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Influence of the Stepsize on Hyers–Ulam Stability of First-Order Homogeneous Linear Difference Equations

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Abstract

This paper is concerned with Hyers–Ulam stability of the first-order linear difference equation $\Delta_h x(t) - ax(t) = 0$ on $h\mathbb{Z}$, where a is a real number, $\Delta_h x(t) = (x(t+h) - x(t))/h$ and $h\mathbb{Z} = \{hk|\, k \in \mathbb{Z}\}$ for the stepsize $h > 0$. It is well known that this equation is an approximation of the ordinary differential equation $x' - ax = 0$. The obtained results are divided into the small stepsize case and the large stepsize case. The main purpose of this paper is to clarify the following: in the small stepsize case, the minimum of HUS constants of difference equation is the same as that of ODE; in the large stepsize case, the minimum of HUS constants of difference equation is different from that of ODE. To illustrate the obtained results, some applications to perturbed linear difference equations are included. Furthermore, some suitable examples are also attached for a deeper understanding.

AMS Subject Classifications: 39A06, 39A30, 34A30.
Keywords: Hyers–Ulam stability, HUS constant, linear difference equation.

1 Introduction

Hyers–Ulam stability originated from a stability problem in the field of functional equations which was posed by Ulam [31,32] in 1940. This problem was solved by Hyers [9] in 1941. After that, many researchers have studied Hyers–Ulam stability in the field of functional equations (see [1, 5, 10, 15, 19]). In 1998, Alsina and Ger [2] considered Hyers–Ulam stability of the linear differential equation $x' - x = 0$. They proved that if a differentiable function $\phi : I \to \mathbb{R}$ satisfies $|\phi'(t) - \phi(t)| \le \varepsilon$ for all $t \in I$, then there exists a solution $x : I \to \mathbb{R}$ of $x' - x = 0$ such that $|\phi(t) - x(t)| \le 3\varepsilon$ for all

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$t \in I$, where $\varepsilon > 0$ is a given arbitrary constant and I is a nonempty open interval of $\mathbb{R}$. Later, this result has been extended to various linear differential equations by many researchers (see [3,6,8,11–14,17,18,22,24,27–30,33]). Moreover, Hyers–Ulam stability of the difference equations can be found in [4,16,23,25,26] and the references therein.

In this paper, we consider the first-order homogeneous linear difference equation

$$\Delta_h x(t) - ax(t) = 0 \tag{1.1}$$

on $h\mathbb{Z}$, where a is a real number and

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h} \quad \text{and} \quad h\mathbb{Z} = \{hk \mid k \in \mathbb{Z}\}$$

for given $h > 0$. We call h the "*stepsize*". Let $T = h\mathbb{Z} \cap I$, where I is a nonempty open interval of $\mathbb{R}$. We define

$$T^* = \begin{cases} T \setminus \{\max T\} & \text{if the maximum of } T \text{ exists,} \\ T & \text{otherwise.} \end{cases}$$

Note here that if a function $x(t)$ exists on T, then $\Delta_h x(t)$ exists on T^*, and if $a = -1/h$ holds, then we no longer have a first-order difference equation. Throughout this paper, we assume that T and T^* are nonempty sets of $\mathbb{R}$, and $a \neq -1/h$.

We say that (1.1) has the "*Hyers–Ulam stability*" on T if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi : T \to \mathbb{R}$ satisfies $|\Delta_h \phi(t) - a\phi(t)| \leq \varepsilon$ for all $t \in T^*$, then there exists a solution $x : T \to \mathbb{R}$ of (1.1) such that $|\phi(t) - x(t)| \leq K\varepsilon$ for all $t \in T$. We call such K a "*HUS constant*" for (1.1) on T. From the following remark, we will treat only the case of $a \neq 0$ and $a \neq -2/h$ in this paper.

Remark 1.1. If $a = 0$ or $a = -2/h$, then (1.1) does not have Hyers–Ulam stability on $h\mathbb{Z}$. For example, in the case $a = 0$, we consider the function $\phi(t) = \varepsilon t$ satisfying $|\Delta_h \phi(t)| = \varepsilon$ for all $t \in T^*$. Since $x(t) \equiv c$ is the general solution of $\Delta_h x(t) = 0$, we have $|\phi(t) - x(t)| \to \infty$ as $t \to \infty$; that is, (1.1) does not have Hyers–Ulam stability on $h\mathbb{Z}$ when $a = 0$. In the case $a = -2/h$, the function $\phi(t) = \varepsilon t(-1)^{t/h}$ satisfies $|\Delta_h \phi(t) + 2\phi(t)/h| = \varepsilon$ for all $t \in T^*$. Since the general solution of $\Delta_h x(t) + 2x(t)/h = 0$ is $x(t) = c(-1)^{t/h}$, we see that $|\phi(t) - x(t)| \to \infty$ as $t \to \infty$; that is, (1.1) does not have Hyers–Ulam stability on $h\mathbb{Z}$ when $a = -2/h$.

It is well known that (1.1) is an approximation of the first-order homogeneous ordinary differential equation

$$x' - ax = 0 \tag{1.2}$$

on I, where I is a nonempty open interval of $\mathbb{R}$, and a is a non-zero real number. In fact, for the sake of simplicity, we consider (1.2) on $[0, b]$, where $b > 0$ and $[0, b] \subset I$. Let us divide the interval $[0, b]$ into n parts, where $n \in \mathbb{N}$. Letting $h = b/n$, and using Euler's method, we get (1.1) on $[0, b]$ since $x'(t)$ can be approximated by $\Delta_h x(t)$ (see [7, pp. 20–27]). Hyers–Ulam stability of (1.2) and its generalized equations on I has been studied by [11–13, 22, 24, 27, 29, 30, 33]. We say that (1.2) has the "*Hyers–Ulam stability*" on I if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a differentiable function $\phi : I \to \mathbb{R}$ satisfies $|\phi'(t) - a\phi(t)| \le \varepsilon$ for all $t \in I$, then there exists a solution $x : I \to \mathbb{R}$ of (1.2) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in I$. We call such K a "*HUS constant*" for (1.2) on I.

The following result is obtained by using one of the results presented by Jung [13] et al.

Theorem 1.2 (See [13, 22, 24, 29])**.** *Suppose that $a \neq 0$. Then* (1.2) *has Hyers–Ulam stability with a HUS constant $1/|a|$ on $\mathbb{R}$. Furthermore, the solution $x(t)$ of* (1.2) *satisfying $|\phi(t) - x(t)| \le \varepsilon/|a|$ for all $t \in \mathbb{R}$ is the only one (unique), where $\phi(t)$ is a differentiable function satisfying $|\phi'(t) - a\phi(t)| \le \varepsilon$ for all $t \in \mathbb{R}$.*

A natural question now arises. Can we find an explicit solution corresponding to the solution $x(t)$ of (1.2) in Theorem 1.2? Onitsuka and Shoji [24] gave the answer to this question. The obtained result is as follows.

Theorem 1.3 (See [24, Theorem 1])**.** *Let $\varepsilon > 0$ be a given arbitrary constant. Suppose that a differentiable function $\phi : I \to \mathbb{R}$ satisfies $|\phi'(t) - a\phi(t)| \le \varepsilon$ for all $t \in I$, where $a \neq 0$. Then one of the following holds:*

(i) if $a > 0$ and $\sup I$ exists, then $\lim_{t\to\tau-0} \phi(t)$ exists where $\tau = \sup I$, and any solution $x(t)$ of (1.2) *with $\left|\lim_{t\to\tau-0} \phi(t) - x(\tau)\right| < \varepsilon/a$ satisfies that $|\phi(t) - x(t)| < \varepsilon/a$ for all $t \in I$;*

(ii) if $a > 0$ and $\sup I$ does not exist, then $\lim_{t\to\infty} \phi(t)e^{-at}$ exists, and there exists exactly one solution $x(t) = \left(\lim_{t\to\infty} \phi(t)e^{-at}\right) e^{at}$ of (1.2) *such that $|\phi(t) - x(t)| \le \varepsilon/a$ for all $t \in I$;*

(iii) if $a < 0$ and $\inf I$ exists, then $\lim_{t\to\sigma+0} \phi(t)$ exists where $\sigma = \inf I$, and any solution $x(t)$ of (1.2) *with $\left|\lim_{t\to\sigma+0} \phi(t) - x(\sigma)\right| < \varepsilon/|a|$ satisfies that $|\phi(t) - x(t)| < \varepsilon/|a|$ for all $t \in I$;*

(iv) if $a < 0$ and $\inf I$ does not exist, then $\lim_{t\to-\infty} \phi(t)e^{-at}$ exists, and there exists exactly one solution $x(t) = \left(\lim_{t\to-\infty} \phi(t)e^{-at}\right) e^{at}$ of (1.2) *such that $|\phi(t) - x(t)| \le \varepsilon/|a|$ for all $t \in I$.*

Using (ii) or (iv) in Theorem 1.3, we can find an explicit solution corresponding to the solution $x(t)$ of (1.2) in Theorem 1.2.

We will recall that (1.1) is an approximation of (1.2). In the case that $h = 1$, the following theorem is obtained by using a result presented by Brzdęk, Popa and Xu [4].

Theorem 1.4 (See [4, Theorem 3])**.** *Suppose that $a \neq 0$, -1, -2 and $h = 1$. Then* (1.1) *has Hyers–Ulam stability with a HUS constant $1/||a+1|-1|$ on $\mathbb{Z}$. Furthermore, the solution $x(t)$ of* (1.1) *satisfying $|\phi(t) - x(t)| \leq \varepsilon/||a+1|-1|$ for all $t \in \mathbb{Z}$ is the only one, where $\phi(t)$ is a function satisfying $|\Delta_1\phi(t) - a\phi(t)| \leq \varepsilon$ for all $t \in \mathbb{Z}$.*

Comparing Theorem 1.2 with Theorem 1.4, we see that if $a \neq 0$ and $a > -1$, then the minimums of HUS constants in Theorems 1.2 and 1.4 are the same. However, if $a \neq -2$ and $a < -1$, then they are different. Now, important questions arise:

(Q1) What is the minimum of HUS constants for (1.1) on $h\mathbb{Z}$?

(Q2) Can we find an explicit solution corresponding to the solution $x(t)$ of (1.1) in Theorem 1.4?

(Q3) How does the stepsize influence the minimum of HUS constants for (1.1) on $h\mathbb{Z}$?

The purpose of this paper is to answer the above questions. The answers to these questions are derived in Section 4 from the following two theorems.

Theorem 1.5. *Let $\varepsilon > 0$ be a given arbitrary constant. Suppose that a function $\phi : T \to \mathbb{R}$ satisfies $|\Delta_h\phi(t) - a\phi(t)| \leq \varepsilon$ for all $t \in T^*$, where $a \neq 0$ and $a > -1/h$. Then one of the following holds:*

(i) if $a > 0$ and $\max T$ exists, then any solution $x(t)$ of (1.1) *with $|\phi(\tau) - x(\tau)| < \varepsilon/a$ satisfies that $|\phi(t) - x(t)| < \varepsilon/a$ for all $t \in T$, where $\tau = \max T$;*

(ii) if $a > 0$ and $\max T$ does not exist, then $\lim_{t\to\infty} \phi(t)(ah+1)^{-t/h}$ exists, and there exists exactly one solution

$$x(t) = \left\{\lim_{t\to\infty} \phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}}$$

of (1.1) *such that $|\phi(t) - x(t)| \leq \varepsilon/a$ for all $t \in T$;*

(iii) if $-1/h < a < 0$ and $\min T$ exists, then any solution $x(t)$ of (1.1) *with $|\phi(\sigma) - x(\sigma)| < \varepsilon/|a|$ satisfies that $|\phi(t) - x(t)| < \varepsilon/|a|$ for all $t \in T$, where $\sigma = \min T$;*

(iv) if $-1/h < a < 0$ and $\min T$ does not exist, then $\lim_{t\to-\infty} \phi(t)(ah+1)^{-t/h}$ exists, and there exists exactly one solution

$$x(t) = \left\{\lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}}$$

of (1.1) *such that $|\phi(t) - x(t)| \leq \varepsilon/|a|$ for all $t \in T$.*

Remark 1.6. If the stepsize $h > 0$ is sufficiently small, then we can choose a h so that $0 < h < 1/|a|$; that is, $-1/h < a$ holds in the case that a is negative. From this fact, we can conclude that the assertions (iii) and (iv) in Theorem 1.5 are the results when the stepsize is sufficiently small.

Theorem 1.7. *Let $\varepsilon > 0$ be a given arbitrary constant. Suppose that a function $\phi : T \to \mathbb{R}$ satisfies $|\Delta_h \phi(t) - a\phi(t)| \le \varepsilon$ for all $t \in T^*$, where $a \neq -2/h$ and $a < -1/h$. Then one of the following holds:*

(i) if $-2/h < a < -1/h$ and $\min T$ exists, then any solution $x(t)$ of (1.1) with $|\phi(\sigma) - x(\sigma)| < \varepsilon/(a + 2/h)$ satisfies that $|\phi(t) - x(t)| < \varepsilon/(a + 2/h)$ for all $t \in T$, where $\sigma = \min T$;

(ii) if $-2/h < a < -1/h$ and $\min T$ does not exist, then $\lim_{t\to-\infty} \phi(t)(ah+1)^{-t/h}$ exists, and there exists exactly one solution

$$x(t) = \left\{ \lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}}$$

of (1.1) such that $|\phi(t) - x(t)| \le \varepsilon/(a + 2/h)$ for all $t \in T$;

(iii) if $a < -2/h$ and $\max T$ exists, then any solution $x(t)$ of (1.1) with $|\phi(\tau) - x(\tau)| < \varepsilon/|a + 2/h|$ satisfies that $|\phi(t) - x(t)| < \varepsilon/|a + 2/h|$ for all $t \in T$, where $\tau = \max T$;

(iv) if $a < -2/h$ and $\max T$ does not exist, then $\lim_{t\to\infty} \phi(t)(ah+1)^{-t/h}$ exists, and there exists exactly one solution

$$x(t) = \left\{ \lim_{t\to\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}}$$

of (1.1) such that $|\phi(t) - x(t)| \le \varepsilon/|a + 2/h|$ for all $t \in T$.

Remark 1.8. If the stepsize $h > 0$ is suitably large, then we can choose a h so that either $a < -2/h$ or $-2/h < a < -1/h$. From this fact, we can conclude that the assertions in Theorem 1.7 are the results when the stepsize is suitably large.

In the next section, we will consider small stepsize linear difference equations, and present the proof of Theorem 1.5. In Section 3, we will consider large stepsize linear difference equations, and present the proof of Theorem 1.7. Using Theorems 1.5 and 1.7, we will give the answers to questions (Q1)–(Q3) in Section 4. In the final section, we will discuss the applications to perturbed linear difference equations corresponding to linear difference equation (1.1). For illustration of the obtained results, we will take some concrete examples.

2 Small Stepsize Linear Difference Equations

In this section, we consider the case of $a \neq 0$ and $a > -1/h$. First we give some preparations for the proof of Theorem 1.5.

Lemma 2.1. *Suppose that $a \neq 0$ and $a > -1/h$. Let $\varepsilon > 0$ be a given arbitrary constant and let $\phi(t)$ be a real-valued function on T. Then the inequality $|\Delta_h \phi(t) - a\phi(t)| \leq \varepsilon$ holds for all $t \in T^*$ if and only if the inequality*

$$0 \leq \Delta_h \left\{ \left(\phi(t) - \frac{\varepsilon}{a} \right) (ah+1)^{-\frac{t}{h}} \right\} \leq 2\varepsilon (ah+1)^{-\frac{t+h}{h}}$$

holds for all $t \in T^*$.

Proof. The statement of Lemma 2.1 is clearly true since the equality

$$\begin{aligned} &\Delta_h \left\{ \left(\phi(t) - \frac{\varepsilon}{a} \right) (ah+1)^{-\frac{t}{h}} \right\} \\ &= \frac{1}{h} \left\{ \left(\phi(t+h) - \frac{\varepsilon}{a} \right) - (ah+1) \left(\phi(t) - \frac{\varepsilon}{a} \right) \right\} (ah+1)^{-\frac{t+h}{h}} \\ &= (\Delta_h \phi(t) - a\phi(t) + \varepsilon)(ah+1)^{-\frac{t+h}{h}} \end{aligned}$$

holds for all $t \in T^*$. □

Proposition 2.2. *Let $\varepsilon > 0$ be a given arbitrary constant. Suppose that a function $\phi : T \to \mathbb{R}$ satisfies $|\Delta_h \phi(t) - a\phi(t)| \leq \varepsilon$ for all $t \in T^*$, where $a \neq 0$ and $a > -1/h$. Then there exist a nondecreasing function $u : T \to \mathbb{R}$ and a nonincreasing function $v : T \to \mathbb{R}$ such that*

$$\phi(t) = u(t)(ah+1)^{\frac{t}{h}} + \frac{\varepsilon}{a} = v(t)(ah+1)^{\frac{t}{h}} - \frac{\varepsilon}{a} \tag{2.1}$$

and one of the following hold:

(i) if $a > 0$ and $\max T$ exists, then the inequality

$$u(t) \leq u(\tau) < v(\tau) \leq v(t) \tag{2.2}$$

holds for all $t \in T$, where $\tau = \max T$;

(ii) if $a > 0$ and $\max T$ does not exist, then $\lim_{t\to\infty} u(t)$ and $\lim_{t\to\infty} v(t)$ exist, and

$$u(t) \leq \lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) \leq v(t) \tag{2.3}$$

holds for all $t \in T$;

(iii) if $-1/h < a < 0$ *and* $\min T$ *exists, then the inequality*

$$v(t) \le v(\sigma) < u(\sigma) \le u(t) \tag{2.4}$$

holds for all $t \in T$*, where* $\sigma = \min T$*;*

(iv) if $-1/h < a < 0$ *and* $\min T$ *does not exist, then* $\lim_{t\to-\infty} u(t)$ *and* $\lim_{t\to-\infty} v(t)$ *exist, and*

$$v(t) \le \lim_{t\to-\infty} v(t) = \lim_{t\to-\infty} u(t) \le u(t) \tag{2.5}$$

holds for all $t \in T$.

Proof. Suppose that $a \neq 0$ and $a > -1/h$. Define the functions $u(t)$ and $v(t)$ as follows:

$$u(t) = \left(\phi(t) - \frac{\varepsilon}{a}\right)(ah+1)^{-\frac{t}{h}} \quad \text{and} \quad v(t) = \left(\phi(t) + \frac{\varepsilon}{a}\right)(ah+1)^{-\frac{t}{h}}$$

for $t \in T$. Then (2.1) holds, and therefore, we obtain

$$u(t) = v(t) - \frac{2\varepsilon}{a}(ah+1)^{-\frac{t}{h}} \tag{2.6}$$

and

$$u(t) \begin{cases} < v(t) & \text{if } \ a > 0, \\ > v(t) & \text{if } \ -\dfrac{1}{h} < a < 0 \end{cases} \tag{2.7}$$

for $t \in T$. Using the assertion in Lemma 2.1, we have the inequality

$$0 \le \Delta_h u(t) \le 2\varepsilon(ah+1)^{-\frac{t+h}{h}}$$

for $t \in T^*$. Noticing that $\Delta_h(ah+1)^{-t/h} = -a(ah+1)^{-(t+h)/h}$ and using (2.6) and the above inequality, we get

$$-2\varepsilon(ah+1)^{-\frac{t+h}{h}} \le \Delta_h v(t) \le 0$$

for $t \in T^*$. Therefore, we see that $u(t)$ is nondecreasing and $v(t)$ is nonincreasing.

First we prove assertion (i). From above-mentioned facts and the assumptions in (i), $u(\tau)$ and $v(\tau)$ become the maximum of $u(t)$ and the minimum of $v(t)$ on T, respectively. Thus, using (2.7), we have (2.2), and therefore, assertion (i) is true.

We next prove assertion (ii). Let $s \in I$ be a fixed number. From (2.7) with $a > 0$, we have

$$u(t) < v(s)$$

for all $t \in T$; that is, $u(t)$ is bounded above. Hence, we see that $\lim_{t\to\infty} u(t)$ exists. Using (2.6) and $a > 0$, we get $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t)$, and therefore, (2.3) is satisfied for $t \in T$ since $u(t)$ is nondecreasing and $v(t)$ is nonincreasing. Thus, assertion (ii) is true.

Using the same arguments in the proofs of (i) and (ii), we can easily see that assertions (iii) and (iv) are also true. The proof is now complete. □

Next we will prove Theorem 1.5 by using the idea of the proof of Theorem 1.3 (see [24]).

Proof of Theorem 1.5. First we prove case (i). From assertion (i) in Proposition 2.2, we can find two functions $u : T \to \mathbb{R}$ and $v : T \to \mathbb{R}$ such that (2.1) and (2.2) hold for $t \in T$. We now consider the solution $x(t)$ of (1.1) with $|\phi(\tau) - x(\tau)| < \varepsilon/a$ and $a > 0$, where $\tau = \max T$. Then, this solution is expressed as

$$x(t) = x(\tau)(ah+1)^{\frac{t-\tau}{h}}$$

for $t \in T$. Since $|\phi(\tau) - x(\tau)| < \varepsilon/a$ and (2.1) hold, we have

$$u(\tau) < x(\tau)(ah+1)^{-\frac{\tau}{h}} < v(\tau).$$

Using (2.1), (2.2) and this inequality, we get

$$\phi(t) - x(t) \leq \left\{u(\tau) - x(\tau)(ah+1)^{-\frac{\tau}{h}}\right\}(ah+1)^{\frac{t}{h}} + \frac{\varepsilon}{a} < \frac{\varepsilon}{a}$$

and

$$\phi(t) - x(t) \geq \left\{v(\tau) - x(\tau)(ah+1)^{-\frac{\tau}{h}}\right\}(ah+1)^{\frac{t}{h}} - \frac{\varepsilon}{a} > -\frac{\varepsilon}{a}$$

for $t \in T$. Therefore, we obtain the inequality $|\phi(t) - x(t)| < \varepsilon/a$ for $t \in T$.

Next we prove case (ii). From assertion (ii) in Proposition 2.2, we can find two functions $u : T \to \mathbb{R}$ and $v : T \to \mathbb{R}$ such that (2.1) and (2.3) hold for $t \in T$. Since $\lim_{t\to\infty} u(t)$ exists and (2.1) holds for $t \in T$, the function $\phi(t)(ah+1)^{-t/h}$ also has the same limiting value $\lim_{t\to\infty} \phi(t)(ah+1)^{-t/h} = \lim_{t\to\infty} u(t)$. We consider the function

$$x(t) = \left\{\lim_{t\to\infty} \phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}} \tag{2.8}$$

for $t \in T$. Then, this function is a solution of (1.1). From (2.1) and (2.3), we have

$$\phi(t) - x(t) = \left\{u(t) - \lim_{t\to\infty} \phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}} + \frac{\varepsilon}{a} \leq \frac{\varepsilon}{a} \tag{2.9}$$

and

$$\phi(t) - x(t) = \left\{v(t) - \lim_{t\to\infty} \phi(t)(ah+1)^{-\frac{t}{h}}\right\}(ah+1)^{\frac{t}{h}} - \frac{\varepsilon}{a} \geq -\frac{\varepsilon}{a} \tag{2.10}$$

for $t \in T$. Hence, we get the inequality $|\phi(t) - x(t)| \leq \varepsilon/a$ for $t \in T$. Noticing that if we choose a constant c so that $c \neq \lim_{t\to\infty} \phi(t)(ah+1)^{-\frac{t}{h}}$, then the function $x(t) = c(ah+1)^{t/h}$ becomes a solution of (1.1), however, it does not satisfy (2.9) or (2.10) for t sufficiently large. Thus, (2.8) is exactly one solution of (1.1) satisfying $|\phi(t) - x(t)| \leq \varepsilon/a$ for $t \in T$.

We will prove case (iii). From assertion (iii) in Proposition 2.2, we can find two functions $u : T \to \mathbb{R}$ and $v : T \to \mathbb{R}$ such that (2.1) and (2.4) hold for $t \in T$. We consider the solution

$$x(t) = x(\sigma)(ah+1)^{\frac{t-\sigma}{h}}$$

of (1.1) with $|\phi(\sigma) - x(\sigma)| < \varepsilon/|a|$ and $-1/h < a < 0$, where $\sigma = \min T$. Since $|\phi(\sigma) - x(\sigma)| < \varepsilon/|a|$ and (2.1) hold, we get

$$v(\sigma) < x(\sigma)(ah+1)^{-\frac{\sigma}{h}} < u(\sigma).$$

Using (2.1), (2.4) and this inequality, we obtain $|\phi(t) - x(t)| < \varepsilon/|a|$ for $t \in T$.

Finally we prove case (iv). By means of assertion (iv) in Proposition 2.2, there exist two functions $u(t)$ and $v(t)$ satisfying (2.1) and (2.5) for $t \in T$. Since $\lim\limits_{t\to-\infty} u(t)$ exists and (2.1) holds for $t \in T$, we have $\lim\limits_{t\to-\infty} \phi(t)(ah+1)^{-t/h} = \lim\limits_{t\to-\infty} u(t)$. Consider the function

$$x(t) = \left\{ \lim_{t\to\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}}, \tag{2.11}$$

which becomes a solution of (1.1). From (2.1) and (2.5), we get the inequality $|\phi(t) - x(t)| \le \varepsilon/|a|$ for $t \in T$. Using the same argument as in the proof of case (ii), we can conclude that (2.11) is exactly one solution of (1.1) satisfying $|\phi(t) - x(t)| \le \varepsilon/|a|$ for $t \in T$. This completes the proof of Theorem 1.5. □

3 Large Stepsize Linear Difference Equations

In this section, we consider the case of $a \neq -2/h$ and $a < -1/h$. First we give some preparations for the proof of Theorem 1.7.

Lemma 3.1. *Suppose that $a \neq -2/h$ and $a < -1/h$. Let $\varepsilon > 0$ be a given arbitrary constant and let $\phi(t)$ be a real-valued function on T. Then the inequality $|\Delta_h \phi(t) - a\phi(t)| \le \varepsilon$ holds for all $t \in T^*$ if and only if the inequality*

$$0 \le \Delta_h \left\{ \left(\phi(t) + \frac{\varepsilon(-1)^{\frac{t}{h}}}{a+2/h} \right) (ah+1)^{-\frac{t}{h}} \right\} \le 2\varepsilon |ah+1|^{-\frac{t+h}{h}}$$

holds for all $t \in T^*$.

Proof. Since the equality

$$\begin{aligned}
&\Delta_h \left\{ \left(\phi(t) + \frac{\varepsilon(-1)^{\frac{t}{h}}}{a+2/h} \right) (ah+1)^{-\frac{t}{h}} \right\} \\
&= \frac{1}{h} \left\{ \phi(t+h) + \frac{\varepsilon(-1)^{\frac{t+h}{h}}}{a+2/h} - (ah+1) \left(\phi(t) + \frac{\varepsilon(-1)^{\frac{t}{h}}}{a+2/h} \right) \right\} (ah+1)^{-\frac{t+h}{h}} \\
&= \left\{ \Delta_h \phi(t) - a\phi(t) + \varepsilon(-1)^{\frac{t+h}{h}} \right\} (ah+1)^{-\frac{t+h}{h}}
\end{aligned}$$

holds for all $t \in T^*$, the statement of Lemma 3.1 is clearly true. $\square$

Proposition 3.2. *Let $\varepsilon > 0$ be a given arbitrary constant. Suppose that a function $\phi : T \to \mathbb{R}$ satisfies $|\Delta_h \phi(t) - a\phi(t)| \le \varepsilon$ for all $t \in T^*$, where $a \ne -2/h$ and $a < -1/h$. Then there exist a nondecreasing function $u : T \to \mathbb{R}$ and a nonincreasing function $v : T \to \mathbb{R}$ such that*

$$\phi(t) = u(t)(ah+1)^{\frac{t}{h}} - \frac{\varepsilon(-1)^{\frac{t}{h}}}{a+2/h} = v(t)(ah+1)^{\frac{t}{h}} + \frac{\varepsilon(-1)^{\frac{t}{h}}}{a+2/h} \tag{3.1}$$

and one of the following hold:

(i) if $-2/h < a < -1/h$ and $\min T$ exists, then the inequality

$$v(t) \le v(\sigma) < u(\sigma) \le u(t) \tag{3.2}$$

holds for all $t \in T$, where $\sigma = \min T$;

(ii) if $-2/h < a < -1/h$ and $\min T$ does not exist, then $\lim_{t\to-\infty} u(t)$ and $\lim_{t\to-\infty} v(t)$ exist, and

$$v(t) \le \lim_{t\to-\infty} v(t) = \lim_{t\to-\infty} u(t) \le u(t) \tag{3.3}$$

holds for all $t \in T$;

(iii) if $a < -2/h$ and $\max T$ exists, then the inequality

$$u(t) \le u(\tau) < v(\tau) \le v(t) \tag{3.4}$$

holds for all $t \in T$, where $\sigma = \min T$;

(iv) if $a < -2/h$ and $\max T$ does not exist, then $\lim_{t\to\infty} u(t)$ and $\lim_{t\to\infty} v(t)$ exist, and

$$u(t) \le \lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) \le v(t) \tag{3.5}$$

holds for all $t \in T$.

Proof. Define the functions

$$u(t) = \left(\phi(t) + \frac{\varepsilon(-1)^{\frac{t}{h}}}{a+2/h}\right)(ah+1)^{-\frac{t}{h}}$$

and

$$v(t) = \left(\phi(t) - \frac{\varepsilon(-1)^{\frac{t}{h}}}{a+2/h}\right)(ah+1)^{-\frac{t}{h}}$$

for $t \in T$, where $a \neq -2/h$ and $a < -1/h$. Then (3.1) and

$$u(t) = v(t) + \frac{2\varepsilon}{a + 2/h}|ah + 1|^{-\frac{t}{h}} \tag{3.6}$$

hold for $t \in T$. Therefore, we have

$$u(t) \begin{cases} > v(t) & \text{if } \; -\dfrac{2}{h} < a < -\dfrac{1}{h}, \\ < v(t) & \text{if } \; a < -\dfrac{2}{h} \end{cases} \tag{3.7}$$

for $t \in T$. From the assertion in Lemma 3.1, we see that

$$0 \leq \Delta_h u(t) \leq 2\varepsilon |ah + 1|^{-\frac{t+h}{h}}$$

for $t \in T^*$. Moreover, using equalities (3.6) and

$$\Delta_h |ah + 1|^{-\frac{t}{h}} = \frac{1 - |ah + 1|}{h}|ah + 1|^{-\frac{t+h}{h}} = \left(\frac{2}{h} + a\right)|ah + 1|^{-\frac{t+h}{h}},$$

and the above inequality, we get

$$-2\varepsilon|ah + 1|^{-\frac{t+h}{h}} \leq \Delta_h v(t) \leq 0$$

for $t \in T^*$. Therefore, we see that $u(t)$ is nondecreasing and $v(t)$ is nonincreasing.

First we prove assertion (i). Since $u(t)$ is nondecreasing and $v(t)$ is nonincreasing, and using (3.7) with $-2/h < a < -1/h$, we obtain (3.2) for $t \in T$; that is, assertion (i) is true.

We next prove assertion (ii). Let $s \in I$ be a fixed number. From (3.7) with $-2/h < a < -1/h$, we have $u(t) > v(s)$ for all $t \in T$; that is, $u(t)$ is bounded below. Since $u(t)$ is a nondecreasing function, we can conclude that $\lim_{t \to -\infty} u(t)$ exists. From (3.6) and $-2/h < a < -1/h$, we get $\lim_{t \to -\infty} u(t) = \lim_{t \to -\infty} v(t)$, and therefore, (3.3) is satisfied for $t \in T$ since $u(t)$ is nondecreasing and $v(t)$ is nonincreasing. Thus, assertion (ii) is true.

Using the same arguments in the proofs of (i) and (ii), we can easily see that assertions (iii) and (iv) are also true. The proof is now complete. $\square$

We next give the proof of Theorem 1.7.

Proof of Theorem 1.7. First we prove case (i). By means of assertion (i) in Proposition 3.2, we can find two functions $u : T \to \mathbb{R}$ and $v : T \to \mathbb{R}$ such that (3.1) and (3.2) hold for $t \in T$. Consider the solution $x(t)$ of (1.1) with $|\phi(\sigma) - x(\sigma)| < \varepsilon/(a + 2/h)$ and $-2/h < a < -1/h$, where $\sigma = \min T$. Then, this solution is expressed as

$$x(t) = x(\sigma)(ah + 1)^{\frac{t-\sigma}{h}}$$

for $t \in T$. From the initial condition $|\phi(\sigma) - x(\sigma)| < \varepsilon/(a + 2/h)$, we have

$$\phi(\sigma) - \frac{\varepsilon}{a + 2/h} < x(\sigma) < \phi(\sigma) + \frac{\varepsilon}{a + 2/h}. \tag{3.8}$$

In the case that σ/h is odd, multiplying (3.8) by $(ah + 1)^{-\sigma/h}$, we get

$$\begin{aligned} u(\sigma) &= \left(\phi(\sigma) - \frac{\varepsilon}{a + 2/h}\right)(ah + 1)^{-\frac{\sigma}{h}} \\ &> x(\sigma)(ah + 1)^{-\frac{\sigma}{h}} \\ &> \left(\phi(\sigma) + \frac{\varepsilon}{a + 2/h}\right)(ah + 1)^{-\frac{\sigma}{h}} = v(\sigma). \end{aligned}$$

On the other hand, in the case that σ/h is even, multiplying (3.8) by $(ah + 1)^{-\sigma/h}$, we have

$$\begin{aligned} v(\sigma) &= \left(\phi(\sigma) - \frac{\varepsilon}{a + 2/h}\right)(ah + 1)^{-\frac{\sigma}{h}} \\ &< x(\sigma)(ah + 1)^{-\frac{\sigma}{h}} \\ &< \left(\phi(\sigma) + \frac{\varepsilon}{a + 2/h}\right)(ah + 1)^{-\frac{\sigma}{h}} = u(\sigma). \end{aligned}$$

Consequently, the inequality

$$v(\sigma) < x(\sigma)(ah + 1)^{-\frac{\sigma}{h}} < u(\sigma) \tag{3.9}$$

holds. Next, we consider the two cases: (a) t/h is odd; (b) t/h is even. In case (a), using (3.1), (3.2) and (3.9), we get

$$\phi(t) - x(t) \leq \left\{u(\sigma) - x(\sigma)(ah + 1)^{-\frac{\sigma}{h}}\right\}(ah + 1)^{\frac{t}{h}} + \frac{\varepsilon}{a + 2/h} < \frac{\varepsilon}{a + 2/h}$$

and

$$\phi(t) - x(t) \geq \left\{v(\sigma) - x(\sigma)(ah + 1)^{-\frac{\sigma}{h}}\right\}(ah + 1)^{\frac{t}{h}} - \frac{\varepsilon}{a + 2/h} > -\frac{\varepsilon}{a + 2/h}$$

for $t \in T$. On the other hand, in case (b), using (3.1), (3.2) and (3.9), we have

$$\phi(t) - x(t) \geq \left\{u(\sigma) - x(\sigma)(ah + 1)^{-\frac{\sigma}{h}}\right\}(ah + 1)^{\frac{t}{h}} - \frac{\varepsilon}{a + 2/h} > -\frac{\varepsilon}{a + 2/h}$$

and

$$\phi(t) - x(t) \leq \left\{v(\sigma) - x(\sigma)(ah + 1)^{-\frac{\sigma}{h}}\right\}(ah + 1)^{\frac{t}{h}} + \frac{\varepsilon}{a + 2/h} < \frac{\varepsilon}{a + 2/h}$$

for $t \in T$. Thus, $|\phi(t) - x(t)| < \varepsilon/a$ holds for all $t \in T$. Therefore, assertion (i) is true.

Next we prove case (ii). By means of assertion (ii) in Proposition 3.2, we can find two functions $u : T \to \mathbb{R}$ and $v : T \to \mathbb{R}$ such that (3.1) and (3.3) hold for $t \in T$. Since $\lim_{t\to-\infty} u(t)$ exists, (3.1) holds for $t \in T$ and $ah + 1 < 0$ is satisfied, we see that

$$\lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}} = \lim_{t\to-\infty} u(t).$$

Then the function

$$x(t) = \left\{ \lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}} \tag{3.10}$$

is a solution of (1.1). We will consider the two cases: (a) t/h is odd; (b) t/h is even. In case (a), using (3.1), (3.3) and $ah + 1 < 0$, we obtain

$$\begin{aligned} \phi(t) - x(t) &= \left\{ u(t) - \lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}} + \frac{\varepsilon}{a+2/h} \\ &\leq \frac{\varepsilon}{a+2/h} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \phi(t) - x(t) &= \left\{ v(t) - \lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}} - \frac{\varepsilon}{a+2/h} \\ &\geq -\frac{\varepsilon}{a+2/h} \end{aligned} \tag{3.12}$$

for $t \in T$. On the other hand, in case (b), using (3.1), (3.3) and $ah + 1 < 0$, we get

$$\begin{aligned} \phi(t) - x(t) &= \left\{ u(t) - \lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}} - \frac{\varepsilon}{a+2/h} \\ &\geq -\frac{\varepsilon}{a+2/h} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \phi(t) - x(t) &= \left\{ v(t) - \lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}} + \frac{\varepsilon}{a+2/h} \\ &\leq \frac{\varepsilon}{a+2/h} \end{aligned} \tag{3.14}$$

for $t \in T$. Consequently, $|\phi(t) - x(t)| \leq \varepsilon/(a + 2/h)$ holds for all $t \in T$. Noticing that if we choose a constant c so that $c \neq \lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}}$, then the function $x(t) = c(ah + 1)^{t/h}$ becomes a solution of (1.1), however, it does not satisfy (3.11), (3.12), (3.13) or (3.14) for t sufficiently large. Thus, (3.10) is exactly one solution of (1.1) satisfying $|\phi(t) - x(t)| \leq \varepsilon/(a + 2/h)$ for $t \in T$.

Next, we prove case (iii). From assertion (iii) in Proposition 3.2, we can find two functions $u : T \to \mathbb{R}$ and $v : T \to \mathbb{R}$ such that (3.1) and (3.4) hold for $t \in T$. Consider the solution

$$x(t) = x(\tau)(ah + 1)^{\frac{t-\tau}{h}}$$

of (1.1) with $|\phi(\tau) - x(\tau)| < \varepsilon/|a + 2/h|$ and $a < -2/h$, where $\tau = \max T$. Since $|\phi(\tau) - x(\tau)| < \varepsilon/|a + 2/h|$ and (3.1) hold, we get

$$u(\tau) < x(\tau)(ah + 1)^{-\frac{\tau}{h}} < v(\tau).$$

Hence, together with (3.1) and (3.4), we obtain $|\phi(t) - x(t)| < \varepsilon/|a + 2/h|$ for $t \in T$.

Finally we prove case (iv). By means of assertion (iv) in Proposition 3.2, there exist two functions $u(t)$ and $v(t)$ satisfying (3.1) and (3.5) for $t \in T$. Since $\lim_{t\to\infty} u(t)$ exists, (3.1) holds for $t \in T$ and $ah + 1 < 0$ is satisfied, we get $\lim_{t\to\infty} \phi(t)(ah + 1)^{-t/h} = \lim_{t\to\infty} u(t)$. Consider the function

$$x(t) = \left\{ \lim_{t\to\infty} \phi(t)(ah + 1)^{-\frac{t}{h}} \right\} (ah + 1)^{\frac{t}{h}}, \tag{3.15}$$

which becomes a solution of (1.1). From (3.1) and (3.5), we see that $|\phi(t) - x(t)| \leq \varepsilon/|a+2/h|$ holds for $t \in T$. Using the same argument as in the proof of case (ii), we can conclude that (3.15) is exactly one solution of (1.1) satisfying $|\phi(t)-x(t)| \leq \varepsilon/|a+2/h|$ for $t \in T$. This completes the proof of Theorem 1.7. □

4 Influence of the Stepsize on Hyers–Ulam Stability

In this section, we will present the answers to questions (Q1)–(Q3), respectively. From Theorems 1.5 and 1.7, we can establish the following simple results.

Corollary 4.1. *If $a \neq 0$ and $a > -1/h$, then* (1.1) *has Hyers–Ulam stability with a HUS constant* $1/|a|$ *on* T.

Corollary 4.2. *If $a \neq -2/h$ and $a < -1/h$, then* (1.1) *has Hyers–Ulam stability with a HUS constant* $1/|a + 2/h|$ *on* T.

In the case that $I = \mathbb{R}$, we can state the following result from the assertions (ii) and (iv) in Theorem 1.5.

Corollary 4.3. *Suppose that $a \neq 0$ and $a > -1/h$. Then* (1.1) *has Hyers–Ulam stability with a HUS constant* $1/|a|$ *on* $h\mathbb{Z}$. *Furthermore, the solution $x(t)$ of* (1.1) *satisfying* $|\phi(t) - x(t)| \leq \varepsilon/|a|$ *for all $t \in h\mathbb{Z}$ is the only one, which written as*

$$x(t) = \left\{ \lim_{t\to\infty} \phi(t)(ah + 1)^{-\frac{t}{h}} \right\} (ah + 1)^{\frac{t}{h}}$$

if $a > 0$ *(resp.,* $x(t) = \left\{ \lim_{t\to-\infty} \phi(t)(ah+1)^{-t/h} \right\} (ah+1)^{t/h}$ *if* $-1/h < a < 0$*).*

Remark 4.4. Let $\varepsilon > 0$ be a given arbitrary constant. We consider the nonhomogeneous difference equation

$$\Delta_h x(t) - ax(t) = \varepsilon \tag{4.1}$$

on $h\mathbb{Z}$, where a is a non-zero real number. We can easily see that the function $\phi(t) = c(ah+1)^{t/h} - \varepsilon/a$ for $t \in h\mathbb{Z}$ is the general solution of (4.1), where c is an arbitrary constant. Since $c(ah+1)^{t/h}$ is a solution of (1.1), $|\phi(t) - x(t)| = \varepsilon/|a|$ holds for all $t \in h\mathbb{Z}$. From this fact and the assertion in Corollary 4.3, we can conclude that $1/|a|$ is the minimum of HUS constants for (1.1) on $h\mathbb{Z}$ when $a \neq 0$ and $a > -1/h$. Moreover, this example shows that it is not possible to weaken the condition $|\phi(\tau) - x(\tau)| < \varepsilon/a$ in (i) of Theorem 1.5 to $|\phi(\tau) - x(\tau)| \leq \varepsilon/a$, whenever $|\phi(t) - x(t)| < \varepsilon/a$ holds for $t \in T$.

In the case $I = \mathbb{R}$, we can establish the following result from the assertions (ii) and (iv) in Theorem 1.7.

Corollary 4.5. *Suppose that* $a \neq -2/h$ *and* $a < -1/h$*. Then* (1.1) *has Hyers–Ulam stability with a HUS constant* $1/|a+2/h|$ *on* $h\mathbb{Z}$*. Furthermore, the solution* $x(t)$ *of* (1.1) *satisfying* $|\phi(t) - x(t)| \leq \varepsilon/|a+2/h|$ *for all* $t \in h\mathbb{Z}$ *is the only one, which written as*

$$x(t) = \left\{ \lim_{t\to-\infty} \phi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}}$$

if $-2/h < a < -1/h$ *(resp.,* $x(t) = \left\{ \lim_{t\to\infty} \phi(t)(ah+1)^{-t/h} \right\} (ah+1)^{t/h}$ *if* $a < -2/h$*).*

Remark 4.6. Let $\varepsilon > 0$ be a given arbitrary constant. Consider the nonhomogeneous difference equation

$$\Delta_h x(t) - ax(t) = \varepsilon(-1)^{\frac{t}{h}} \tag{4.2}$$

on $h\mathbb{Z}$, where $a \neq -2/h$. Then the function

$$\phi(t) = c(ah+1)^{\frac{t}{h}} + \frac{\varepsilon(-1)^{\frac{t-h}{h}}}{a+2/h}$$

for $t \in h\mathbb{Z}$ becomes the general solution of (4.2), where c is an arbitrary constant. Since $c(ah+1)^{t/h}$ is a solution of (1.1), we obtain $|\phi(t) - x(t)| = \varepsilon/|a+2/h|$ for all $t \in h\mathbb{Z}$. From this fact and the assertion in Corollary 4.5, we can conclude that $1/|a+2/h|$ is the minimum of HUS constants for (1.1) on $h\mathbb{Z}$. Moreover, this example implies that it is not possible to weaken the condition $|\phi(\sigma) - x(\sigma)| < \varepsilon/(a+2/h)$ in (i) of Theorem 1.7 to $|\phi(\sigma) - x(\sigma)| \leq \varepsilon/(a+2/h)$, whenever $|\phi(t) - x(t)| < \varepsilon/(a+2/h)$ holds for $t \in T$.

Let

$$H(h) = \frac{1}{||a+1/h| - 1/h|} \tag{4.3}$$

for $h > 0$, where $a \neq 0, -1/h, -2/h$. From Corollaries 4.3, 4.5, Remarks 4.4 and 4.6, we see that the minimum of HUS constants for (1.1) on $h\mathbb{Z}$ is $H(h)$. This is the answer to our question (Q1) raised in Section 1. Moreover, Corollaries 4.3 and 4.5 imply the answer to question (Q2). Namely, we can find an explicit solution corresponding to the solution $x(t)$ of (1.1) in Theorem 1.4. The following remark is an answer to final question (Q3).

Remark 4.7. In the case that h is a small stepsize or a is positive, Corollary 4.3 and Remark 4.4 mean that the minimum of HUS constants is the same as that of ordinary differential equation (1.2). On the other hand, in the case that h is a large stepsize, Corollary 4.5 and Remark 4.6 mean that the minimum of HUS constants is different from that of ordinary differential equation (1.2).

The following remark is also an answer to question (Q3).

Remark 4.8. When restricted to the case $a < 0$, we can rewrite $H(h)$ as

$$H(h) = \begin{cases} \dfrac{1}{|a|} & \text{if} \quad 0 < h < -\dfrac{1}{a}, \\ \dfrac{1}{|a + 2/h|} & \text{if} \quad -\dfrac{1}{a} < h \ \text{ and } \ h \neq -\dfrac{2}{a} \end{cases}$$

for $h > 0$. In Figure 4.1, we give the graph of $H(h)$ which depends on the stepsize.

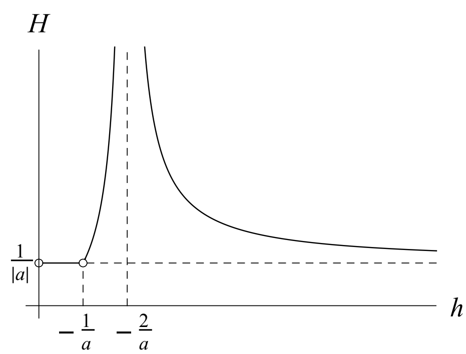

Figure 4.1: The graph of $H(h)$ when $a < 0$.

Next we consider the case that I is a finite interval. Using the assertions in Theorems 1.5 and 1.7, we can verify the following facts.

Corollary 4.9. *Suppose that $a \neq 0$ and $a > -1/h$. Let I be a finite nonempty open interval of $\mathbb{R}$ and $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi : T \to \mathbb{R}$ satisfies $|\Delta_h \phi(t) - a\phi(t)| \leq \varepsilon$ for all $t \in T^*$, then there exists a solution $x : T \to \mathbb{R}$ of* (1.1) *such that $|\phi(t) - x(t)| < \varepsilon/|a|$ for all $t \in T$.*

Corollary 4.10. *Suppose that $a \neq -2/h$ and $a < -1/h$. Let I be a finite nonempty open interval of $\mathbb{R}$ and $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi : T \to \mathbb{R}$ satisfies $|\Delta_h \phi(t) - a\phi(t)| \leq \varepsilon$ for all $t \in T^*$, then there exists a solution $x : T \to \mathbb{R}$ of* (1.1) *such that $|\phi(t) - x(t)| < \varepsilon/|a + 2/h|$ for all $t \in T$.*

5 Applications: Perturbed Linear Difference Equations

In this section, we give some applications to illustrate the main results. We consider the first-order perturbed linear difference equation

$$\Delta_h x(t) - ax(t) = f(t, x(t)), \tag{5.1}$$

which corresponding to unperturbed linear difference equation (1.1), where $f(t,x)$ is the real-valued function on $T^* \times \mathbb{R}$. Under the assumption that $\max T$ does not exist and $f(t,x)$ is bounded on $T^* \times \mathbb{R}$, we see that $0 < |ah+1| < 1$ implies the uniform-ultimate boundedness; that is, there exists a $B > 0$ and, for any $\alpha > 0$, there exists an $S(\alpha) > 0$ such that $t_0 \in T^*$ and $|\phi_0| < \alpha$ imply that $|\phi(t)| < B$ for all $t \geq t_0 + S(\alpha)$ and $t \in T$, where $\phi(t)$ is a solution of (5.1) satisfying $\phi(t_0) = \phi_0$ and $(t_0, \phi_0) \in T^* \times \mathbb{R}$. If this property holds, then we say that the solutions of (5.1) are uniform-ultimately bounded for bound B (see [20, 21, 34, 35]). Using Theorems 1.5 and 1.7, we can establish the following result.

Corollary 5.1. *Let $\delta > 0$ be an arbitrary constant. Suppose that $\max T$ does not exist, and there exists an $L > 0$ such that $|f(t,x)| \leq L$ for all $(t,x) \in T^* \times \mathbb{R}$. If $0 < |ah+1| < 1$, then the solutions of* (5.1) *are uniform-ultimately bounded for bound $LH(h) + \delta$, where $H(h)$ is the minimum of HUS constants for* (1.1) *on $h\mathbb{Z}$ which given by* (4.3).

Proof. Let $B = LH(h) + \delta$. We consider the solution $\phi(t)$ of (5.1) with $\phi(t_0) = \phi_0$ and $|\phi_0| < \alpha$, where $(t_0, \phi_0) \in T^* \times \mathbb{R}$ and α is any positive constant. Using (iii) in Theorem 1.5 and (i) in Theorem 1.7, there exists a solution $x(t)$ of (1.1) with $|\phi_0 - x(t_0)| < LH(h)$ such that $|\phi(t) - x(t)| < LH(h)$ for $t \geq t_0$ and $t \in T$. Since $x(t) = x(t_0)(ah+1)^{(t-t_0)/h}$ for $t \in T$, we have

$$|x(t)| < (LH(h) + |\phi_0|)|ah+1|^{\frac{t-t_0}{h}} < (LH(h) + \alpha)|ah+1|^{\frac{t-t_0}{h}} \tag{5.2}$$

for $t \geq t_0$. If $LH(h) + \alpha \leq \delta$, then

$$|\phi(t)| < LH(h) + |x(t)| < LH(h) + \delta = B$$

holds for $t \geq t_0$. Next, we have only to consider that the case of $LH(h) + \alpha > \delta$. Let

$$S(\alpha) = \frac{h \log(\delta/(LH(h) + \alpha))}{\log|ah+1|} > 0.$$

From $LH(h) + \alpha > \delta$ and $0 < |ah+1| < 1$, we see that

$$(LH(h) + \alpha)|ah+1|^{\frac{t-t_0}{h}} \leq \delta$$

holds for all $t \geq t_0 + S(\alpha)$. Hence, combining this with (5.2), we get $|x(t)| < \delta$ for all $t \geq t_0 + S(\alpha)$. Since $|\phi(t) - x(t)| < LH(h)$ holds for $t \in T$, we obtain $|\phi(t)| < B$ for all $t \geq t_0 + S(\alpha)$. This completes the proof of Corollary 5.1. $\square$

Remark 5.2. Let $I = (0, \infty)$. When $h = 5$, $a = -1/3$ and $\varepsilon = 1$, (4.2) is reduced to the nonhomogeneous linear difference equation

$$\Delta_5 x(t) + \frac{1}{3} x(t) = (-1)^{\frac{t}{5}} \tag{5.3}$$

on $T^* = 5\mathbb{Z} \cap (0, \infty)$. Since $|ah + 1| = 2/3$ and $|f(t, x)| = \left|(-1)^{t/5}\right| = 1$ for all $(t, x) \in T^* \times \mathbb{R}$, all assumptions in Corollary 5.1 are satisfied. Therefore, the solutions of (5.3) are uniform-ultimately bounded for bound

$$LH(h) + \delta = \frac{L}{|a + 2/h|} + \delta = 15 + \delta.$$

Since

$$\phi(t) = 15(-1)^{\frac{t-5}{5}} \left\{ 3 \left(\frac{2}{3} \right)^{\frac{t}{5}} + 1 \right\}$$

is a solution of (5.3) with $\phi(5) = 45$, it is easy to check that $\phi(t) < -15 = -LH(h)$ on $10\mathbb{Z} \cap (0, \infty)$. This means that $\delta > 0$ in Corollary 5.1 cannot be removed. Let $\delta = 3$. Figure 5.1 indicates a solution of (5.3). The starting point is $(5, 45) \in T^* \times \mathbb{R}$. Moreover, broken lines are $\phi = -LH(h) - \delta = -18$ and $\phi = LH(h) + \delta = 18$, and dotted lines are $\phi = -LH(h) = -15$ and $\phi = LH(h) = 15$, respectively.

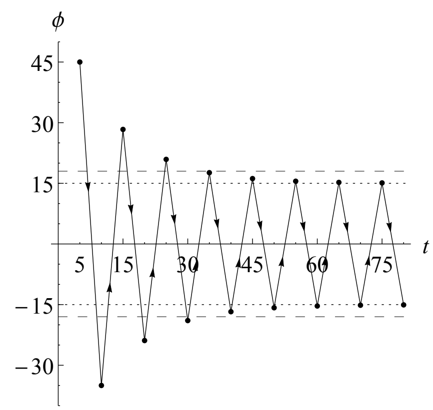

Figure 5.1: A solution of (5.3).

Furthermore, we can clarify the asymptotic behavior of any solution of (5.1) by using Theorems 1.5 and 1.7.

Corollary 5.3. *Suppose that* $a \neq 0, -1/h, -2/h$. *Suppose also that* $\max T$ *does not exist, and there exists an* $L > 0$ *such that* $|f(t, x)| \leq L$ *for all* $(t, x) \in T^* \times \mathbb{R}$. *Then any solution* $\phi(t)$ *of* (5.1) *satisfies one of the following:*

(i) if $a > 0$ *or* $a < -2/h$ *then* $\lim_{t \to \infty} \phi(t)(ah + 1)^{-t/h}$ *exists;*

(ii) if $0 < |ah+1| < 1$ *then* $\limsup_{t\to\infty} |\phi(t)| \le LH(h)$*,*

where $H(h)$ *is the minimum of HUS constants for* (1.1) *on* $h\mathbb{Z}$ *which given by* (4.3).

Proof. Assertion (i) is an immediate consequence from (ii) in Theorem 1.5 and (iv) in Theorem 1.7. The proof of assertion (ii) is by contradiction. Suppose that there exists a solution $\phi(t)$ of (5.1) satisfying $\limsup_{t\to\infty} |\phi(t)| > LH(h)$ and $\phi(t_0) = \phi_0$, where $(t_0, \phi_0) \in T^* \times \mathbb{R}$. Using (iii) in Theorem 1.5 and (i) in Theorem 1.7, there exists a solution $x(t)$ of (1.1) with $|\phi(t_0) - x(t_0)| < LH(h)$ such that $|\phi(t) - x(t)| < LH(h)$ for $t \ge t_0$ and $t \in T$. Since $0 < |ah+1| < 1$ and $x(t) = x(t_0)(ah+1)^{(t-t_0)/h}$, we obtain

$$\limsup_{t\to\infty} |\phi(t)| = \limsup_{t\to\infty} |\phi(t) - x(t)| \le LH(h).$$

This is a contradiction. The proof is now complete. □

To illustrate Corollary 5.3, we give a concrete example.

Example 5.4. We consider the nonlinear difference equation

$$\Delta_h x(t) + \frac{1}{3} x(t) = \sin(t + x(t)) \tag{5.4}$$

on $T^* = h\mathbb{Z} \cap (0, \infty)$, where $h \ne 3$, 6. By $I = (0, \infty)$, the maximum of T does not exist. Moreover, since $|\sin(t + x(t))| \le 1$ for all $(t, x) \in T^* \times \mathbb{R}$, we choose $L = 1$. Using Corollary 5.3, we see that any solution $\phi(t)$ of (5.4) satisfies either of the following: $\lim_{t\to\infty} \phi(t)(1 - h/3)^{-t/h}$ exists if $h > 6$; $\limsup_{t\to\infty} |\phi(t)| \le 3/(6/h - 1)$ if $0 < h < 6$. In Figure 5.2, we draw a solution of (5.4) with $h = 8$ starting from the point $(8, 45) \in T^* \times \mathbb{R}$. This solution diverges, and oscillates infinitely. In Figure 5.3, we draw a solution of (5.4) with $h = 5$ starting from the point $(5, 45) \in T^* \times \mathbb{R}$. Since $3/(6/h - 1) = 15$ when $h = 5$, any solution $\phi(t)$ of (5.4) with $h = 5$ satisfies $\limsup_{t\to\infty} |\phi(t)| \le 15$.

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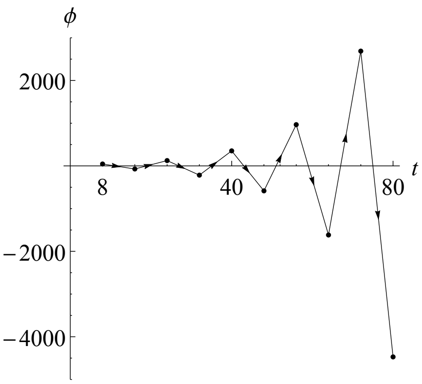

Figure 5.2: A solution of (5.4) with $h = 8$.

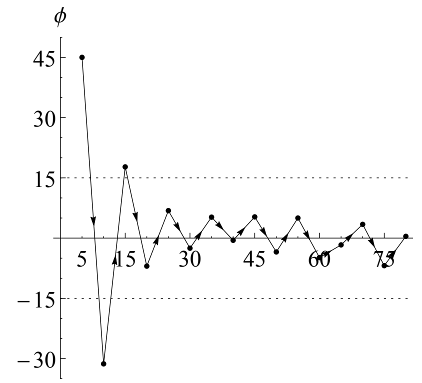

Figure 5.3: A solution of (5.4) with $h = 5$.

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