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Spacelike Surfaces on Time Scales

Hatice Kuşak Samancı
Bitlis Eren University
Department of Mathematics
Bitlis, 1300, Turkey
hkusak@beu.edu.tr

Abstract

An arbitrary, nonempty closed subset of real numbers is defined as a time scale. Time scales are used to unify discrete and nondiscrete mathematics. In our work, the geometric properties of the spacelike surfaces on time scales are studied.

AMS Subject Classifications: 53A35, 53C57, 34N99.
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1 Introduction

The concept of Minkowski space was first developed by H. Minkowski in 1907, and its components include Euclidean space and time. The fundamental properties of spacelike surfaces and Minkowski space may be obtained in [7, 11]. The time scale concept was first defined by Stefan Hilger in his PhD thesis in 1988 to create a theory that can unify discrete and continuous analyses. M. Bohner and A. Peterson published a book for basic time scale calculus and dynamical systems in [9]. Bohner and Guseinov studied partial differentiation on time scales in [8]. G. Guseinov and E. Ozyilmaz studied the tangent lines of generalized regular curves in [1]. J.L. Cieslinski investigated pseudospherical surfaces on time scales in [6]. H. Kusak and A. Caliskan obtained some geometric properties of time scales in [2–5]. The directional nabla-derivative and curves on n-dimensional time scales were presented by N. Aktan et al. in [10]. S.P. Atmaca and O. Akguller studied the surfaces on time scales in [12].

In this paper, some metric properties of the spacelike surfaces on time scales are studied. First, we introduce these spacelike surfaces parametrized by two arbitrary time scales in Minkowski 3 space. Second, the normal vector of a spacelike surface on a time scale is calculated. Finally, local computations of curvatures on a time scale are performed.

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2 Preliminaries

The set $\mathbb{R}^3_1 = \left(\mathbb{R}^3, \langle x, y\rangle_L\right)$ is called a Minkowski 3-space with the Lorentzian inner product $\langle u, v\rangle_L = u_1v_1+u_2v_2-u_3v_3$ where u and v are two vectors in $\mathbb{R}^3_1$. In Minkowski 3-space, a vector u is called spacelike vector if and only if $\langle u, u\rangle_L > 0$ or $u = 0$, u is called timelike vector if and only if $\langle u, u\rangle_L < 0$, u is called null vector if and only if $\langle u, u\rangle_L = 0$. The length of any u is defined by $|\langle u, u\rangle_L|^{1/2} < 0$. Let M be a surface in the Minkowski space $\mathbb{R}^3_1$. The tangent plane at the point P on the surface is T_PM. If $\langle u, v\rangle_L$ is positively definite, then the tangent plane T_PM and the surface M are called a spacelike plane and a spacelike surface, respectively. If $\langle u, v\rangle_L$ is a metric with index 1 (or a degenerate metric), then the tangent plane T_PM and the surface M are called a timelike (or lightlike) plane and a timelike (or lightlike) surface, respectively. Assume that $x : M \to \mathbb{R}^3_1$ is spacelike immersion. Therefore, all normal vector fields on the surface M are timelike vector fields. If the immersion x is spacelike immersion, then the surface M is orientable. It should be point out that there exists no closed spacelike surface, [7,11].

A time scale $\mathbb{T}$ is an arbitrary, nonempty closed subset of real numbers. Let $\mathbb{T}_i$ denote time scales for each $i \in \{1, \ldots, n\}$. Λ^n is called an n-dimensional time scale and defined by

$$\Lambda^n = \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_n = \{t = (t_1, t_2, \ldots, t_n) : t_i \in \{1, \ldots, n\}\}.$$

The set Λ^n is a complete metric space with the metric d described by

$$d(t, s) = \left(\sum_{i=1}^{n} |t_i - s_i|^2\right)^{1/2},$$

for $t, s \in \Lambda^n$. The forward jump operator $\sigma_i : \mathbb{T}_i \to \mathbb{T}_i$ denotes

$$\sigma_i(u) = \inf\{v \in \mathbb{T}_i : v > u,\ u \in \mathbb{T}_i\}.$$

If $\sigma_i(u) > u$ and $\sigma_i(u) = u$, then u is considered right-scattered and right-dense in $\mathbb{T}_i$, respectively. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. As $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$ and $\mu(t) = 0$. As $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$ and $\mu(t) = 1$. If $\mathbb{T}_i$ has a right-scattered minimum m, then we define $\mathbb{T}^\kappa_i = \mathbb{T}_i \backslash \{m\}$; otherwise, $\mathbb{T}^\kappa_i = \mathbb{T}_i$. The partial derivative of $f : \Lambda^n \to \mathbb{T}^\kappa_i$ for $s_i \neq \sigma_i(t_i)$ is defined as

$$\lim_{s_i \to t_i} \frac{f(t_1, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i}$$

provided that this limit exists as a finite number, and is denoted by any of the symbols $\dfrac{\partial f(t_1, \ldots, t_n)}{\Delta_i t_i}$, $\dfrac{\partial f(t)}{\Delta_i t_i}$ or $f^{\Delta_i}_{t_i}(t)$, [9]. Let V and W be a vector field on the space Λ^2.

By considering the delta covariant derivative of W with respect to V, which implies the following equation $(\Delta_V W)(p) = \Delta_{V(p)} W$, then the mapping

$$\begin{aligned} \Delta : \quad & T(\Lambda^2) \times T(\Lambda^2) \to T(\Lambda^2), \\ & (V, W) \to \Delta_V W \end{aligned}$$

is described "the delta nature connection" on a time scale. In [3], some properties of the delta connection are given as

1. $\Delta_V(fY + gZ) = f\Delta_V Y + g\Delta_V Z$
2. $\Delta_{fV+gW} Y = f\Delta_V Y + g\Delta_W Y$
3. $$\begin{aligned} \Delta_V(fY) = & \frac{\partial f}{\Delta V} Y(\sigma_1(t^0), s^0) + f(\sigma_1(t^0), \sigma_1(s^0))\Delta_V Y \\ & + \{-\mu_1 V_1 \frac{\partial f}{\Delta_1 V_1} + \mu_2 V_1 \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 V_2}\} \sum_{i=1}^{2} \frac{\partial y_i}{\Delta_1 V_1} \frac{\partial}{\partial x_i} \end{aligned}$$
4. $$\begin{aligned} \Delta_V < Y, Z > = & < \Delta_V Y, Z > + < \Delta_V Z, Y > \\ & - \mu_1 V_1 \sum_{i=1}^{2} \frac{\partial y_i}{\Delta_1 V_1} \frac{\partial z_i}{\Delta_1 V_1} \frac{\partial}{\partial x_i} - \mu_2 V_1 \sum_{i=1}^{2} \frac{\partial z_i}{\Delta_1 V_1} \frac{\partial y_i(\sigma_1(t^0), s^0)}{\Delta_2 V_2} \frac{\partial}{\partial x_i}. \end{aligned}$$

The delta derivation of the inner and outer products for the two vector–valued functions $x(t)$ and $y(t)$ are defined by

$$\begin{aligned} \frac{\partial}{\Delta t} < x(t), y(t) > &= \frac{\partial x(t)}{\Delta t} y(t) + x^\sigma(t). \frac{\partial y(t)}{\Delta t}, \\ \frac{\partial}{\Delta t}(x(t) \times y(t)) &= \frac{\partial x(t)}{\Delta t} \times y(t) + \vec{x}^\sigma(t) \times \frac{\partial y(t)}{\Delta t}, \end{aligned}$$

[12]. Some other basic properties of time scales can be examined in [8, 9].

3 Main Results

Let $\Lambda_1^2 = (\mathbb{T}_1 \times_L \mathbb{T}_2, \langle , \rangle_L)$ be a time scale with a Lorentzian metric and S be a surface in a three-dimensional Minkowski space.

Definition 3.1. The tangent plane $T_P S$ of the surface S passing through the point $P_0 = (t_0, s_0, f((t_0, s_0)))$ where f is a real-valued continuous function is defined as the spacelike tangent plane at the point P_0 in the Minkowski space on time scales.

Definition 3.2. Let $r : S \subset \Lambda_1^2 \to \mathbb{R}_1^3$ be an immersion. If any tangent plane is spacelike, the immersion r is called spacelike and the surface S is called a spacelike surface on time scale.

Definition 3.3. Let $r : S \subset \Lambda_1^2 \to \mathbb{R}_1^3$ be a spacelike immersion on time scales. Let $\chi(t,s) = r(t,s)$ be a completely Δ_1 and Δ_2 -differentiable local parametrization and

$$\left\{ \frac{\partial \chi(t,s)}{\Delta_1 t}, \frac{\partial \chi(t,s)}{\Delta_2 s} \right\}$$

be a local base of the tangent plane at each point.

Definition 3.4. Let $\chi = r(t,s)$ be completely Δ-differentiable, then

$$\frac{\partial \chi(t,s)}{\Delta_1 t} \times_L \frac{\partial \chi(t,s)}{\Delta_2 s} \neq 0.$$

Therefore, χ is Δ-regular. $\chi = r(t,s)$ is called a spacelike surface patch on the time scale and S is called a spacelike smooth surface.

Definition 3.5. Let $r : S \subset \Lambda_1^2 \to \mathbb{R}_1^3$ be a spacelike immersion. On time scales, the normal vector field of the spacelike surface is timelike and defined by

$$N = \frac{\frac{\partial \chi(t,s)}{\Delta_1 t} \times_L \frac{\partial \chi(t,s)}{\Delta_2 s}}{\left\| \frac{\partial \chi(t,s)}{\Delta_1 t} \times_L \frac{\partial \chi(t,s)}{\Delta_2 s} \right\|_L},$$

where $\dfrac{\partial \chi(t,s)}{\Delta_1 t} \times_L \dfrac{\partial \chi(t,s)}{\Delta_2 s} \neq 0.$

Theorem 3.6. *Let S be a surface. If $r : S \subset \Lambda_1^2 \to \mathbb{R}_1^3$ is a spacelike immersion on time scales, then S is an orientable surface.*

Proof. Each tangent plane $T_P S$ is spacelike for spacelike surfaces. If the spacelike immersion $\chi = r(t,s)$ is taken, then the vectors and are tangent to the spacelike surface. Thus, these vectors are spacelike vectors. As the unit vector N_P is orthogonal to the tangent plane, the vector

$$\frac{\partial \chi(t,s)}{\Delta_1 t} \times_L \frac{\partial \chi(t,s)}{\Delta_2 s}$$

is a timelike vector. For the Lorentzian inner product, the vector $e_3 = (0,0,1)$ is a timelike vector. As this vector is timelike, we have $|\langle N_P, e_3 \rangle_L| \geq 1$. Therefore, $\langle N_P, e_3 \rangle_L \leq -1$ or $\langle N_P, e_3 \rangle_L \geq 1$. This definition allows us to identify a globally normal vector field to the spacelike surface on time scales and shows us that this field is an orientable surface. □

Definition 3.7. S is a spacelike surface on a time scale in the space Λ_1^3. Let $\{e_1, e_2\}$ be the orthonormal basis of the surface unit normal N_P at the point P. If $\det(e_1, e_2, N_P) > 0$, then the base is oriented positively. If $\det(e_1, e_2, N_P) < 0$, then the base is oriented negatively.

Theorem 3.8. *$r : S \subset \Lambda_1^2 \to \mathbb{R}_1^3$ is a spacelike immersion, and Δ_1 and Δ_2 are the delta differential operators of the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Let $t : \tilde{\mathbb{T}} \to \mathbb{R}$ and $s : \tilde{\mathbb{T}} \to \mathbb{R}$ be the coordinate functions, where $x \in [a,b] \subset \tilde{\mathbb{T}}$ and $\tilde{\Delta}$ is the delta differential operator of the time scale $\tilde{\mathbb{T}}$. The first fundamental form is defined by*

$$\begin{aligned} I_P : &\quad T_PS \times_L T_PS \to \mathbb{R}, \\ I_P(u,v) &= <u,v>_L|_P, \\ I_P &= E\left(\frac{dt}{\tilde{\Delta}x}\right)^2 + 2F\left(\frac{dt}{\tilde{\Delta}x}\right)\left(\frac{ds}{\tilde{\Delta}x}\right) + G\left(\frac{ds}{\tilde{\Delta}x}\right)^2 \end{aligned}$$

where

$$\begin{aligned} E &= \left\langle \frac{\partial\chi(t,s)}{\Delta_1 t}, \frac{\partial\chi(t,s)}{\Delta_1 t} \right\rangle_L, \\ F &= \left\langle \frac{\partial\chi(t,s)}{\Delta_1 t}, \frac{\partial\chi(t,s)}{\Delta_2 s} \right\rangle_L, \\ G &= \left\langle \frac{\partial\chi(t,s)}{\Delta_2 s}, \frac{\partial\chi(t,s)}{\Delta_2 s} \right\rangle_L. \end{aligned}$$

Proof. If we take the complete Δ-differentiable of $\chi = r(t,s)$, then we get $\Delta\chi = \frac{\partial\chi}{\Delta_1 t}.\frac{dt}{\tilde{\Delta}x} + \frac{\partial\chi}{\Delta_2 s}.\frac{ds}{\tilde{\Delta}x}$. This concludes the proof. □

$$\begin{aligned} (I) &= <\Delta\chi, \Delta\chi>_L \\ &= \left\langle \frac{\partial\chi}{\Delta_1 t}, \frac{\partial\chi}{\Delta_1 t} \right\rangle_L . \left(\frac{dt}{\tilde{\Delta}x}\right)^2 + 2\left\langle \frac{\partial\chi}{\Delta_1 t}, \frac{\partial\chi}{\Delta_2 s} \right\rangle_L . \left(\frac{dt}{\tilde{\Delta}x}\right)\left(\frac{ds}{\tilde{\Delta}x}\right) \\ &\quad - \left\langle \frac{\partial\chi}{\Delta_2 s}, \frac{\partial\chi}{\Delta_2 s} \right\rangle_L \left(\frac{ds}{\tilde{\Delta}x}\right)^2 \\ &= E\left(\frac{dt}{\tilde{\Delta}x}\right)^2 + 2F\left(\frac{dt}{\tilde{\Delta}x}\right)\left(\frac{ds}{\tilde{\Delta}x}\right) + G\left(\frac{ds}{\tilde{\Delta}x}\right)^2. \end{aligned}$$

Definition 3.9. The matrix representation of the first fundamental form with respect to the base $\left\{ \frac{\partial\chi(t,s)}{\Delta_1 t}, \frac{\partial\chi(t,s)}{\Delta_2 s} \right\}$ can be also written as

$$(I) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

Theorem 3.10. *If the surface S is spacelike, then the inequality $F^2 < E.G$ is satisfied.*

Proof. If the surface S is spacelike, then the tangent plane is spacelike. Thus, the unit normal is a timelike vector and satisfies

$$\langle N, N\rangle_L = F^2 - E.G < 0.$$

Then, we get $F^2 < E.G$. This deduces the proof. $\square$

Theorem 3.11. *If* $\det(I) > 0$*, the surface is a spacelike surface on time scales.*

Proof. The determination of the first fundamental form is

$$\det(I) = E.G - F^2 = -\langle N, N\rangle_L > 0.$$

Thus, we get $\langle N, N\rangle_L < 0$. If the unit normal is a timelike vector, then the surface S is spacelike. This finalizes the proof. $\square$

Definition 3.12. Let us consider $r : S \subset \Lambda_1^2 \to \mathbb{R}_1^3$ to be a spacelike immersion on time scales at the point P. The second fundamental form at P is

$$II_P : T_PS \times_L T_PS \to \mathbb{R}_1^3,$$
$$II_P(t, s) = -\left\langle \frac{\partial N}{\Delta_1 t}|_P, \frac{\partial \chi}{\Delta_2 s}|_P \right\rangle_L$$

Definition 3.13. Let S be a surface and T_PS be a tangent plane of the surface on a time scale. Let

$$\langle , \rangle : T_PS \times_L T_PS \to \mathbb{R}_1^3$$

be a Riemannian metric on the time scale. On the spacelike surface S, we have the Levi–Civita connection denoted by $\tilde{\nabla}$ corresponding to the induced Lorentzian metric denoted by ∇. If $\mathrm{X}, \Upsilon, \mathrm{Z}$ are the vector fields on the surface S, then the Levi–Civita connection on the time scale is

$$\tilde{\nabla}_{\mathrm{Z}}(\mathrm{X}, \Upsilon) = \nabla_{\mathrm{X}}\Upsilon + II(\mathrm{X}, \Upsilon).$$

Therefore, the second fundamental form can be defined as

$$II(\mathrm{X}, \Upsilon) = \tilde{\nabla}_Z(\mathrm{X}, \Upsilon) - \nabla_{\mathrm{X}}\Upsilon$$

by the Levi–Civita connection.

Definition 3.14. Let N be the unit timelike vector of the spacelike surface on a time scale. The Weingarten map of the spacelike surface on the time scale is defined by

$$A_N(V) = - < \Delta_V N, N >_L .$$

Theorem 3.15. *The equation*

$$< \Delta_V N, N >_L = \mu_1 \delta_1 + \mu_2 \delta_2$$

is satisfied, where μ_1 and μ_2 are graininess functions on the time scale. This finishes the proof.

Proof. If the Levi–Civita connection of the unit normal vector N on a time scale is taken, using the delta connection properties, then we get

$$\begin{aligned}\tilde{\nabla}_V(N,N) &= < \Delta_V N, N >_L + < N, \Delta_V N >_L \\ &\quad -\mu_1 V_1 \sum_{i=1}^{2} \frac{\partial x_i}{\Delta_1 V_1} \frac{\partial y_i}{\Delta_1 V_1} \frac{\partial}{\partial x_i} - \mu_2 V_1 \sum_{i=1}^{2} \frac{\partial y_i}{\Delta_1 V_1} \frac{\partial x_i(\sigma_1(t^0), s^0)}{\Delta_2 V_2} \frac{\partial}{\partial x_i}.\end{aligned}$$

$\Delta_V(N,N) = \Delta_V(1) = 0$ and $< \Delta_V N, N >_L = < N, \Delta_V N >_L$, we get

$$< \Delta_V N, N >_L = \mu_1 \frac{1}{2} V_1 \sum_{i=1}^{2} \frac{\partial x_i}{\Delta_1 V_1} \frac{\partial y_i}{\Delta_1 V_1} \frac{\partial}{\partial x_i} + \mu_2 \frac{1}{2} V_1 \sum_{i=1}^{2} \frac{\partial y_i}{\Delta_1 V_1} \frac{\partial x_i(\sigma_1(t^0), s^0)}{\Delta_2 V_2} \frac{\partial}{\partial x_i}.$$

Hence, the equation $< \Delta_V N, N >_L = \mu_1 \delta_1 + \mu_2 \delta_2$ is desired. □

Definition 3.16. The matrix representation of II_P with respect to the base

$$\left\{ \frac{\partial \chi(t,s)}{\Delta_1 t}, \frac{\partial \chi(t,s)}{\Delta_2 s} \right\}$$

is defined by

$$(II) = \begin{pmatrix} e & f \\ f & f \end{pmatrix},$$

where the elements of the matrix are

$$\begin{aligned} e &= -\left\langle \frac{\partial \chi}{\Delta_1 t}, \frac{\partial N}{\Delta_1 t} \right\rangle_L, \\ f &= -\left\langle \frac{\partial \chi}{\Delta_1 t}, \frac{\partial N}{\Delta_2 s} \right\rangle_L, \\ g &= -\left\langle \frac{\partial \chi}{\Delta_2 s}, \frac{\partial N}{\Delta_2 s} \right\rangle_L. \end{aligned}$$

Corollary 3.17. *The respective mean and Gauss curvatures for the spacelike surface on the time scale are*

$$H = -\frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}, \quad \textit{and} \quad K = -\frac{eg - f^2}{EG - F^2}.$$

4 Conclusion

In our study, the spacelike surface on time scales was introduced for the first time and some metric properties of the spacelike surface on time scales were obtained. Our main aim is to unify the discrete and continuous cases of spacelike surfaces. The results of this work are expected to contribute to future research on this topic.

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