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Preserving Dynamic Behaviors of a Delay-Induced Discrete Predator-Prey Model

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Abstract

Preserving dynamic behaviors of a continuous system in its corresponding discrete system is a difficult task and it becomes more difficult if there is a delay in the continuous system. Here we discretize the interaction between prey and predator represented by continuous-time nonlinear differential equations and then analyze it in presence and absence of a time delay. It is shown that the dynamics of the discrete model in presence and absence of delay is consistent with that of the continuous model. Numerical simulations are provided to substantiate the analytical findings.

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1 Introduction

Researchers are frequently interested to construct discrete model which are dynamically consistent with its continuous counter part. Conventional discretization schemes, such as Euler method, Runge–Kutta method, show dynamic inconsistency and exhibit dynamic behaviours which are not observed in the corresponding continuous system [16]. For example, Euler forward method, the most frequently used discretize technique, produces spurious solutions which are not observed in its parent model and its dynamics depend on the step-size. Another important drawback of the conventional discrete model is that positivity of its solutions do not hold for all positive initial values. Any finite-difference scheme that allows negative solutions will have numerical instabilities [22]. One alternative to remove these drawbacks is using of non-standard finite difference (NSFD) method to discretize a continuous-time model [8, 9]. It is to be mentioned that NSFD method shows dynamic consistency with its continuous counterpart [19]. This means that all the dynamic properties of the continuous system are preserved by the NSFD system. Moreover, the dynamic properties are preserved for all step-size as opposed to conventional discretization techniques. Nonstandard finite difference scheme has been successfully used to discretize a continuous system in different fields of physical and biological sciences [1, 3–5, 9, 11, 20, 21, 23–25]. However, dynamics preserving discretization of a delay-induced continuous system is rarely observed in the literature. Here we discretize a continuous-time predator-prey model and then analyze it in presence and absence of a time delay. We prove that all the qualitative behaviors of the continuous system in absence and presence of delay are preserved in the discrete model.

The Lotka–Volterra predator-prey model is represented by the following system of nonlinear differential equations:

$$\begin{aligned} \frac{dP}{dt} &= Q(P) - R(P, Z), \\ \frac{dZ}{dt} &= \kappa R(P, Z) - S(Z), \end{aligned} \tag{1.1}$$

where P and Z represent, respectively, the densities of phytoplankton and zooplankton populations at time t. Here $Q(P)$ is the intrinsic growth rate of prey, $R(P, Z)$ is the predator's feeding rate, κ $(0 < \kappa < 1)$ is the conversion efficiency of predator and $S(Z)$ is predator's natural death rate. In planktonic ecosystem, zooplankton feeds on the primary producer phytoplankton. There are many examples that some phytoplankton species liberate toxic chemical to escape or reduce predation pressure of zooplankton [2, 12, 13]. These toxin producing phytoplankton (TPP) may cause additional death to zooplankton and supposed to be one of the regulatory mechanisms in planktonic dynamics [14, 15]. Assuming that $T(P, Z)$ is the TPP-dependent additional death rate of zooplankton, system (1.1) then can be represented by

$$\frac{dP}{dt} = Q(P) - R(P, Z), \tag{1.2}$$

$$\frac{dZ}{dt} = \kappa R(P,Z) - S(Z) - T(P,Z).$$

Considering $Q(P) = rP\left(1 - \frac{P}{k}\right)$, $R(P,Z) = \alpha PZ$, $S(Z) = \mu Z$ and $T(P,Z) = \frac{\theta PZ}{\gamma + P}$, Chattopadhyay et al. [7] proposed the following mathematical model for the interaction of toxic phytoplankton (*Noctilucca Scintillans*) and zooplankton (*Paracalanus*):

$$\begin{aligned} \frac{dP}{dt} &= rP\left(1 - \frac{P}{k}\right) - \alpha PZ, \\ \frac{dZ}{dt} &= \beta PZ - \mu Z - \frac{\theta PZ}{\gamma + P}, \end{aligned} \tag{1.3}$$

where $\beta = \kappa\alpha$ and all other parameters are positive. For further illustration of the model, readers are referred to [7]. Liberation of the toxic substances by phytoplankton species is not an instantaneous process but mediated by some time lag required for the maturity of species. There are also several reports that zooplankton mortality due to the bloom of toxic phytoplankton bloom occurs after some time lag. In laboratory experiment. Chattopadhyay et al. [6] observed that the abundance of *Paracalanus* (zooplankton) population reduces after some time lags of the bloom of toxic phytoplankton *Noctilucca scintillans*. Assuming that τ be the time required for the maturity of toxic phytoplankton, the model (1.3) can be extended to a delay differential equations model as follows:

$$\begin{aligned} \frac{dP}{dt} &= rP\left(1 - \frac{P}{k}\right) - \alpha PZ, \\ \frac{dZ}{dt} &= \beta PZ - \mu Z - \frac{\theta P(t-\tau)Z}{\gamma + P(t-\tau)}. \end{aligned} \tag{1.4}$$

It is shown that the system (1.4) produces oscillations around the interior equilibrium point depending on the length of delay and thus mimics the cyclic nature of the phytoplankton-zooplankton system [6]. In this study, we first discretize the phytoplankton-zooplankton model (1.3) by nonstandard finite difference method and compare their dynamic properties with the corresponding continuous-time model. We show that NSFD model preserves positivity of solutions and other dynamic properties of the continuous-time model. In the second part, we discretize the delay-induced continuous model (1.4) and study its dynamic properties to show that the delay has no effect on the dynamics of trivial and semi-trivial fixed points. However, the stability of the coexistence fixed point depends on the length of delay and the results are shown to be consistent with that of the delay-induced continuous system.

The rest of the paper is organized as follows. Section 2 is devoted for the discretization of the non-delayed continuous model. Its dynamic consistency is also proved there. In Section 3, we discretize the delay-induced continuous-time system and present its stability results. Comprehensive simulations of both non-delayed and delayed models are performed to validate our analytical results in Section 4. The study ends with a discussion in Section 5.

2 Nondelayed Discrete Model

Here we propose and study the dynamic behaviors of the discrete model corresponding to the continuous system (1.3) formulated by nonstandard finite difference (NSFD) technique [18].

2.1 Nonstandard finite difference model

For convenience, we express system (1.3) as

$$\begin{aligned} \frac{dP}{dt} &= rP - \frac{rP^2}{K} - \alpha PZ, \\ \frac{dZ}{dt} &= \beta PZ - \mu Z - \frac{P}{\gamma + P} Z. \end{aligned} \tag{2.1}$$

With nonlocal approximations, we write the continuous-time system (2.1) as

$$\begin{aligned} \frac{P_{n+1} - P_n}{h} &= rP_n - \frac{r}{K} P_n P_{n+1} - \alpha p_{n+1} Z_n, \\ \frac{Z_{n+1} - Z_n}{h} &= \beta P_{n+1} Z_n - \mu Z_{n+1} - \frac{\theta P_{n+1}}{\gamma + P_{n+1}} Z_{n+1}, \end{aligned} \tag{2.2}$$

where $h\ (> 0)$ is the step-size. This system can be expressed as

$$\begin{aligned} P_{n+1} &= F(P_n, Z_n), \\ Z_{n+1} &= G(P_{n+1}, Z_n), \end{aligned} \tag{2.3}$$

with $F(P,Z) = \dfrac{(1+rh)P_n}{[1 + \frac{rh}{K}P_n + h\alpha Z_n]}$ and $G(P,Z) = \dfrac{(1 + h\beta P_{n+1})(\gamma + P_{n+1})Z_n}{[(1+h\mu)(\gamma + P_{n+1}) + h\theta P_{n+1}]}$. Note that solutions of the discrete-time system (2.3) will remain positive for any step-size if it starts with positive initial values.

2.1.1 Existence and stability of fixed points

At the fixed point, we have $P_{n+1} = P_n = P$ and $Z_{n+1} = Z_n = Z$. The discrete-time system (2.2) has three fixed points. One trivial fixed point $E_0 = (0,0)$, one semi-trivial fixed point $E_1 = (K, 0)$ and one interior fixed point $E^* = (P^*, Z^*)$, where $P^* = \dfrac{-(\beta\gamma - \mu - \theta) + \sqrt{(\beta\gamma - \mu - \theta)^2 + 4\beta\gamma\mu}}{2\beta}$, $Z^* = \dfrac{r}{\alpha}\big(1 - \dfrac{P^*}{K}\big)$. Observe that the first two fixed points always exist and the interior fixed point exists if $\theta < \theta^*$ and $P^* < K$, where $\theta^* = \dfrac{1}{K}(K + \gamma)(\beta K - \mu)$.

Definition 2.1 (See [9])**.** A finite difference method is called elementary stable if for any value of the step size, the fixed points of the discrete system are those of the corresponding differential system and the linear stability properties of each fixed point being the same for both the differential and discrete systems.

Definition 2.2 (See [10])**.** Let X^* be a fixed point of the discrete system (2.3) and $J(X^*)$ be the Jacobian of the system (2.3) at X^*. If λ_1 and λ_2 be two eigenvalues of $J(X^*)$, then the fixed point X^* is called stable if $|\lambda_1| < 1$, $|\lambda_2| < 1$ and a source if $|\lambda_1| > 1$, $|\lambda_2| > 1$. It is called a saddle if $|\lambda_1| < 1$, $|\lambda_2| > 1$ or $|\lambda_1| > 1$, $|\lambda_2| < 1$ and a nonhyperbolic fixed point if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Theorem 2.3 (See [10])**.** *Let λ_1 and λ_2 be the eigenvalues of the 2×2 matrix $J = (a_{ij})$. Then $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff the following conditions holds:*

$(i)\ 1-\det(J) > 0,\ (ii)\ 1-\mathrm{trace}(J)+\det(J) > 0$ *and* $(iii)\ 1+\mathrm{trace}(J)+\det(J) > 0.$

We have the following theorem for the stability of different fixed points of (2.3).

Theorem 2.4. *(i) The fixed point $E_0 = (0,0)$ is always a saddle. (ii) The fixed point $E_1 = (K,0)$ is stable if $\theta > \theta^*$ and it can not be a source. It is a saddle point if $\theta < \theta^*$. (iii) The coexistence fixed point E^* is stable if $\theta < \theta^*$, where $\theta^* = \dfrac{1}{K}(K+\gamma)(\beta K-\mu)$.*

Proof. The variational matrix of system (2.3) evaluated at an arbitrary fixed point (P, Z) is

$$J(P,Z) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{cases} a_{11} = \dfrac{(1+rh)}{[1+\frac{rh}{K}P+h\alpha Z]} - \dfrac{(1+rh)P(\frac{rh}{K})}{[1+\frac{rh}{K}P+h\alpha Z]^2}, \\ a_{12} = -\dfrac{(1+rh)Ph\alpha}{[1+\frac{rh}{K}P+h\alpha Z]^2}, \\ a_{21} = [\dfrac{(h\beta\gamma+1+2h\beta P)}{[(1+h\mu)(\gamma+P)+h\theta P]} - \dfrac{(1+h\beta P)(\gamma+P)(1+h\mu+h\theta)}{[(1+h\mu)(\gamma+P)+h\theta P]^2}]a_{11}Z, \\ a_{22} = \dfrac{(1+h\beta P)(\gamma+P)}{[(1+h\mu)(\gamma+P)+h\theta P]} + [\dfrac{(h\beta\gamma+1+2h\beta P)}{[(1+h\mu)(\gamma+P)+h\theta P]} - \\ \dfrac{(1+h\beta P)(\gamma+P)(1+h\mu+h\theta)}{[(1+h\mu)(\gamma+P)+h\theta P]^2}]a_{12}Z. \end{cases} \tag{2.4}$$

It is easy to check that the eigenvalues at E_0 are $\lambda_1 = 1+rh$, $\lambda_2 = \dfrac{1}{1+h\mu}$ and therefore the fixed point E_0 is always saddle.

At the fixed point E_1, eigenvalues are evaluated as $\lambda_1 = 1 - \dfrac{rh}{1+rh}$ and $\lambda_2 = \dfrac{(1+h\beta K)(\gamma+K)}{(1+h\mu)(\gamma+K)+h\theta K}$. Here λ_1 is always less than unity for any step size $h > 0$

and λ_2 is always positive. Thus, for any $h > 0$, $\lambda_2 < 1$ if $\theta > \dfrac{(K+\gamma)(\beta K-\mu)}{K}$ and therefore E_1 is stable. In this case, however, E^* does not exist. Since λ_1 is always less than unity, E_1 can not be a source. On the other hand, $\lambda_2 > 1$ if $\theta < \dfrac{(K+\gamma)(\beta K-\mu)}{K}$. Therefore, existence of the interior fixed point implies that the axial fixed point is saddle.

At the interior fixed point E^*, elements of the variational matrix $J(P^*, Z^*)$ are $a_{11} = 1 - \left(\dfrac{rh}{1+rh}\right)\left(\dfrac{P^*}{K}\right)$, $a_{12} = -\dfrac{h\alpha P^*}{1+rh}$, $a_{21} = \dfrac{h[\beta\gamma + 2\beta P^* - \mu - \theta]}{(1+h\beta P^*)(\gamma + P^*)}a_{11}Z^*$, $a_{22} = 1 + \dfrac{h[\beta\gamma + 2\beta P^* - \mu - \theta]}{(1+h\beta P^*)(\gamma + P^*)}a_{12}Z^*$. Clearly, a_{11} is positive and less than unity. Simple algebraic computations show that $\det(J) = a_{11}$. Therefore, $0 < \det(J) < 1$. After some algebraic computations, one can show that $1 - \text{trace}(J) + \det(J) = -\dfrac{h[\beta\gamma + 2\beta P^* - \mu - \theta]}{(1+h\beta P^*)(\gamma + P^*)}a_{12}Z^*$. As $a_{12} < 0$, condition (ii) of Theorem 2.3 will hold if $P^* > \dfrac{\mu + \theta - \beta\gamma}{2\beta}$. One can easily verify that this last inequality is always satisfied whenever P^* exists. Straight forward calculations show that $1 + \text{trace}(J) + \det(J)$= $2a_{11} + 2\left[1 - \left(\dfrac{h\beta P^*}{1+h\beta P^*}\right)\left(\dfrac{rh}{1+rh}\right)\left(1 - \dfrac{P^*}{K}\right)\left(\dfrac{\beta\gamma + 2\beta P^* - \mu - \theta}{2\beta\gamma + 2\beta P^*}\right)\right]$. Note that terms in each first bracket are positive and less than unity. Thus, condition (iii) of Theorem 2.3 is satisfied. Therefore, following Definition 2.2, E^* is stable whenever it exists. This completes the proof of the theorem. □

It is noted that above analysis shows that the fixed points of the discrete system (2.3) are those of the continuous system (1.3) and the stability properties of each fixed point for both the continuous and discrete systems are identical. As the solution of the discrete system always remain positive if it starts with positive initial values, the discrete model (2.3), following Definition 2.1, is therefore positive and elementary stable.

3 Delay-Induced Discrete Model

In this section, we first discretize the delay-induced continuous system (1.4) and then analyze its dynamic properties. We here want to analyze the stability of each fixed point of the discrete system by considering τ as the bifurcation parameter. Following [1, 24], we consider $P(t) = x(t\tau)$ and $Z(t) = y(t\tau)$ so that the continuous system (1.4) can be transferred to

$$\begin{aligned} \frac{dx}{dt} &= \tau r x(1 - \frac{x}{k}) - \tau\alpha xy, \\ \frac{dy}{dt} &= \tau\beta xy - \tau\mu y - \tau\frac{\theta x(t-1)y}{\gamma + x(t-1)}. \end{aligned} \tag{3.1}$$

We employ the following nonlocal approximations termwise to discretize the system (3.1):

$$\begin{cases} \dfrac{dx}{dt} \to \dfrac{x_{n+1}-x_n}{h}, & \dfrac{dy}{dt} \to \dfrac{y_{n+1}-y_n}{h}, \\ x \to x_n, & y \to y_{n+1}, \\ xy \to y_n x_{n+1}, & xy \to x_{n+1}y_{n+1}, \\ x^2 \to x_n x_{n+1}, & \dfrac{x(t-1)y}{\gamma + x(t-1)} \to \dfrac{x_{n-m}y_{n+1}}{\gamma + x_{n-m}}. \end{cases} \tag{3.2}$$

Here the step-size is of the form $h = \dfrac{1}{m}$, where m is a positive integer. x_n and y_n are numerical approximations of $x(t_0 + nh)$ and $y(t_0 + nh)$, where $n = 0, 1, 2, \ldots$. The initial conditions are assumed to be $x_m = \phi_m$ and $y_0 = y(t_0)$, where $\phi_m = \phi(t_m)$, for $m = -i, -i+1, -i+2, \ldots, 0$.

Using (3.2) in (3.1), we obtain the discrete system

$$\begin{aligned} x_{n+1} &= \frac{(1+rh\tau)x_n}{[1+\frac{rh\tau}{K}x_n + h\alpha\tau y_n]}, \\ y_{n+1} &= \frac{(1+h\tau\beta x_{n+1})(\gamma + x_{n-m})y_n}{[(1+\mu h\tau)(\gamma + x_{n-m}) + h\tau\theta x_{n-m}]}. \end{aligned} \tag{3.3}$$

It is to be noted that all terms in the right hand side are positive and hence all solutions of the delay-induced discrete system (3.3) remain positive for all positive initial values.

3.1 Existence and stability of fixed points

As before, one can compute the fixed points of the system (3.3). It is to be mentioned that the fixed points of all three systems (1.3), (2.3) and (3.3) are the same. For stability analysis, we linearize the system (3.3) about an arbitrary fixed point $(\overline{x}, \overline{y})$ by substituting $x_n = \overline{x} + u_n$ and $y_n = \overline{y} + v_n$, where $u_n << 1$ and $v_n << 1$. The linearized system is then given by

$$\begin{aligned} u_{n+1} &= a_{11}u_n + a_{12}v_n, \\ v_{n+1} &= a_{21}u_n + a_{22}v_n + a_{23}u_{n-m}, \end{aligned} \tag{3.4}$$

where

$$\begin{cases} a_{11} = \dfrac{(1+rh\tau)}{[1+\frac{rh\tau}{K}\overline{x}+h\tau\alpha\overline{y}]} - \dfrac{(1+rh\tau)\frac{rh\tau}{K}\overline{x}}{[1+\frac{rh\tau}{K}\overline{x}+h\tau\alpha\overline{y}]^2}, \\ a_{12} = -\dfrac{(1+rh\tau)\overline{x}\tau h\alpha}{[1+\frac{rh\tau}{K}\overline{x}+h\alpha\tau\overline{y}]^2}, \\ a_{21} = \dfrac{h\beta\tau(\gamma+\overline{x})}{[(1+\mu h\tau)(\gamma+\overline{x})+h\tau\theta\overline{x}]}a_{11}\overline{y}, \\ a_{22} = \dfrac{(1+h\beta\tau\overline{x})(\gamma+\overline{x})}{[(1+\mu h\tau)(\gamma+\overline{x})+h\tau\theta\overline{x}]} + \dfrac{h\beta\tau(\gamma+\overline{x})}{[(1+\mu h\tau)(\gamma+\overline{x})+h\tau\theta\overline{x}]}a_{11}\overline{y}, \\ a_{23} = \dfrac{(1+h\beta\tau\overline{x})}{[(1+\mu h\tau)(\gamma+\overline{x})+h\tau\theta\overline{x}]} + \dfrac{(1+h\beta\tau\overline{x})(\gamma+\overline{x})\overline{y}[1+h\mu\tau+h\theta\tau]}{[(1+\mu h\tau)(\gamma+\overline{x})+h\tau\theta\overline{x}]^2}. \end{cases} \tag{3.5}$$

System (3.4) can be expressed as

$$U_{n+1} = MU_n, \tag{3.6}$$

where $U_n = (u_n, u_{n-1}, \ldots, u_{n-m}, v_n)^T$ and M is a $(m+2)\times(m+2)$ matrix given by

$$M = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & 0 & a_{12} \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ a_{21} & 0 & \cdots & 0 & a_{23} & a_{22} \end{bmatrix}.$$

The characteristic equation of the matrix M is

$$\lambda^{m+2} + A\lambda^{m+1} + B\lambda^m + C = 0, \tag{3.7}$$

where $A = -(a_{11}+a_{22}),\ B = a_{11}a_{22} - a_{12}a_{21},\ C = -a_{12}a_{23}$.

We now prove the following theorems for the stability of different fixed points.

Theorem 3.1. *(i) The fixed point $E_0 = (0,0)$ is always a saddle. (ii) The fixed point $E_1 = (K,0)$ is stable if $\theta > \theta^*$ and it can not be a source. It is a saddle point if $\theta < \theta^*$, where $\theta^* = \dfrac{(K+\gamma)(\beta K-\mu)}{K}$.*

Proof. At the fixed point E_0, (3.7) becomes

$$\lambda^k[\lambda^2 - (1+th\tau+\frac{1}{1+h\mu\tau})\lambda + (1+th\tau)\frac{1}{1+h\mu\tau}] = 0.$$

It is easy to check that the eigenvalues are $\lambda_1 = (1+th\tau)$, $\lambda_2 = \dfrac{1}{1+h\mu\tau}$ and $k-$ fold roots $\lambda = 0$. Since all eigenvalues are positive but one is greater than unity and others are less than unity, the fixed point E_0 is always a saddle point.

At the fixed point E_1, the characteristic equation (3.7) becomes

$$\lambda^k[\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22}] \;\; = 0,$$

where $a_{11} = 1 - \dfrac{rh\tau}{1 + rh\tau}$ and $a_{22} = \dfrac{(1 + h\tau\beta K)(\gamma + K)}{(1 + h\tau\mu)(\gamma + K) + h\tau\theta K}$. The eigenvalues are $\lambda_1 = a_{11} = 1 - \dfrac{rh\tau}{1 + rh\tau}$, $\lambda_2 = a_{22} = \dfrac{(1 + h\tau\beta K)(\gamma + K)}{(1 + h\tau\mu)(\gamma + K) + h\tau\theta K}$ and $k-$ fold roots $\lambda = 0$. Observe that, for any step size $h > 0$, λ_1 is always less than unity and $\lambda_2 < 1$ if $\theta > \dfrac{(K + \gamma)(\beta K - \mu)}{K}$. Thus, the fixed point E_1 is stable if $\theta > \dfrac{(K + \gamma)(\beta K - \mu)}{K}$. In this case, however, E^* does not exist. Since λ_1 is always less than unity, so E_1 can not be a source. On the other hand, $\lambda_2 > 1$ if $\theta < \dfrac{(K + \gamma)(\beta K - \mu)}{K}$, which is the existence condition of E^*. In this later case, E_1 is a saddle. Hence the theorem is proven. □

It is to be noted that the stability properties of E_0 and E_1 remain unchanged in presence of delay and the results are consistent with that of the delay-induced continuous system (1.4).

Now we observe whether stability of the coexistence fixed point E^* is altered in presence of delay. First we state the following lemma.

Lemma 3.2 (See [25]). *Suppose that $B \subset R$ is a bounded, closed and connected set and $f(\lambda, \tau) = \lambda^k + p_1(\tau)\lambda^{k-1}(\tau) + \ldots + p_k(\tau)$ is continuous in $(\lambda, \tau) \in C \times B$, where $\tau \geq 0$ is a parameter, $\tau \in B$. Then as τ varies, the sum of the order of the zeros of $f(\lambda, \tau)$ out of the unit circle $\{\lambda \in C : |\lambda| > 1\}$ can change only if a zero appears on or crosses the unit circle.*

Theorem 3.3. *There exists a critical value τ^* of τ such that for $0 < \tau < \tau^*$, the coexistence fixed point E^* is stable whenever it exists.*

Proof. When $\tau = 0$, the characteristic equation (3.7) at E^* is

$$\lambda^{k+2} - 2\lambda^{k+1} + \lambda^k \;\; = 0. \tag{3.8}$$

This equation has $k-$ fold roots $\lambda = 0$ and double roots $\lambda = 1$. Consider the root $\lambda(\tau)$ such that $|\lambda(0)| = 1$. This root depends continuously on τ and is a differentiable function of τ. Note that $\dfrac{d\lambda}{d\tau}$ satisfies the following equation

$$\left(\frac{d\lambda}{d\tau}\right)^2 + \frac{rhx^*}{K}\left(\frac{d\lambda}{d\tau}\right) + h^2\alpha\beta x^* y^* \;\; = 0. \tag{3.9}$$

Thus, whenever, $rx^* = 4K\beta(K - x^*)$ is satisfied, we have

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=0,\lambda=1} = -\frac{rhx^*}{K} < 0. \tag{3.10}$$

Then all the roots of (3.7) lie inside the open unit circle for sufficiently small $\tau > 0$ and the existence of maximal τ^* follows. Hence the theorem. □

A Neimark–Sacker bifurcation occurs when a complex conjugate pair of eigenvalues of the matrix M cross the unit circle as τ varies. Let $\lambda = e^{i\omega}$ be the root of (3.7) when $\tau = \tau^*$. Substituting $\lambda = e^{i\omega}$ in (3.7), we have

$$e^{(m+2)i\omega} + A^* e^{(m+1)i\omega} + B^* e^{mi\omega} + C^* = 0, \tag{3.11}$$

where A^*, B^*, C^* are the values of A, B, C evaluated at the interior fixed point E^* with $\tau = \tau^*$. Separating real and imaginary parts, we have

$$\begin{gathered}\cos(m+2)\omega^* + A^* \cos(m+1)\omega^* + B^* \cos m\omega^* + C^* = 0,\\ \sin(m+2)\omega^* + A^* \sin(m+1)\omega^* + B^* \sin \omega^* = 0.\end{gathered} \tag{3.12}$$

Thus, there exists an infinite sequence of τ such that $0 < \tau_1 < \tau_2 < \ldots \tau_i < \ldots$ and satisfies (3.12). From (3.7), one can calculate

$$\frac{d\lambda}{d\tau} = \frac{2N}{D}, \tag{3.13}$$

where

$$\begin{cases} N &= X + Y(1-\cos\omega),\\ D &= \left|(m+2)C + A\lambda^{m+1} + 2B\lambda^m\right|^2\\ &= (m+2)^2C^2 + A^2 + 4B^2 - (m+2)A^2 + 2(m+2)BA_1\\ &\quad + (2AB - (m+2)A(B+1) + 2(m+2)A)\cos\omega,\\ X &= \left[(m+2)C + \dfrac{A^2}{C} + \dfrac{2A_1B}{C} - \dfrac{A(B+1)}{C} + \dfrac{2A}{C}\right]\dfrac{dC}{d\tau}\\ &\quad + \left[2B + (m+2)A_1 + (m+2)AB + A\right]\dfrac{dB}{d\tau}\\ &\quad + \left[A - (m+2)A - (m+2)(B+1) + 2B\right]\dfrac{dA}{d\tau},\\ A_1 &= (1-B) + \dfrac{A^2 + (B-1)^2 - C^2}{2B},\\ Y &= -\left[2B - (m+2)(B+1)\right]\dfrac{dA}{d\tau} + \left[(m+2)AB + A\right]\dfrac{dB}{d\tau}\\ &\quad + \left[\dfrac{2A}{B} - \dfrac{A(B+1)}{C}\right]\dfrac{dC}{d\tau} \end{cases} \tag{3.14}$$

and $\cos\omega = \dfrac{A(B+1) + \sqrt{(B-1)^2(A^2-4B) + 4BC^2}}{4B}$. Here D is always positive and

$$A = 2 - \frac{rh\tau x^*}{(1+rh\tau)K} - \frac{h^2\tau^2\alpha\beta x^* y^*}{(1+h\beta\tau x^*)(1+rh\tau)}, \quad B = 1 - \frac{rh\tau x^*}{(1+rh\tau)K},$$

$$C = \frac{h^2\tau^2\alpha x^* y^*[\beta x^* - \mu - \theta]}{(1+h\beta\tau x^*)^2(1+rh\tau)^2(\gamma+x^*)},$$
$$\frac{dA}{d\tau} = \frac{rhx^*}{(1+rh\tau)^2 K} + \frac{h^2\tau\alpha\beta x^* y^*[2+h\tau\beta x^* + rh\tau]}{[(1+rh\tau)(1+h\beta\tau)]^2},$$
$$\frac{dB}{d\tau} = -\frac{rhx^*}{(1+rh\tau)^2 K},$$
$$\frac{dC}{d\tau} = \frac{h^2\tau\alpha x^* Y^*(\beta x^* - \mu - \theta)[2+rh\tau + h\beta\tau x^*]}{(1+rh\tau)^2(1+h\beta\tau x^*)^2(\gamma+x^*)}.$$

Note that all A, B, C are actually A^*, B^*, C^* which satisfy equation (3.12). Substituting all these values in N, one has to show that $\frac{d\lambda}{d\tau}$ is positive at $\tau = \tau^*$. Analytically it is very difficult to show the positivity of $\frac{d\lambda}{d\tau}$. In numerical section we however show that $\frac{d\lambda}{d\tau} > 0$ at $\tau = \tau^*$. Thus, we have the following theorem assuming that $\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau^*} > 0$.

Theorem 3.4. *Assume that the equation* (3.12) *is satisfied. There exists a sequence of values of the delay parameter τ, $0 < \tau^* < \tau_1 < \tau_2 < \ldots \tau_i < \ldots$, such that the interior fixed point E^* of the system* (3.3) *is asymptotically stable for $\tau \in [0, \tau^*)$ and unstable for $\tau > \tau^*$. A Neimark-Sacker bifurcation occurs around E^* at $\tau = \tau_i$, $i = *, 1, 2, \ldots$.*

4 Numerical Simulations

In this section we will validate and illustrate our theoretical results with the parameter values as in [6]. The same parameter set $r = 0.2$, $\alpha = 0.9$, $\beta = 0.3$, $\gamma = 0.06$, $\mu = 0.02$, $\theta = 0.9$, except $K = 60$, is considered for the numerical simulations of both the nondelayed and delayed systems. Initial point $I = (0.6, 0.2)$ is kept fixed for all simulations [6]. It is to be mentioned that this parameter set satisfies the condition of Theorem 2.3(iii) with $\theta^* = 17.9980$. To observe the dynamic consistency about the interior equilibrium E^*, we plot phase portraits of both the continuous system (1.3) and discrete system (2.3) in Figure 4.1 with $\theta = 0.9$. It shows that trajectory in each system reaches to the interior equilibrium $E^* = (3.0080, 0.2111)$ and therefore the systems are dynamically equivalent. Figure 4.2 shows that time series solutions of the continuous system (1.3) and the discrete system (2.3) converges to the predator-free equilibrium E_1 for $\theta = 18(> \theta^*)$, following Theorem 2.3(ii). It again shows dynamic consistency of both systems.

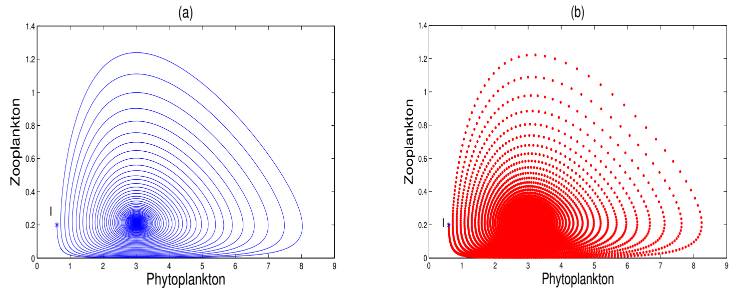

Figure 4.1: Phase portraits of the continuous system (1.3) (a) and NSFD system (2.3) (b). These figures show that solution starting from $I = (0.6, 0.2)$ reaches to the interior fixed point $E^* = (3.0080, 0.2111)$ for both systems. Parameters are $r = 0.2$, $\alpha = 0.9$, $\beta = 0.3$, $\gamma = 0.06$, $\mu = 0.02$, $K = 60$, $\theta = 0.9$. Step-size is $h = 0.1$ for the NSFD system.

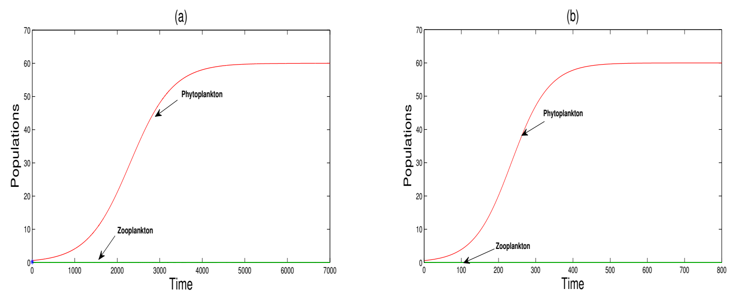

Figure 4.2: These figures show that phytoplankton population reaches to its carrying capacity and zooplankton population goes to extinction, indicating stability of the equilibrium E_1, for the continuous system (1.3) (a) and the NSFD system (2.3) (b). All parameters and initial points are as in Figure 4.1 except $\theta = 18$.

It is interesting to observe that the corresponding Euler-discrete system

$$\begin{aligned} P_{n+1} &= P_n + h[rP_n(1 - \frac{P_n}{k}) - \alpha P_n Z_n], \\ Z_{n+1} &= Z_n + h[\beta P_n Z_n - \mu Z_n - \frac{\theta P_n Z_n}{\gamma + P_n}], \end{aligned} \tag{4.1}$$

behaves differently. To compare the behavior of the Euler discrete system (4.1) and NSFD discrete system (2.3) of the same continuous system (1.3), we present bifurcation diagrams of both discrete systems with step-size h as the bifurcation parameter (Figure 4.3). It shows that the dynamical behavior of the NSFD system is independent of the step-size (Figure 4.3 (a)) but the dynamics of the Euler system depends on the step-size h (Figure 4.3 (b)). Thus, the NSFD system is dynamically consistent with its continuous counterpart, but the Euler system is dynamically inconsistent.

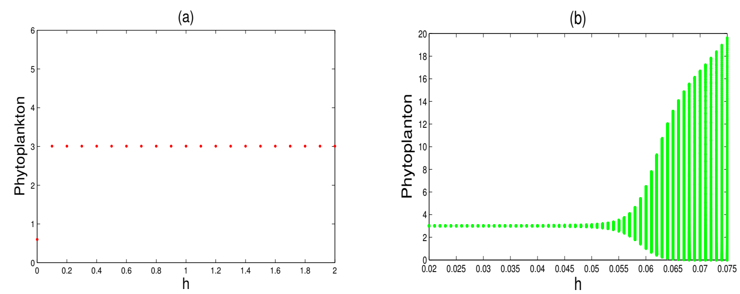

Figure 4.3: Bifurcation diagram of phytoplankton population of the NSFD system (2.3) (a) and that of the Euler system (4.1) (b). The first figure shows that phytoplankton population is stable for all step-size h.The second figure shows that phytoplankton population is stable for small values of h and unstable for higher values. Parameters and initial value are same as in Figure 4.1.

Now we show that our delayed NSFD system (3.3) is dynamically consistent with the delayed-continuous system (3.1). We consider the same initial point and parameter set as before. Figure 4.4 compares the time evolutions of the delayed continuous system (3.1) (Figure 4.4 (a)) and delayed discrete system (3.3) (Figure 4.4 (b)). These figures show that both the systems are stable for lower delay $\tau = 5$. However, the systems are unstable for higher value $\tau = 15$ (Figures 4.4 (c) and (d)).

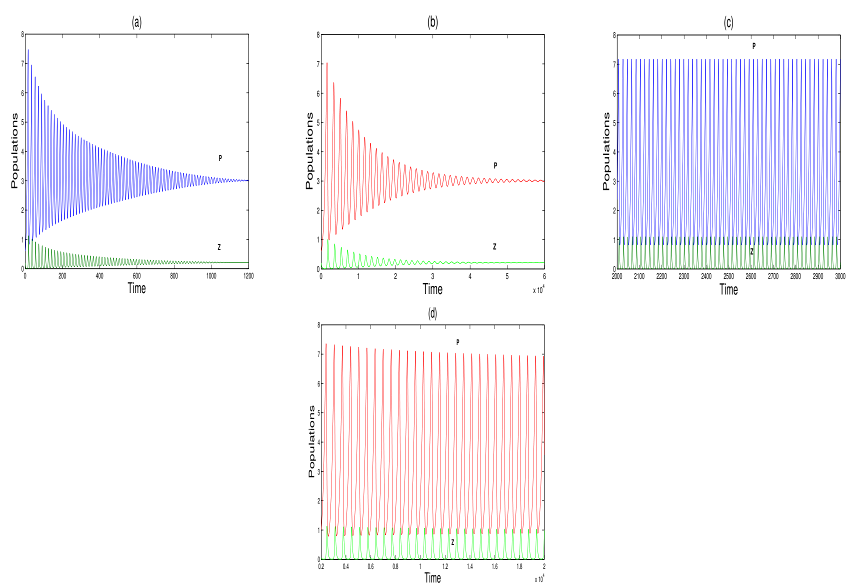

Figure 4.4: (a) and (b) show stable behavior of the delayed continuous system (3.1) and the delayed discrete system (2.3) for $\tau = 5$. (c) and (d) show the unstable behavior of these systems for $\tau = 15$. All parameters and initial point are same as in Figure 4.1 except $m = 500$ (i.e., $h = 0.002$), $x_{n-m} = 0.6$ for $n = 1, 2, 3, \ldots 500, 501$ and other parameters are as in Figure 4.1.

We compare our delayed continuous system and delayed NSFD system with the delayed Euler system

$$\begin{aligned} x_{n+1} &= x_n + h[\tau r x_n(1 - \frac{x_n}{k}) - \tau\alpha x_n y_n], \\ y_{n+1} &= y_n + h[\tau\beta x_n y_n - \tau\mu y_n - \tau\frac{\theta x_{n-m} y_n}{\gamma + x_{n-m}}], \end{aligned} \tag{4.2}$$

where h and m have the same meanings as before. The delayed Euler system (4.2) has been constructed following [1]. In Figure 4.5, we give bifurcation diagrams of the delayed continuous system (3.1), delayed NSFD system (3.3) and delayed Euler system (4.2) with τ as the bifurcation parameter. Figures 4.5 (a) and (b) show that the continuous system and the NSFD system are stable around E^* for $\tau < 14.52$ and unstable for $\tau > 14.52$. Therefore, a Neimark–Sacker bifurcation occurs at $\tau = \tau^* = 14.52$. This result supports Theorem 3.4 and also proves the dynamic consistency of the delayed NSFD system (3.3) with that of the delayed-continuous system (3.1). However, the delayed Euler system (4.2) is unstable around the fixed point E^* for significantly lower value of τ, depicting its inconsistency with the continuous system.

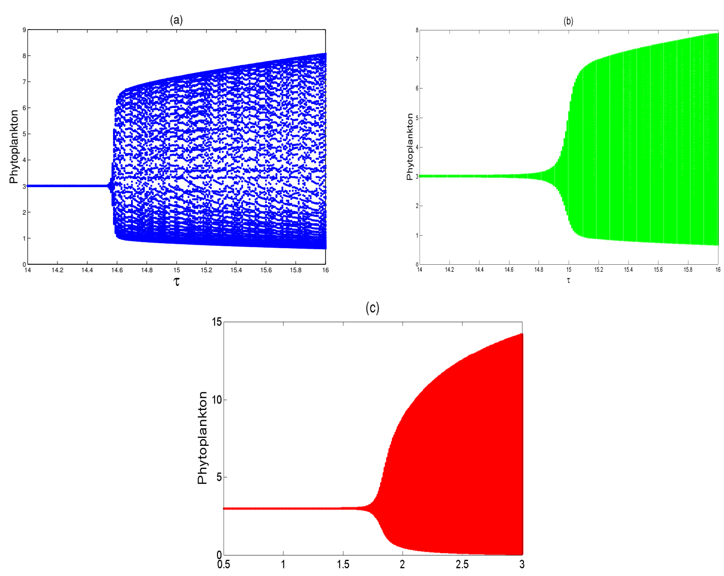

Figure 4.5: Bifurcation diagrams of phytoplankton population of the delayed continuous system (3.1) (a), delayed NSFD system (3.3) (b) and delayed Euler system (4.2) (c) with τ as the bifurcation parameter. (a) and (b) show that the phytoplankton population is stable for $\tau < 14.525$ and unstable as τ crosses 14.52. (c) shows that phytoplankton population is stable for $\tau < 1.78$ and unstable for all $\tau > 1.78$. Parameters and initial point are same as in Figure 4.4.

5 Summary and Discussion

In this paper, we discretized a continuous-time phytoplankton-zooplankton (respectively, prey and predator) model studied in [6] where phytoplankton produces some harmful chemical which is toxic to zooplankton. We first discretized the continuous time nondelayed model following nonstandard finite difference scheme so that the discrete model shows dynamic consistency with its continuous counterpart. Considering that the negative effect of toxic chemical on zooplankton is not instantaneous, we incorporated a delay in the system which measures the delayed negative effect on the zooplankton. The delay-induced NSFD discrete system is then analyzed to show that the delay has no effect on the dynamics of trivial and zooplankton-free fixed points. However, the stability of the coexistence fixed point depends on the length of delay. There exists some critical length of the delay parameter such that the system is stable if the length is smaller than it and unstable if it is higher. A Neimark–Sacker bifurcation occurs at that critical value of the delay. We numerically show that the critical value of the delay parameter is same with that of the delay-induced continuous system. It is also shown that solutions of both the nondelayed and delayed discrete systems are always positive with all feasible initial points. Our simulations results show that the corresponding delayed Euler system is dynamically inconsistent and exhibits unstable behavior for significantly lower value of delay.

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