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Optimal Control of Quadrotor Unmanned Aerial Vehicles on Time Scales

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Abstract

In this paper, an optimal controller is developed for quadrotor unmanned aerial vehicles (UAVs) on time scales. The UAVs are assumed to have desired positions and orientations and the proposed controller is used to bring the UAVs to the desired positions and orientations by minimizing a cost function on time scales. The proposed controller will be able to work for general time scales such as the discrete time intervals with time varying sampling interval and the bounded graininess. This will provide several benefits such as computational cost reduction in real time applications. The effectiveness of our optimal controller of quadrotor UAVs is demonstrated in a simulation, which validates our theoretical claims.

AMS Subject Classifications: 34N05, 37N35, 37N40, 39A13, 39A60, 93B05, 93C05, 93C10, 93C15.
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1 Introduction

During recent decays, quadrotor helicopters have quickly become one of the most popular unmanned aerial vehicle (UAV) platform. Its popularity comes mainly from its

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simple construction when compared to the conventional helicopters. Also, its suitability for applications like surveillance and search and rescue can be mentioned as benefits of UAVs. Mechanically, a quadrotor UAV employs fixed pitch rotors so that its rotor speed can be adjusted to achieve control as opposed to mechanical control linkages used in conventional helicopters. Thus, a quadrotor UAV is easier to build and maintain, see [7].

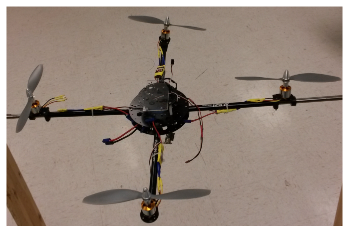

Figure 1.1: Quadrotor UAV

We consider the quadrotor UAV shown in Figure 1 modeled as

$$\begin{cases} x^{\Delta\Delta}(t) = -\dfrac{u(t)\sin\theta(t)}{m} \\ y^{\Delta\Delta}(t) = \dfrac{u(t)\cos\theta(t)\sin\phi(t)}{m} \\ z^{\Delta\Delta}(t) = \dfrac{u(t)\cos\theta(t)\cos\phi(t)}{m} - g \\ \phi^{\Delta\Delta}(t) = u_\phi(t) \\ \theta^{\Delta\Delta}(t) = u_\theta(t) \\ \psi^{\Delta\Delta}(t) = u_\psi(t), \end{cases} \tag{1.1}$$

where (x, y, z) are position coordinates, m is the total mass of the quadrotor, g is the gravity, (ϕ, θ, ψ) are orientations of UAV referred as roll, pitch and yaw, respectively, and u, u_ϕ, u_θ and u_ψ are described as controllers. For the real time controller of Quadrotor UAVs, computation cost can be significant since the embedded microprocesssors have limited capacity. Therefore, reducing computation of the built–in controller of unmanned vehicles is more desirable than the systems those have bigger processors. Motivated with that, in the recent years, some works addressed resource–aware implementations of the control law using event–triggered sampling, where the control value is updated only when some events occur. An event is usually generated by an event–function that indicates if the control signal must be updated or not. Typical event–detection mechanisms are functions on the variation of the state (or at least the output) of system (1.1), see [8]. In particular, the idea is to show that an event–triggered scheme could reduce the number of samples even in such a case where rotor blades have to be actively controlled in [8]. However, checking the event trigger condition can also increase the computation cost. Therefore, we propose a novel optimal control strategy in order to stabilize the quadrotor in attitude and position on time scales. Choosing a suitable time scale for applications yields a significant computation reduction. This control law incorporated by applying the Linear Quadratic Regulator approach and using the nonlinear dynamical model by an exact linearization. The Linear Quadratic Regulator on time scales is an optimal control approach used in order to minimize the cost function proposed in advance. More information about the linear quadratic regulator can be found in [15].

A time scale, denoted by $\mathbb{T}$, is a nonempty closed subset of real numbers and was introduced by Stefan Hilger in 1988 in his PhD thesis in order to harmonize discrete and continuous analyses to combine them in one comprehensive theory and eliminate obscurity from both. The time scale theory was published in a series of two books by Bohner and Peterson in 2001 and 2003, see [3] and [4]. System (1.1) is reduced to the system of differential equations and difference equations, see [10, 13] respectively.

In [14], it is used the continuous linear optimal control for a fixed wing UAVs while it is applied continuous suboptimal control to quadrotors by using Control Lyapunov functions in [12]. In [1, 5] , it can also be found other approaches applied to mini helicopters and four-rotor helicopters, respectively.

The remainder of the paper is organized as follows: In Section 2, we stabilize the quadrotor states by using the optimal control laws by means of performance index. In Section 3, we give a couple of examples on one of the most–well known time scales for simulations, and finally we give a conclusion and open problems in the last section.

2 Preliminaries

2.1 Time Scale Calculus

This section represents the basic definitions and theorems in order for the interested readers to understand the basis of the time scale theory.

Definition 2.1 (See [3, Definition 1.1])**.** Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we have the following definitions:

(i) The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \quad s > t\} \quad \text{for all} \quad t \in \mathbb{T}.$$

(ii) The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : \quad s < t\} \quad \text{for all} \quad t \in \mathbb{T}.$$

(iii) The graininess function $\mu : \mathbb{T} \to [0, \infty)$ by

$$\mu(t) := \sigma(t) - t \quad \text{for all} \quad t \in \mathbb{T}.$$

We also need the set $\mathbb{T}^\kappa$ that is defined as follows:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty \end{cases}$$

Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 2.2 (See [3, Definition 1.10])**.** For any ϵ, if there exists a $\delta > 0$ such that

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in (t - \delta, t + \delta) \cap \mathbb{T},$$

then f is called *delta (or Hilger) differentiable* on $\mathbb{T}^\kappa$ and f^Δ is called *delta derivative* of f.

Theorem 2.3 (See [3, Theorem 1.16])**.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a function with $t \in \mathbb{T}^\kappa$. Then*

a. If f is differentiable at t, f is continuous at t.

b. If f is continuous at t and t is right-scattered, then f is differentiable at t and

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

c. If t is right dense, then f is differentiable at t iff

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number.

In this paper, we assume that $\mathbb{T}$ is unbounded above, i.e., $\sup \mathbb{T} = \infty$ and the graininess function μ is bounded because of the definition of stability on time scales, e.g., [6].

2.2 Controllability on Time Scales

A *control system* is a system of devices that manages, commands, or regulates the behaviors of other systems in order to achieve desired results. *Controllability* is a crucial feature for control systems to make them perform the way we want, e.g., stabilization of unstable systems by feedback control or optimal control.

Definition 2.4 (See [15, Definition 2.45])**.** A symmetric matrix–valued function A is said to positive definite (denoted $A > 0$) if $x^T Ax > 0$ for any nonzero vector x. A symmetric matrix-valued function A is said to positive semi-definite (denoted $A \geq 0$) if $x^T Ax \geq 0$ for any nonzero vector x.

Let us consider the state equation

$$x^{\Delta}(t) = Ax(t) + Bu(t), \tag{2.1}$$

where $x \in \mathbb{R}^n$ is a state, $u \in \mathbb{R}^m$ is the input variable(controller), A and B are real valued matrices of dimensions $n \times n$ and $n \times m$, respectively and A is assumed to be regressive. Throughout the paper, we assume $t_0, T \in \mathbb{T}$, where t_0 is the initial time while T is the final time. The following definition introduces us how controllability on time scales is defined, see [15].

Definition 2.5. The state equation (2.1) is *completely controllable* on $[t_0, T]$ if there exists a controller u such that the solution x of equation (2.1) with $x(t_0) = x_0$ satisfies $x(T) = 0$.

In other words, Definition 2.5 tells us Equation (2.1) is controllable if there exists at least one controller input that drives the state vector to the origin for any given initial value.

Next, we give the controllability criteria of state equation (2.1) and how the Riccati equation is defined on time scales see [9, 15].

Lemma 2.6 (See [9, Theorem 3.3])**.** *The state equation* (2.1) *is completely controllable if and only if the* $n \times (nm)$ *controllability matrix* $\Gamma_C[A, B]$ *has a full rank* n*, where* $\Gamma_C[A, B] = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$.

Optimal control is a process of determining control and state inputs for a system over a time period to minimize a cost function(which sometimes known as a performance index). Let us consider state equation (2.1) with the associated cost function

$$J(t_0) = \frac{1}{2}\left(x^T(T)s(T)x(T) + \int_{t_0}^{T}(x^TQx + u^TRu)(\tau)\Delta\tau\right), \tag{2.2}$$

where $S \geq 0$, $Q \geq 0$ and $R > 0$. Then as discussed in [15, Section 4.4], the control input

$$u^*(t) = (-R + \mu(t)B^TS^{\sigma}(t)B)^{-1}B^TS^{\sigma}(t)(I + \mu(t)A)x(t), \quad t \geq t_0 \tag{2.3}$$

minimizes the performance index (2.2). Here, S is the solution of the *Riccati equation* (second form) on time scales given by

$$-S^{\Delta} = Q + A^TS^{\sigma} + (I + \mu(t)A^T)S^{\sigma}(I + \mu(t)BR^{-1}B^TS^{\sigma})^{-1}(A - BR^{-1}B^TS^{\sigma}) \tag{2.4}$$

provided $BR^{-1}B^TS^{\sigma}(t)$ is regressive, see [15, Theorem 4.15].

2.3 Stability on Time Scales

One of the most important problems arise for dynamical systems on time scales is the stability and instability of their equilibrium solutions. Consider the following linear system of dynamic equations

$$x^{\Delta}(t) = Ax(t), \tag{2.5}$$

where $t \in \mathbb{T}$ and $x \in \mathbb{R}^n$. Next, we give the following definitions and theorem for the stability on time scales, see [2, 11].

Definition 2.7. The equilibrium point $x = 0$ is *stable* if for any ϵ and $t_0 \in \mathbb{T}$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that the condition $\|x(t_0)\| < \delta$ implies the inequality $\|x(t, t_0, x(t_0))\| < \epsilon$ for all $t \geq t_0$.

Definition 2.8. System (2.5) is *exponentially stable* if there exists a constant $\alpha > 0$ such that for every $t_0 \in \mathbb{T}$, there exists $K = K(t_0) \geq 1$ with

$$\|e_A(t, t_0)x(t_0)\| \leq Ke^{-\alpha(t-t_0)}\|x(t_0)\|$$

for $t \geq t_0$.

Definition 2.9. System (2.1) is *stabilizable* if there exists $u(t) = Kx(t)$ for $K \in \mathbb{R}^{m \times n}$ such that the closed loop system $x^{\Delta}(t) = (A + BK)x(t)$ is exponentially stable.

Theorem 2.10. *Suppose the graininess function $\mu(t)$ is bounded. Then system* (2.1) *is stabilizable if it is controllable.*

3 Optimal Controller Designs

3.1 Optimal Control of Altitude

In this section, we start to stabilizing the state z by using the exact linearization for the third equation of system (1.1) and find the optimal control law by means of the Riccati equation (2.4). Therefore, we consider the following subsystem of system (1.1)

$$z^{\Delta\Delta}(t) = \frac{u(t)\cos\theta(t)\cos\phi(t)}{m} - g.$$

Let $\xi_z = [\xi_{z1} \quad \xi_{z2}]^T = [z - z_d \quad z^{\Delta}]^T$, where z_d is defined as the desired elevation of quadrotor. So, the state space representation of z can be written as

$$\begin{cases} \xi_{z1}^{\Delta}(t) = \xi_{z2} \\ \xi_{z2}^{\Delta}(t) = \dfrac{u(t)\cos\theta(t)\cos\phi(t)}{m} - g. \end{cases} \tag{3.1}$$

Theorem 3.1. *Consider system* (3.1) *along with nonlinear controller*

$$u(t) = \frac{m(v_1(t) + g)}{\cos\theta(t)\cos\phi(t)}, \tag{3.2}$$

where $\cos\theta(t)\cos\phi(t) \neq 0$ *for* $\theta, \phi \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, v_1 *is a parameter. Then the equilibrium point* $\xi_z = 0$ *of* (3.1) *is stable.*

Proof. Suppose that u is a controller defined as in (3.2) and that $\cos\theta(t)\cos\phi(t) \neq 0$ for $\theta, \phi \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, which is a tolerable assumption for the quadrotor, see [13]. By plugging (3.2) into system (3.1), we have

$$\begin{bmatrix}\xi_{z1}\\ \xi_{z2}\end{bmatrix}^{\Delta} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\xi_{z1}\\ \xi_{z2}\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix} v_1. \tag{3.3}$$

By Lemma 2.6, we obtain that system (3.3) is controllable since controllability matrix $\Gamma_C[A, B] = [B \;\; AB]$ has a rank of 2, where

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}0\\ 1\end{bmatrix},$$

which also implies (1.1) is stabilizable by Theorem 2.10. Therefore, we can find a control law v_1 that minimizes the performance index

$$J_z(t_0) = \frac{1}{2}\int_{t_0}^{\infty} (\xi_z^T Q_z \xi_z + v_1^T R_z v_1)(\tau)\Delta\tau, \tag{3.4}$$

where $Q_z \geq 0$, $R_z > 0$. By the discussion in Section 2.2, we can choose an optimal controller

$$v_1^*(t) = -(R_z + \mu(t)B^T S_z^{\sigma}(t)B)^{-1}B^T S_z^{\sigma}(t)(I + \mu(t)A)\xi_z(t), \quad t \geq t_0, \tag{3.5}$$

where S_z is the solution of the Riccati equation given as in (2.4). Therefore, system (3.3) is stable with controller (3.5) that minimizes the cost function (3.4). This completes the proof. □

3.2 Optimal Control of Yaw

Consider $\psi^{\Delta\Delta} = u_\psi(t)$. Define $\xi_\psi = [\xi_{\psi1} \;\; \xi_{\psi2}]^T$, where $\xi_{\psi1} = \psi$ and $\xi_{\psi2} = \psi^{\Delta}$. Then we have the system

$$\begin{cases}\xi_{\psi1}^{\Delta}(t) = \xi_{\psi2}(t)\\ \xi_{\psi2}^{\Delta}(t) = u_\psi(t),\end{cases} \tag{3.6}$$

which can be written as

$$\begin{bmatrix} \xi_{\psi 1} \\ \xi_{\psi 2} \end{bmatrix}^{\Delta}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_{\psi 1} \\ \xi_{\psi 2} \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{\psi}(t). \tag{3.7}$$

By the same discussion in Section 3.1, system (3.7) is controllable by defining the performance index as

$$J_{\psi}(t_0) = \frac{1}{2}\int_{t_0}^{\infty} (\xi_{\psi}^T Q_{\psi}\xi_{\psi} + u_{\psi}^T R_{\psi} u_{\psi})(\tau)\Delta\tau. \tag{3.8}$$

The optimal controller u_{ψ}^* is also given by

$$u_{\psi}^*(t) = -(R_{\psi} + \mu(t)B^T S_{\psi}^{\sigma}(t)B)^{-1} B^T S_{\psi}^{\sigma}(t)(I + \mu(t)A)\xi_{\psi}(t), \quad t \geq t_0, \tag{3.9}$$

respectively, where $Q_{\psi} \geq 0$, $R_{\psi} > 0$, and S_{ψ} is the solution of the Riccati equation given as in (2.4).

3.3 Optimal Control of y and ϕ

Consider the subsystem

$$\begin{cases} y^{\Delta\Delta}(t) = \dfrac{u(t)\cos\theta(t)\sin\phi(t)}{m} \\ \phi^{\Delta\Delta}(t) = u_{\phi}(t) \end{cases}$$

and the state space representation

$$\begin{cases} \xi_{y1}^{\Delta}(t) = \xi_{y2}(t) \\ \xi_{y2}^{\Delta}(t) = \dfrac{u(t)\cos\theta(t)\sin\xi_{\phi 3}(t)}{m} \\ \xi_{\phi 3}^{\Delta}(t) = \xi_{\phi 4}(t) \\ \xi_{\phi 4}^{\Delta}(t) = u_{\phi}(t), \end{cases} \tag{3.10}$$

where $\xi_{y1} = y, \xi_{y2} = y^{\Delta}, \xi_{\phi 3} = \phi$ and $\xi_{\phi 4} = \phi^{\Delta}$.

Theorem 3.2. *Consider* (3.10) *with an optimal controller* u_{ϕ}^* *such that* $\tan\xi_{\phi 3}(t) \to \xi_{\phi 3}(t)$. *Then system* (3.10) *is stable.*

Proof. Consider system (3.10). By plugging the controller (3.2) in system (3.10), we have the following new system:

$$\begin{cases} \xi_{y1}^{\Delta}(t) = \xi_{y2}(t) \\ \xi_{y2}^{\Delta}(t) = (v_1(t) + g)\tan\xi_{\phi 3}(t) \\ \xi_{\phi 3}^{\Delta}(t) = \xi_{\phi 4}(t) \\ \xi_{\phi 4}^{\Delta}(t) = u_{\phi}(t). \end{cases} \tag{3.11}$$

It is well–known that the optimal controller $v_1^*(t) \to 0$ as $t \to \infty$, e.g., see [13]. Then there exists $t_1 \in \mathbb{T}$ so large that $|v_1^*(t)|$ is bounded and neglected. Therefore, the second equation of system (3.11) can be rewritten as $\xi_{y2}^{\Delta}(t) = g \tan \xi_{\phi 3}(t)$. However, system (3.11) is a nonlinear system. So, we are looking for an optimal control u_ϕ^* such that $\tan \xi_{\phi 3}(t) \to \xi_{\phi 3}(t)$. This is possible by using small–angle approximation for tangent functions. Then we can rewrite system (3.11) as

$$\xi_{y,\phi}^{\Delta}(t) = A_{y,\phi}\xi_{y,\phi}(t) + B_{y,\phi}u_\phi(t), \tag{3.12}$$

where

$$A_{y,\phi} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \xi_{y,\phi} = \begin{bmatrix} \xi_{y1} \\ \xi_{y2} \\ \xi_{\phi 3} \\ \xi_{\phi 4} \end{bmatrix}, \tag{3.13}$$

which is controllable by Lemma 2.6. By the similar discussion as in Section 3.1, the optimal controller is given by

$$u_\phi^*(t) = -(R_{y,\phi} + \mu(t)B_{y,\phi}^T S_{y,\phi}^\sigma(t)B_{y,\phi})^{-1}B_{y,\phi}^T S_{y,\phi}^\sigma(t)(I + \mu(t)A_{y,\phi})\xi_{y,\phi}(t) \tag{3.14}$$

for $t \geq t_1$, which minimizes the cost function

$$J_{y,\phi}(t_1) = \frac{1}{2}\int_{t_1}^{\infty} (\xi_{y,\phi}^T Q_{y,\phi}\xi_{y,\phi} + u_\phi^T R_{y,\phi}u_\phi)(\tau)\Delta\tau, \tag{3.15}$$

where $Q_{y,\phi} \geq 0$, $R_{y,\phi} > 0$ and $S_{y,\phi}$ is the solution of the Riccati equation (2.4). □

3.4 Optimal Control of x and θ

Consider

$$\begin{cases} x^{\Delta\Delta}(t) = -\dfrac{u(t)\sin\theta(t)}{m} \\ \theta^{\Delta\Delta}(t) = u_\theta(t) \end{cases}$$

and the state space representation

$$\begin{cases} \xi_{x1}^{\Delta}(t) = \xi_{x2}(t) \\ \xi_{x2}^{\Delta}(t) = \dfrac{u(t)\sin\xi_{\theta 3}(t)}{m} \\ \xi_{\theta 3}^{\Delta}(t) = \xi_{\theta 4}(t) \\ \xi_{\theta 4}^{\Delta}(t) = u_\theta(t), \end{cases}$$

where $\xi_{x1} = x, \xi_{x2} = x^{\Delta}, \xi_{\theta 3} = \theta$ and $\xi_{\theta 4} = \theta^{\Delta}$.

By the same discussion as in Section 3.3 for the optimal controls $v_1^*(t)$ and $u(t)$, where $t \geq t_1 \in \mathbb{T}$, we have the following system:

$$\xi_{x,\theta}^{\Delta}(t) = A_{x,\theta}\xi_{x,\theta}(t) + B_{x,\theta}u_\theta(t), \tag{3.16}$$

where $A_{x,\theta}$ and $B_{x,\theta}$ are given as in equation (3.13), and $\xi_{x,\theta} = [\xi_{x1}\ \xi_{x2}\ \xi_{\theta 3}\ \xi_{\theta 4}]^T$.

Let us define the optimal control as

$$u_\theta^*(t) = -(R_{x,\theta} + \mu(t) B_{x,\theta}^T S_{x,\theta}^\sigma(t) B_{x,\theta})^{-1} B_{x,\theta}^T S_{x,\theta}^\sigma(t)(I + \mu(t) A_{x,\theta})\xi_{x,\theta}(t)$$

for $t \geq t_1$, which minimizes the cost function

$$J_{x,\theta}(t_1) = \frac{1}{2}\int_{t_1}^{\infty} (\xi_{x,\theta}^T Q_{x,\theta} \xi_{x,\theta} + u_\theta^T R_{x,\theta} u_\theta)(\tau)\Delta\tau,$$

where $Q_{x,\theta} \geq 0$, $R_{x,\theta} > 0$ and $S_{x,\theta}$ is the solution of the Riccati equation (2.4).

4 Simulation Results

To illustrate of the effectiveness of our proposed controller u and v_1^*, a quadrotor UAV with the dynamics of system (1.1) is considered. Initial states of the UAV are chosen arbitrarily. The controllers force altitude of the quadrotor to a desired point. The simulation is evaluated for two different time scales in Example 4.2 and Example 4.3 and following proposition gives us how we define derivatives for those time scales, see [3, Theorem 1.16].

Proposition 4.1. *(i) Let $\mathbb{T} = h\mathbb{Z}$, where $h > 0$. Then the delta derivative of f is given by*

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}, \tag{4.1}$$

where $\sigma(t) = t + h$, $\mu(t) = h$ and $f^\sigma(t) = f(\sigma(t)) = f(t+h)$ for $t \in h\mathbb{Z}$.

(ii) Let $\mathbb{T} = q^{\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $q > 1$. Then the derivative of f is defined as in equation (4.1) *where $\sigma(t) = tq$, and $\mu(t) = (q-1)t$.*

Example 4.2. Consider system (1.1) when $\mathbb{T} = h\mathbb{Z}$, where $h = 0.01$, $m = 4\,kg, g = 9.8m/sec^2$, given by

$$\begin{cases} x_{n+0.02} = -\dfrac{u_n \sin\theta_n}{40000} + 2x_{n+0.01} - x_n \\ y_{n+0.02} = \dfrac{u_n \cos\theta_n \sin\phi_n}{40000} + 2y_{n+0.01} - y_n \\ z_{n+0.02} = \dfrac{u_n \cos\theta_n \cos\phi_n}{40000} - 0.00098 + 2z_{n+0.01} - z_n \\ \phi_{n+0.02} = 0.0001u_{\phi_n} + 2\phi_{n+0.01} - \phi_n \\ \theta_{n+0.02} = 0.0001u_{\theta_n} + 2\theta_{n+0.01} - \theta_n \\ \psi_{n+0.02} = 0.0001u_{\psi_n} + 2\psi_{n+0.01} - \psi_n. \end{cases}$$

The following graphs show how the system behave in a long term and the system is stabilized in about 20 seconds.

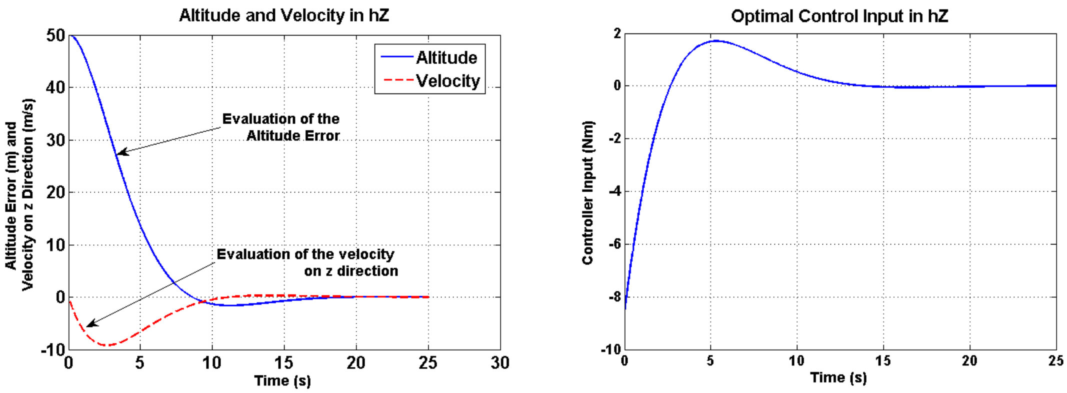

Figure 4.1: Errors and controller for altitude and velocity in hZ

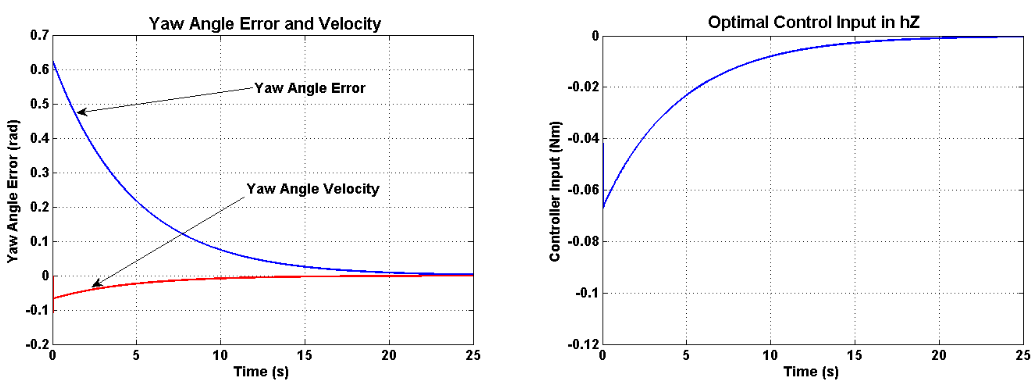

Figure 4.2: Errors and controller for yaw angle in $h\mathbb{Z}$

Example 4.3. Consider system (1.1) when $\mathbb{T} = 2^n$, where $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots, N\}$ for $N = 4$ and consider the same parameters for the initial conditions, g and m as in

Example 4.2. Therefore considering the system

$$\begin{cases} x(4t) = -t^2 \dfrac{u(t)\sin\theta(t)}{2} + 3x(2t) - 2x(t) \\ y(4t) = t^2 \dfrac{u(t)\cos\theta(t)\sin\phi(t)}{2} + 3y(2t) - 2y(t) \\ z(4t) = t^2 \dfrac{u(t)\cos\theta(t)\cos\phi(t)}{2} - 19.6t^2 + 3z(2t) - 2z(t) \\ \phi(4t) = 2t^2 u_\phi(t) + 3\phi(2t) - 2\phi(t) \\ \theta(4t) = 2t^2 u_\theta(t) + 3\theta(2t) - 2\theta(t) \\ \psi(4t) = 2t^2 u_\psi(t) + 3\psi(2t) - 2\psi(t), \end{cases}$$

we have the simulation results as follows:

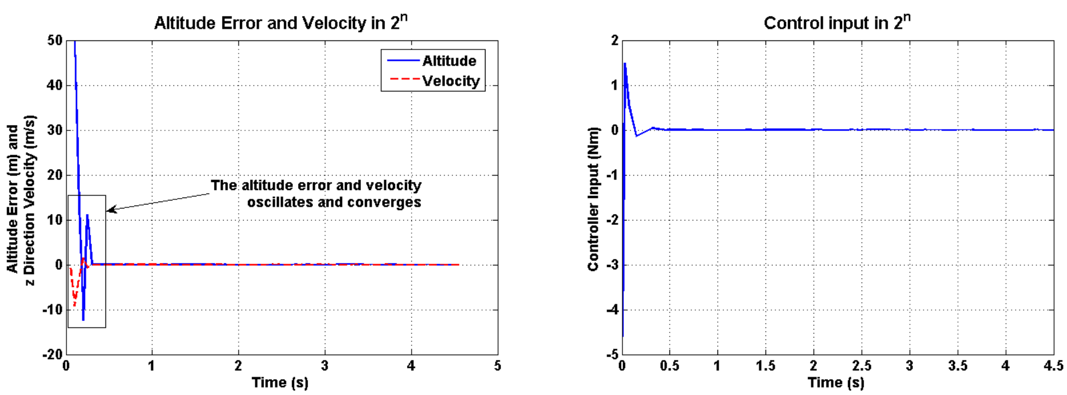

Figure 4.3: Errors and controller in 2^n

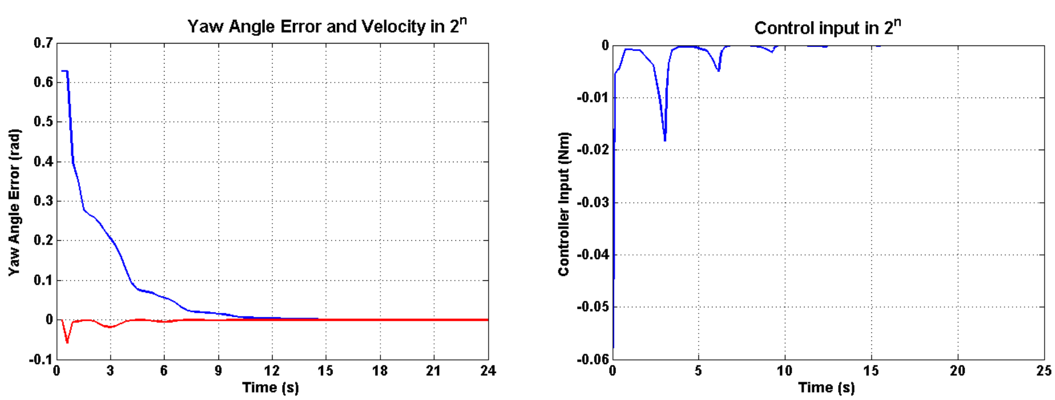

Figure 4.4: Errors and controller of yaw angle in 2^n

In the second example, a novel time scale is defined as truncated 2^n for the UAV control application. The reason that we truncate n is to maintain stability between two sampling time. Keeping the same controller over the larger intervals may cause instability. In this simulation, the maximum exponent is chosen as $n = 4$. Therefore, the sampling interval is increased from $2^0 = 1$ up to $2^4 = 16$ and reduced back to $2^0 = 1$ again. When Figure 4.3 is compared with Figure 4.2, we realize that the system oscillates more in the second example. However, the computation cost of the controller is significantly reduced in the second example.

5 Conclusion and Future Work

A novel regulation control for the elevation of quadrotor UAVs was provided for time scales. Regulation error of the elevation was optimally stabilized by using Kalman gain on time scales. Nonlinear quadrotor dynamics were canceled out through feedback linearizing controller. Finding controllers for x, y positions and the orientations ϕ, θ, ψ can be considered as a desirable future work. However, it would be more challenging due to the in linearizing the dynamics. Therefore, nonlinear controllers has to be utilized in order to control all the states of quadrotor on time scales.

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