\Delta^m}(t)\,\Delta t - \sum_{k=0}^{n-m-1} \left(f^{\Delta^{k+m}}(a) \prod_{\nu=0}^{k} \frac{x - q^\nu a}{\sum\limits_{\mu=0}^{\nu} q^\mu} - f^{\Delta^{k+m}}(b) \prod_{\nu=0}^{k} \frac{x - q^\nu b}{\sum\limits_{\mu=0}^{\nu} q^\mu} \right) \right| \leq$$

$$\left\| f^{\Delta^n} \right\|_{L_1\left([a,b]\cap q^{\overline{\mathbb{Z}}}\right)} \left\{ \int_a^x (t - qa)^{n-m-1}\,\Delta t + \int_x^b (qb - t)^{n-m-1}\,\Delta t \right\}, \tag{4.8}$$

$\forall\, x \in [a, b] \cap q^{\overline{\mathbb{Z}}}$.

Proof. By Theorem 3.5, (3.40), and Remark 4.2. □

One can give many similar applications for other time scales.

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