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Global Dynamics of Higher-Order Transcendental-Type Generalized Beverton–Holt Equations

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Abstract

We investigate generalized Beverton–Holt difference equations of order k of the form

$$x_{n+1} = \frac{af(x_n, x_{n-1}, \ldots, x_{n+1-k})}{1 + f(x_n, x_{n-1}, \ldots, x_{n+1-k})}, \quad n = 0, 1, \ldots, \quad k \geq 1,$$

where f is a function nondecreasing in all arguments, $a > 0$, and $x_0, \ldots, x_{1-k} \geq 0$ such that the solution is defined. We will discuss several interesting examples of such equations involving transcendental functions and present some general theory. In particular, we will analyze the global dynamics of the class of difference equations for which $f(x, \ldots, x)$ is chosen to be a concave function. Moreover, we give sufficient conditions to guarantee this equation has a unique positive and globally attracting fixed point.

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1 Introduction and Preliminaries

Consider the following order-k difference equation:

$$x_{n+1} = \frac{af(x_n, x_{n-1}, \ldots, x_{n+1-k})}{1 + f(x_n, x_{n-1}, \ldots, x_{n+1-k})}, \quad n = 0, 1, \ldots, \quad k \geq 1, \tag{1.1}$$

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where f is a continuous function nondecreasing in all arguments, the parameter a is a positive real number, and the initial conditions $x_0, x_{-1}, \ldots, x_{1-k}$ are arbitrary nonnegative numbers such that the solution is defined. We assume f is never identically equal to the zero function.

Equation (1.1) is a generalization of the first-order Beverton–Holt equation

$$x_{n+1} = \frac{ax_n}{1 + x_n}, \tag{1.2}$$

where $a > 0$ and $x_{-1}, x_0 \geq 0$. Global dynamics are known and may be summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & a \leq 1 \\ a - 1 & a > 1 \text{ and } x_0 > 0. \end{cases}$$

Many variations of Equation (1.2) have been studied. The form of the model actually predates its use by Beverton and Holt; see [16]. German biochemist Leonor Michaelis and Canadian physician Maud Menten used the model in their study of enzyme kinetics in 1913. Additionally, Jacques Monod, a French biochemist, happened upon the model empirically in his study of microorganism growth around 1942. It was not until 1957 that fisheries scientists Ray Beverton and Sidney Holt used the model in their study of population dynamics; see [16].

For instance, the so-called Monod system of differential equations is given by

$$\frac{dS}{dt} = -\frac{1}{\gamma} N \frac{rS}{a+S}, \quad \frac{dN}{dt} = N \frac{rS}{a+S}, \tag{1.3}$$

where $N(t)$ is the concentration of bacteria at time t, $\frac{dN}{dt}$ is the growth rate of the bacteria, $S(t)$ is the concentration of the nutrient, r is the maximum growth rate of the bacteria, k is a half-saturation constant, and the constant γ is called the growth yield; see [16]. Both Equation (1.2) and System (1.3) contain the function $f(x) = rx/(a + x)$ known as the Monod function, Michaelis-Menten function, Beverton–Holt function, or Holling function of the first kind; see [1, 4, 9, 11]. Some global dynamic scenarios of several two-generation models using this function were obtained in [3]. The special case of Equation (1.1) where $f(x_n, x_{n-1}) = ax_n + bx_{n-1}$ and $a, b > 0$ was investigated in [15].

The Beverton–Holt function is an increasing and concave function and we will prove some global attractivity results for general difference equations with a transition function that is increasing and concave along the diagonal. More precisely, we will prove some global attractivity results for Equation (1.1), where $f(x, \ldots, x)$ is an increasing and concave function.

The following theorem from [2] applies to Equation (1.1) when $k = 2$.

Theorem 1.1. *Let I be a set of real numbers and $F : I \times I \to I$ be a function which is nondecreasing in both variables. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = F(x_n, x_{n-1}), \qquad x_{-1}, x_0 \in I, \ n = 0, 1, \ldots, \tag{1.4}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ *and* $\{x_{2n-1}\}_{n=0}^{\infty}$ *of even and odd terms of the solution are eventually monotonic.*

The consequence of Theorem 1.1 is that every bounded solution of Equation (1.4) converges to either an equilibrium, a period-two solution, or to a singular point on the boundary. Notice that Theorem 1.1 does not apply if $k > 2$, but the results from [6,8,12] can give global dynamics in some regions of the parametric space. In the case $k > 2$, Equation (1.1) may have periodic solutions of different periods and even chaos; see [7].

The following theorem from [10] applies to the kth-order Equation (1.1) and will be instrumental in establishing our main result.

Theorem 1.2. *Consider the equation*

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n+1-k}), \quad x_0, x_{-1}, \ldots, x_{1-k} \in I, \quad n = 0, 1, \ldots, \tag{1.5}$$

where $I \subseteq [0, \infty)$ *is some open interval, and assume that* $F \in C[I^k, (0, \infty)]$ *satisfies the following conditions:*

(i) F *is nondecreasing in each of its arguments;*

(ii) Equation (1.5) *has a unique positive equilibrium point* $\bar{x} \in I$ *and the function* F *satisfies the* ***negative feedback condition****:*

$$(x - \bar{x})(F(x, \ldots, x) - x) < 0 \text{ for every } x \in I \backslash \{\bar{x}\}.$$

Then every solution of Equation (1.5) *with initial conditions* $x_0, x_{-1}, \ldots, x_{1-k}$ *in* I *converges to* $\bar{x}$*.*

The rest of this paper is organized as follows. The next section utilizes the concavity and increasing character of the transition function to analyze the local and global stability of the zero and positive equilibrium solutions. In view of the fact that Theorem 1.1 does not hold in higher dimensions, the obtained results are particularly relevant. The third section will provide some examples of global dynamics of Equation (1.1) when the function $f(u_1, \ldots, u_k)$ is defined using either exponential, trigonometric, or linear functions. The obtained results will be interesting from a modeling point of view as they cover a wide range of nonlinear functions such as logistic, inverse tangent, and polynomial functions.

2 General Stability Results and Global Attractivity

Let the function $F : [0, \infty)^k \to [0, a)$ be defined as follows:

$$F(u_1, \ldots, u_k) = \frac{af(u_1, \ldots, u_k)}{1 + f(u_1, \ldots, u_k)}. \tag{2.1}$$

Using Equation (2.1), Equation (1.1) may be rewritten as $x_{n+1} = F(x_n, \ldots, x_{n+1-k})$ for all $n = 0, 1, \ldots,$ where F is a nondecreasing function in all its variables. It is clear that $0 \leq x_n < a$ for all $n \geq 1$.

It will be useful to examine the multivariable functions f and F along the diagonal. For convenience, make the following definitions:

$$g(x) = f(x, \dots, x) \tag{2.2}$$
$$G(x) = F(x, \dots, x). \tag{2.3}$$

An equilibrium $\bar{x}$ of Equation (1.1) satisfies

$$\bar{x}\,(1 + g(\bar{x})) = ag(\bar{x}). \tag{2.4}$$

Clearly $\bar{x}_0 = 0$ is an equilibrium point if and only if $g(0) = f(0, \dots, 0) = 0$.

2.1 Local Stability of an Equilibrium

The linearized equation of Equation (1.1) about an equilibrium $\bar{x}$ is

$$z_{n+1} = F_{u_1}(\bar{x}, \dots, \bar{x})z_n + \dots + F_{u_k}(\bar{x}, \dots, \bar{x})z_{n+1-k}, \quad n = 0, 1, \dots.$$

Set

$$\lambda(\bar{x})_k = \sum_{i=1}^{k} F_{u_i}(\bar{x}, \dots, \bar{x}) = \frac{a\sum_{i=1}^{k} f_{u_i}(\bar{x}, \dots, \bar{x})}{(1 + f(\bar{x}, \dots, \bar{x}))^2}. \tag{2.5}$$

In view of [12, Corollary 2] we have the following result.

Theorem 2.1. *Let $\bar{x}$ be an equilibrium of Equation* (1.1). *Then*

$$\bar{x} \textit{ is } \begin{cases} \textit{locally asymptotically stable} & \textit{if } \lambda(\bar{x})_k < 1 \\ \textit{nonhyperbolic} & \textit{if } \lambda(\bar{x})_k = 1 \\ \textit{unstable} & \textit{if } \lambda(\bar{x})_k > 1. \end{cases}$$

2.2 Existence and Global Attractivity of a Unique Positive Equilibrium

We will now establish several sufficient conditions under which Equation (1.1) will have a unique positive fixed point. Recall the definitions of G and g given in Equations (2.2) and (2.3).

Lemma 2.2. *Suppose G is twice differentiable and satisfies the following three conditions:*
(i) $G(0) = 0$,
(ii) $G'(0) > 1$, and
(iii) $G''(x) < 0$ for all $x \in (0, a)$.
Then Equation (1.1) *has a unique positive equilibrium.*

Remark 2.3. Notice that $G(0) = 0$ if and only if $g(0) = 0$. If indeed $G(0) = g(0) = 0$ then $G'(0) = ag'(0)$. Further, since $x \geq 0$, we interpret derivatives at zero in the right-handed sense.

Proof. First we will show that there exists a positive equilibrium for Equation (1.1). First, let $H(x) = G(x) - x$. Notice that $H(0) = 0$ and $H(a) < 0$, as

$$H(a) = G(a) - a = F(a, \ldots, a) - a = \frac{af(a, \ldots, a)}{1 + f(a, \ldots, a)} - a < a - a = 0.$$

Also, $H'(0) = G'(0) - 1 > 0$ by assumption (ii) and hence H is increasing at $x = 0$; by continuity of H', for any sufficiently small $\delta > 0$ it must be the case that $H(\delta) > 0$. But since $H(\delta) > 0$ and $H(a) < 0$, by the intermediate value theorem there exists some point $p \in (\delta, a)$ such that $H(p) = 0$. But this immediately implies that $G(p) = p$, and hence p is a fixed point of Equation (1.1), as required.

Next we will show this fixed point is unique. Suppose there are two fixed points $p_1, p_2 > 0$ of Equation (1.1) such that $p_1 < p_2$. Since $G''(x) < 0$ for all $x \in (0, a)$, the function is *strictly* concave on this interval; that is, for all $t \in (0, 1)$ and all $x, y \in (0, \infty)$ with $x \neq y$,

$$G(tx + (1 - t)y) > tG(x) + (1 - t)G(y). \tag{2.6}$$

Let $b \in (0, p_1)$ be arbitrary and set $t = \dfrac{p_2 - p_1}{p_2 - b}$. Here $t \in (0, 1)$ since $0 < b < p_1 < p_2$. By Inequality (2.6), if $x = b$ and $y = p_2$, we obtain the following:

$$G\left(\left(\frac{p_2 - p_1}{p_2 - b}\right) b + \left(1 - \frac{p_2 - p_1}{p_2 - b}\right) p_2\right) > \left(\frac{p_2 - p_1}{p_2 - b}\right) G(b) + \left(1 - \frac{p_2 - p_1}{p_2 - b}\right) p_2$$
$$\iff p_1 = G(p_1) > \frac{G(b)\,(p_2 - p_1) + p_2\,(p_1 - b)}{p_2 - b}$$
$$\iff p_1(p_2 - b) - p_2(p_1 - b) > G(b)(p_2 - p_1) \iff b > G(b).$$

Therefore for each $b \in (0, p_1)$, $H(b) = G(b) - b < 0$. However, this contradicts our initial claim that $H(\delta) > 0$ for $\delta > 0$ small enough. □

Lemma 2.4. *Suppose G is twice differentiable and satisfies the following three conditions:*
(i) $G(0) = 0$*,*
(ii) $G'(0) \leq 1$*, and*
(iii) $G''(x) < 0$ *for all* $x \in (0, a)$*.*
Then there exists no positive fixed point for Equation (1.1).

Proof. If $H(x) = G(x) - x$, then $H'(x) = G'(x) - 1$ and $H''(x) = G''(x)$, so in particular $H''(x) < 0$ for all $x \in (0, a)$. For any $x \in (0, a]$ we may apply the mean value theorem to H' over $[0, x]$ to conclude that there exists some $c \in (0, x)$ such that

$$\frac{H'(x) - H'(0)}{x - 0} = H''(c).$$

But since $H''(c) < 0$, we have that $H'(x) < H'(0) \leq 0$ and hence H is strictly decreasing for all $x \in (0,a)$. But since $H(0) = 0$, we have that $H(x) < 0$ (and hence $G(x) < x$) for all $x \in (0,a)$, and therefore in this case there exist no positive fixed points for G. □

Theorem 2.5. *Under the hypotheses of Lemma 2.2, the unique positive equilibrium of* (1.1) *is a global attractor of all solutions with positive initial conditions.*

Proof. By Lemma 2.2, Equation (1.1) has a unique positive fixed point p. Now $H(x) = G(x) - x$ is continuous and has only one positive root (at $x = p$) such that it does not change sign on $(0,p)$ or (p,a); in particular, $H(x) > 0$ for $x \in (0,p)$ and $H(x) < 0$ for $x \in (p,a)$. If $I = (0,a)$, we have that $(x-p)(G(x)-x) < 0$ for all $x \in I\backslash\{p\}$. By Theorem 1.2, we have that every positive solution with initial conditions in I converges to p. Since $(0,a)$ is an attracting, invariant interval for all solutions with positive initial conditions, the proof is complete. □

Remark 2.6. If $(0,a)$ is an attracting interval for *all* nonzero solutions, including those with initial conditions that are not all necessarily positive, then the results of Theorem 2.5 (and later Corollary 2.8) will give a complete classification of global dynamics for any choice of nonnegative initial conditions.

Theorem 2.7. *Under the hypotheses of Lemma 2.4, the zero equilibrium is a global attractor of all solutions.*

Proof. By Lemma 2.4, Equation (1.1) has only the zero equilibrium in the invariant interval $[0,a]$. But then the kth-order extension of [13, Theorems 1.4.8 and A.0.1] or [14, Theorem 4] will apply to this equation. Since $[0,a]$ is an attracting interval, all solutions must converge to the zero equilibrium. □

Corollary 2.8. *Suppose $g(x)$ is a strictly concave function on $(0,a)$.*
(1) Under hypotheses (i) and (ii) of Lemma 2.2, the unique positive equilibrium of Equation (1.1) *is a global attractor of all solutions with positive initial conditions.*
(2) Under hypotheses (i) and (ii) of Lemma 2.4, the zero equilibrium is a global attractor of all solutions.

Proof. Since $g(x) = f(x,\ldots,x)$ is strictly concave for all $x \in (0,a)$, $g''(x) < 0$. An immediate computation yields

$$G''(x) = \frac{a\left[g''(x)\left(1+g(x)\right) - 2\left(g(x)\right)^2\right]}{\left(1+g(x)\right)^3} < 0.$$

Thus condition (iii) is satisfied for Lemmas 2.2 and 2.4, and the proof follows from an application of Theorems 2.5 and 2.7. □

Remark 2.9. Corollary 2.8 shows that $g(x)$ being concave is a sufficient but not necessary condition for $G(x)$ to be concave. For the case $k = 2$, consider $f(u, v) = pu^2 + qv$. If $a = 1, p = 1, q = 2$, then

$$g''(x) = \frac{d^2}{dx^2}(f(x,x)) = 2 > 0 \text{ yet } G''(x) = \frac{d^2}{dx^2}(F(x,x)) = \frac{-6}{(1+x)^4} < 0,$$

so for these values $G(x)$ is concave even though $g(x)$ is convex.

In some cases neither the function $g(x)$ nor $G(x)$ is concave on the interval $(0, a)$. In such situations it is useful to have the following theorem, which provides a sufficient condition to guarantee the existence (or nonexistence) of a unique positive fixed point that is a global attractor of positive solutions.

Theorem 2.10. *Let g be continuously differentiable such that $g'(0) \neq 0$ and $g(x) > 0$ for all $x > 0$. If*

$$xg'(x) < g(x)(g(x) + 1) \tag{2.7}$$

for all $x \in (0, a)$, then Equation (1.1) *has at most one positive fixed point.*
(1) If $G(0) = g(0) = 0$ and $G'(0) = ag'(0) > 1$, then Equation (1.1) *has precisely one positive fixed point, and it is a global attractor of all solutions with positive initial conditions.*
(2) If $G(0) = g(0) = 0$ and $G'(0) = ag'(0) \leq 1$, then Equation (1.1) *has only the zero equilibrium, and it is a global attractor of all solutions.*

Proof. Solve Equation (2.4) for a to find that

$$a = \frac{x}{g(x)} + x.$$

Set $u(x) = \dfrac{x}{g(x)} + x$. If $u(x)$ is an injective (or monotone) function, then it intersects the line $y = a$ at most once. Setting $u'(x) > 0$ and rearranging will establish the main claim.

To prove the remaining claims, suppose $u'(x) > 0$. By l'Hôpital's rule,

$$\lim_{x\to 0^+} u(x) = \lim_{x\to 0^+}\left(\frac{x}{g(x)} + x\right) = \lim_{x\to 0^+}\frac{1}{g'(x)} = \frac{1}{g'(0)}. \tag{2.8}$$

Now $\lim_{x\to 0^+} u(x) < a$ implies there exists exactly one positive fixed point of Equation (1.1), so Equation (2.8) establishes the hypothesis of (1). As in the proof of Theorem 2.5, the global attractivity of the unique fixed point will again follow from Theorem 1.2.

If $\lim_{x\to 0^+} u(x) \geq a$, then Equation (1.1) has only the zero equilibrium since u is increasing, and Equation (2.8) establishes the hypothesis of (2). Again we may employ the order-k generalization of [13, Theorems 1.4.8 and A.0.1] or [14, Theorem 4] to obtain the global attractivity of the zero equilibrium, and the proof is complete. □

Remark 2.11. In some cases the veracity of Inequality (2.7) of Theorem 2.10 may imply the concavity condition required by Theorems 2.5 or 2.7 or Corollary 2.8, but the hypotheses of the latter results may be easier to verify.

3 Examples

In most of the provided examples we will focus on equations of second order for concision. However, all results can be generalized to corresponding equations of any order.

3.1 Exponential Nonlinearity: $f(u,v) = p(1-e^{-u}) + q(1-e^{-v})$

We consider the equation

$$x_{n+1} = \frac{a\left(p(1-e^{-x_n}) + q(1-e^{-x_{n-1}})\right)}{1 + p(1-e^{-x_n}) + q(1-e^{-x_{n-1}})}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where $p, q > 0$. An equilibrium $\bar{x}$ of Equation (3.1) satisfies the following:

$$\bar{x} = \frac{a(p+q)(1-e^{-\bar{x}})}{1 + (p+q)(1-e^{-\bar{x}})}.$$

In particular, $\bar{x}_0 = 0$ has $\lambda(\bar{x}_0) = a(p+q)$, so

$$\bar{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } a(p+q) < 1 \\ \text{nonhyperbolic} & \text{if } a(p+q) = 1 \\ \text{unstable} & \text{if } a(p+q) > 1. \end{cases}$$

The following results give the global dynamics of Equation (3.1).

Theorem 3.1. *(1) If $a(p+q) > 1$, then there exists a unique positive equilibrium $\bar{x}_+$, and it is a global attractor of all nonzero solutions.*
(2) If $a(p+q) \leq 1$, then $\bar{x}_0 = 0$ is a global attractor of all solutions.

Proof. Notice $G(0) = 0$, $G'(0) = ag'(0) = a(p+q)$, and $g''(x) = \dfrac{d^2}{dx^2}(f(x,x)) = -e^{-x}(p+q) < 0$. Moreover, if $x_{-1} + x_0 > 0$, then $x_1 = F(x_0, x_{-1}) > 0$ since $p, q > 0$, and similarly $x_2 > 0$. Thus all solutions enter the attracting, invariant interval $(0, a)$. In view of Remark 2.6, the result follows by a direct application of Corollary 2.8. □

We may also consider the kth-order equation

$$x_{n+1} = \frac{a \sum_{i=0}^{k-1} p_i (1-e^{-x_{n-i}})}{1 + \sum_{i=0}^{k-1} p_i (1-e^{-x_{n-i}})}, \quad n = 0, 1, \ldots, \tag{3.2}$$

where $p_i \geq 0$ for $i = 0, \ldots, k-1$. We can establish global results for Equation (3.2) by immediately applying Corollary 2.8.

Theorem 3.2. *(1) If* $a\sum_{i=0}^{k-1} p_i > 1$*, then there exists a unique positive equilibrium* $\bar{x}_+$*, and it is a global attractor of all solutions with positive initial conditions.*
(2) If $a\sum_{i=0}^{k-1} p_i \leq 1$*, then* $\bar{x}_0 = 0$ *is a global attractor of all solutions.*

However, notice that we cannot necessarily establish global dynamics for all values of the nonnegative parameters and initial conditions. Equation (3.2) may have a variety of periodic solutions in which some of the entries in the periodic cycle equal zero. However, the above result captures the substantial global dynamics for all solutions with positive initial conditions.

3.2 Arctangent Nonlinearity: $f(u,v) = p\arctan(u) + q\arctan(v)$

We next consider the equation

$$x_{n+1} = \frac{a\,(p\arctan(x_n) + q\arctan(x_{n-1}))}{1 + p\arctan(x_n) + q\arctan(x_{n-1})}, \quad n = 0, 1, \ldots, \tag{3.3}$$

where $p, q > 0$. An equilibrium $\bar{x}$ of Equation (3.3) satisfies the following:

$$\bar{x} = \frac{a(p+q)\arctan(\bar{x})}{1 + (p+q)\arctan(\bar{x})}.$$

Again $\bar{x}_0 = 0$ has $\lambda(\bar{x}_0) = a(p+q)$, so

$$\bar{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } a(p+q) < 1 \\ \text{nonhyperbolic} & \text{if } a(p+q) = 1 \\ \text{unstable} & \text{if } a(p+q) > 1. \end{cases}$$

Notice that, as in the previous second-order example, $G(0) = 0$, $G'(0) = a(p+q)$, and $g''(x) = \frac{d^2}{dx^2}\left(f(x,x)\right) = -\frac{2x(p+q)}{(1+x^2)^2} < 0$ for $x > 0$. It is clear that the global dynamics of Equation (3.3) are described exactly by Theorem 3.1.

Remark 3.3. There are a wealth of other functions f such that $g(x)$ is concave and Corollary 2.8 applies to Equation (1.1). Second-order examples include the logarithmic function

$$f_1(u,v) = \log\left((1+u)^p(1+v)^q\right)$$

as well as the shifted sigmoid function

$$f_2(u,v) = \frac{p}{1+e^{-u}} + \frac{q}{1+e^{-v}} - \frac{p+q}{2} = \frac{p}{2}\tanh\left(\frac{u}{2}\right) + \frac{q}{2}\tanh\left(\frac{v}{2}\right),$$

which has a wide range of applications in neural networks.

3.3 Sine Nonlinearity: $f(u,v) = p\,(u+\sin(u)) + q\,(v+\sin(v))$

Consider the equation

$$x_{n+1} = \frac{a\,(p\,(x_n+\sin(x_n)) + q\,(x_{n-1}+\sin(x_{n-1})))}{1+p\,(x_n+\sin(x_n)) + q\,(x_{n-1}+\sin(x_{n-1}))}, \quad n = 0,1,\ldots, \tag{3.4}$$

where $p, q > 0$. Notice that $f_u(u,v) = p(1+\cos(u)) \geq 0$ and $f_v(u,v) = q(1+\cos(v)) \geq 0$. The second-order difference equation $x_{n+1} = \frac{1}{2} f(x_n, x_{n-1})$ for $p = q = 1$ was investigated in [5, Example 1].

The applicability of Corollary 2.8 is limited by the fact that $g(x) = (p+q)(\sin(x)+x)$ is strictly concave only when $\sin(x) > 0$, and therefore global results can only be obtained for $a \leq \pi$. Using the full strength of Theorems 2.5 and 2.7 will also have limitations for any choice of $a > 0$; the interval $[0,a]$ is always invariant for Equation (3.4), but a larger value of a would prescribe the need for a larger interval over which $G(x)$ should be concave. Instead we may consider applying Theorem 2.10.

Theorem 3.4. *Suppose that, for all $x \in (0,a)$,*

$$x\cos(x) < (p+q)(\sin(x)+x)^2 + \sin(x). \tag{3.5}$$

(1) If $2a(p+q) > 1$, then Equation (3.4) *has precisely one positive fixed point, and it is a global attractor of all solutions with positive initial conditions.*
(2) If $2a(p+q) \leq 1$, then Equation (3.4) *has only the zero equilibrium, and it is a global attractor of all solutions.*

Remark 3.5. Verifying Inequality (3.5) in general appears to be difficult, although for specific values of p and q this hypothesis should be able to be easily checked. For example, when $p = q = 1$, this condition is immediately satisfied and leads to a global exchange of stability result as a passes through the critical value $\frac{1}{4}$. In general, *Mathematica* verifies this inequality should hold for all $x > 0$ when approximately $p + q > 0.2015$. For p and q smaller than this threshold, multiple equilibria or even interior periodic solutions may exist.

3.4 Order-k Linear

Consider the equation

$$x_{n+1} = \frac{a\sum_{i=0}^{k-1} c_i x_{n-i}}{1+\sum_{i=0}^{k-1} c_i x_{n-i}}, \quad n = 0,1,\ldots, \tag{3.6}$$

where $c_i \geq 0$ for $i = 1, \ldots, k-1$. An equilibrium $\bar{x}$ of Equation (3.6) satisfies the following:

$$\bar{x} = \frac{a \sum_{i=0}^{k-1} c_i \bar{x}}{1 + \sum_{i=0}^{k-1} c_i \bar{x}}.$$

Using Equation (2.5) we see that the zero equilibrium $\bar{x}_0 = 0$ has $\lambda(\bar{x}_0)_k = a \sum_{i=0}^{k-1} c_i$, so

$$\bar{x}_0 \text{ is } \begin{cases} \text{locally asymptotically stable} & \text{if } a \sum_{i=0}^{k-1} c_i < 1 \\ \text{nonhyperbolic} & \text{if } a \sum_{i=0}^{k-1} c_i = 1 \\ \text{unstable} & \text{if } a \sum_{i=0}^{k-1} c_i > 1. \end{cases}$$

If $a \sum_{i=0}^{k-1} c_i > 1$, then Equation (3.6) has the unique positive equilibrium

$$\bar{x}_+ = \frac{a \left(\sum_{i=0}^{k-1} c_i \right) - 1}{\sum_{i=0}^{k-1} c_i}.$$

Since

$$\lambda(\bar{x}_+)_k = \frac{1}{a \sum_{i=0}^{k-1} c_i} < 1,$$

we have that $\bar{x}_+$ is locally asymptotically stable whenever it exists. The next result, which gives the global dynamics of Equation (3.6), is a simple exchange of stability bifurcation result.

Theorem 3.6. *Consider Equation* (3.6).

(1) If $a \sum_{i=0}^{k-1} c_i \leq 1$*, then* $\bar{x}_0$ *is a global attractor of all solutions.*

(2) If $a \sum_{i=0}^{k-1} c_i > 1$*, then* $\bar{x}_+$ *is a global attractor of all solutions with positive initial conditions.*

Proof. The proof of (1) is the same as that of Theorem 2.7. To prove (2), notice that

$$\begin{aligned} x_{n+1} - \bar{x}_+ &= \frac{a\sum_{i=0}^{k-1} c_i x_{n-i}}{1+\sum_{i=0}^{k-1} c_i x_{n-i}} - \frac{a\sum_{j=0}^{k-1} c_j - 1}{\sum_{j=0}^{k-1} c_j} \\ &= \frac{\sum_{i=0}^{k-1} c_i \left(x_{n-i} - \bar{x}_+\right) + \sum_{j=0}^{k-1} c_j \left(\bar{x}_+ - a\right) + 1}{\left(1+\sum_{i=0}^{k-1} c_i x_{n-i}\right)\sum_{j=0}^{k-1} c_j} \\ &= \frac{\sum_{i=0}^{k-1} c_i \left(x_{n-i} - \bar{x}_+\right)}{\left(1+\sum_{i=0}^{k-1} c_i x_{n-i}\right)\sum_{j=0}^{k-1} c_j}. \end{aligned}$$

Let

$$g_l = \frac{c_l}{\left(1+\sum_{i=0}^{k-1} c_i x_{n-i}\right)\sum_{j=0}^{k-1} c_j}.$$

The substitution $y_n = x_n - \bar{x}_+$ yields

$$y_{n+1} = \sum_{l=0}^{k-1} g_l y_{n-l}.$$

Now we have

$$\sum_{l=0}^{k-1} |g_l| = \frac{1}{1+\sum_{i=0}^{k-1} c_i x_{n-i}} \leq \frac{1}{1+M} < 1$$

for some $M > 0$ so long as $\sum_{i=0}^{k-1} c_i x_{n-i} > 0$. The latter is true by assumption since $c_i > 0$ for at least one i and the initial conditions satisfy $x_{1-j} > 0$ for each $j = 1, \dots, k$. By [12, Theorem 1], $\lim_{n\to\infty} y_n = 0$, and hence $\lim_{n\to\infty} x_n = \bar{x}_+$. □

Remark 3.7. Theorem 3.6 is proven using the powerful linearization technique discussed in [12]. However, we could also use Theorems 2.5 and 2.7 to arrive at the same result.

Proof. Notice that in the case of Equation (3.6) we have $g(x) = x\sum_{i=0}^{k-1} c_i$ and hence

$G(x) = \dfrac{ax \sum_{i=0}^{k-1} c_i}{1 + x \sum_{i=0}^{k-1} c_i}$. Now $G(0) = 0$, $G'(0) = a \sum_{i=0}^{k-1} c_i$, and

$$G''(x) = \frac{-2a \left(\sum_{i=0}^{k-1} c_i \right)^2}{\left(1 + x \sum_{i=0}^{k-1} c_i \right)^3} < 0 \text{ for all } x \geq 0.$$

Thus an application of Theorems 2.5 and 2.7 completes the proof. □

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