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Existence Results for a Semilinear System of Discrete Equations

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Abstract

In this work, we establish several results about the existence and uniqueness of solutions for some classes of semilinear systems of difference equations with initial and boundary conditions. The approach is based on a fixed point theory in vector-valued Banach spaces. Also, we give an abstract formulation to Sadovskii's fixed point theorem in vector-valued Banach space.

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1 Introduction

The continuing interest in the field of difference equations can be attributed to two main factors:

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1. due to the theory's powerful and versatile applications to almost all areas of science, engineering and technology, which teem with discrete phenomena;
2. from the emergence and popularity of computers, where differential equations are solved by utilizing their approximate difference equation formulations.

Therefore, scientific advancements in the area of difference equations are naturally motivated and are of significant interest. The valuable uses of difference equations lead to a deeper theoretical analysis of the subject. The theory of the qualities of solutions to difference equations are particularly interesting, as studies have shown rich distinctions and interesting links between the qualitative theory of solutions to difference equations and the qualitative theory of solutions to differential equations. This has included essential concepts such as: existence, uniqueness and multiplicity of solutions; existence and nonexistence of spurious solutions; a priori bounds on solutions; stability, instability and disconjugacy of solutions. We refer the reader to [1–6] and the references therein.

In this paper, we consider a semilinear discrete system of the form

$$\begin{cases} x(t) & = A(t)x(t) + f_1(t, x(t), y(t)),\ k \in \mathbb{N}(a,b), \\ y(t) & = A(t)y(t) + f_2(t, x(t), y(t)),\ k \in \mathbb{N}(a,b), \\ x(a) & = x_0, \\ y(a) & = y_0, \end{cases} \tag{1.1}$$

where $\mathbb{N}(a,b) = \{a, a+1, \ldots, b+1\}$, $f_1, f_2 : \mathbb{N}(a,b) \times X \to X$ are given functions and with a variable linear operator $A(t)$ in a Banach space X.

Later, we study a boundary value problem of the form

$$\begin{cases} x(t) & = A(t)x(t) + f_1(t, x(t), y(t)),\ k \in \mathbb{N}(0,b), \\ y(t) & = A(t)y(t) + f_2(t, x(t), y(t)),\ k \in \mathbb{N}(0,b), \\ L_1(x(0)) & = l_1 \in X, \\ L_2(y(0)) & = l_2 \in X, \end{cases} \tag{1.2}$$

where $L_1, L_2 : \mathrm{C}(\mathbb{N}(0,b), X) \to X$ are two bounded linear operators.

This paper is organized as follows. In Section 2, we introduce all the background material needed such as generalized metric spaces and some fixed point theorems. In Section 3, by using the measure of noncompactness, we prove some Sadovskii fixed point theorems. The existence and uniqueness of solutions to the problems (1.1) and (1.2) are studied in Sections 4 and 5, respectively.

2 Fixed Point Results

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. If $x, y \in \mathbb{R}^n$ $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, then by $x \leq y$, we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots |x_n|)$, $\max(x, y) = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i > 0\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$.

Definition 2.1. Let X be a nonempty set. By a vector-valued metric on X, we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call (X, d) a generalized metric space with $d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_n(x, y) \end{pmatrix}$, where each $d_i, i = 1, \ldots, n$, is a metric on X. (Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, \ldots, n$ are metrics on X.)

For $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\} = \{x \in X : d_i(x_0, x) < r_i, i = 1, \ldots, n\}$$

the open ball centered in x_0 with radius r, and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\} = \{x \in X : d_i(x_0, x) \leq r_i, i = 1, \ldots, n\}$$

the closed ball centered in x_0 with radius r. We mention that for a generalized metric space, the notions of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 2.2. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, all the eigenvalues of M are in the open unit disc.

Theorem 2.3 (See [8, pages 12, 88])**.** *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) M is convergent towards zero;

(ii) $M^k \to 0$ as $k \to \infty$;

(iii) The matrix $(I - M)$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots;$$

(iv) The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Definition 2.4. Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x), N(y)) \leq Md(x, y) \text{ for all } x, y \in X.$$

For $n = 1$, we recover the classical Banach contraction fixed point result.

Theorem 2.5. *Let (X, d) be a complete generalized metric space with $d : X \times X \to \mathbb{R}^n$ and let $N : X \to X$ be such that*

$$d(N(x), N(y)) \leq Md(x, y)$$

for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix M is convergent to zero, that is $M^k \to 0$ as $k \to \infty$, then N has a unique fixed point $x_* \in X$,

$$d(N^k(x_0), x_*) \leq M^k(I - M)^{-1}d(N(x_0), x_0)$$

for every $x_0 \in X$ and $k \geq 1$.

Theorem 2.6 (See [9]). *Let E be a generalized Banach space, $C \subset E$ be a nonempty closed convex subset of E and $N : C \to C$ be a continuous operator with relatively compact range. Then N has at least one fixed point in C.*

As a consequence of the Schauder fixed point theorem, we present the version of Schaefer's fixed point theorem and nonlinear alternative Leray–Schauder type theorem in generalized Banach space.

Theorem 2.7. *Let E be a generalized Banach space and $N : E \to E$ be a continuous compact mapping. Moreover assume that the set*

$$A = \{x \in E : x = \lambda N(x) \text{ for some } \lambda \in (0, 1)\}$$

is bounded. Then N has a fixed point.

Lemma 2.8. *Let X be a generalized Banach space, $U \subset E$ be a bounded, convex open neighborhood of zero and let $G : \overline{U} \to E$ be a continuous compact map. If G satisfies the boundary condition*

$$x \neq \lambda G(x)$$

for all $x \in \partial U$ and $0 \leq \lambda \leq 1$, then the set $\mathrm{Fix}(G) = \{x \in U : x = G(x)\}$ is nonempty.

Theorem 2.9 (Gronwall inequality, see [1]). *Let $p, q, f, u : \mathbb{N}(a) \to \mathbb{R}_+$ be nonnegative functions such that*

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{l=k-1} f(l)u(l), \text{ for all } k \in \mathbb{N}(a) = \{a, a+1, \ldots\},$$

Then

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{l=k-1} p(l)f(l) \prod_{\tau=l+1}^{\tau=k-1} (1 + q(\tau)f(\tau)).$$

Theorem 2.10 (Arzelà–Ascoli, see [2]). *Let E be a Banach space and Ω be a closed subset of $\mathrm{C}(\mathbb{N}(0, b), E)$. If Ω is uniformly bounded and the set*

$$\{x(k) : x \in \Omega\}$$

is relatively compact for each $k \in \mathbb{N}(0, b)$, then Ω is compact.

3 Measure of Noncompactness

In this section, by using Theorem 2.6 and noncompactness measure in vector-valued Banach space, we obtain a Sadovkii fixed point theorem. First, we give definitions and properties for a measure of noncompactness. Denote by

$$\mathcal{P}(X) \;=\; \{Y \subset X : \; Y \neq \emptyset\}.$$

Definition 3.1. Let X be a generalized Banach space and $(\mathcal{A}, \leq)$ a partially ordered set. A map $\beta : \mathcal{P}(X) \to \mathcal{A} \times \mathcal{A} \ldots \times \mathcal{A}$ is called a generalized measure of noncompactness (m.n.c.) on X, if

$$\beta(\overline{\mathrm{co}}\Omega) = \beta(\Omega) \text{ for every } \Omega \in \mathcal{P}(X),$$

$\Omega \in \mathcal{P}(X)$, where $\beta(\Omega) := \begin{pmatrix} \beta_1(\Omega) \\ \vdots \\ \beta_n(\Omega) \end{pmatrix}.$

Definition 3.2. A measure of noncompactness β is called

(a) Monotone if $\Omega_0, \Omega_1 \in \mathcal{P}(X)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$.

(b) Nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X$, $\Omega \in \mathcal{P}(X)$.

(c) Invariant with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset X$, and $\Omega \in \mathcal{P}(X)$.

(d) Real if $\mathcal{A} = \overline{R}_+$ and $\beta(\Omega) < \infty$ for every $i = 1, \ldots, n$ and every bounded Ω.

(e) Semi-additive if $\beta(\Omega_0 \cup \Omega_1) = \max(\beta(\Omega_0), \beta(\Omega_1))$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(X)$.

(f) Lower-additive if β is real and $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(X)$.

(g) Regular if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

A typical example of an m.n.c. is the Hausdorff measure of noncompactness α defined, for all $\Omega \subset X$, by

$$\alpha(\Omega) := \inf\{\epsilon \in \mathbb{R}_+^n : \text{there exists } n \in \mathbb{N} \text{ such that } \Omega \subseteq \Cup_{i=1}^n B(x_i, \epsilon)\}.$$

Definition 3.3. Let X, Y be two generalized normed spaces and $F : X \to \mathcal{P}(Y)$ be a multivalued map. F is called an M-contraction (with respect to β) if there exists $M \in M_{n\times n}(\mathbb{R})$ converging to zero such that for every $D \in \mathcal{P}(X)$, we have

$$\beta(F(D)) \leq M\beta(D).$$

The next result is concerned with β-condensing or M-contractivity.

Theorem 3.4. *Let $V \subset X$ be a bounded closed convex subset and $N : V \to V$ a generalized β-condensing continuous mapping, where β is a nonsingular measure of noncompactness defined on subsets of X. Then the set*

$$\mathrm{Fix}(N) = \{x \in V : x \in N(x)\}$$

is nonempty.

Proof. Let $\mathcal{M}_1 = V,\ \mathcal{M}_{k+1} = \overline{\mathrm{co}}N(\mathcal{M}_k),\ k \in \mathbb{N}$. It is clear that the sequence $(\mathcal{M}_k)_{k\in\mathbb{N}}$ consists of a decreasing sequence of nonempty closed convex subsets of V. Since N is β-condensing, we have

$$\beta(\overline{\mathrm{co}}N(\mathcal{M}_1)) = \beta(N(\mathcal{M}_1)) \leq M\beta(V).$$

Continuing this process, we get

$$\beta(\mathcal{M}_{k+1}) \leq M^{k+1}\beta(V).$$

Therefore,

$$\lim_{k\to\infty} \beta(\mathcal{M}_k) = 0.$$

Thus

$$C = \bigcap_{k=1}^{\infty} \mathcal{M}_k \neq \emptyset,$$

is convex and compact. Furthermore, by the convexity of $\{\mathcal{M}_k\}_{k\in\mathbb{N}}$ and $N(\mathcal{M}_1) \subseteq \mathcal{M}_1$, we have

$$N(\mathcal{M}_2) \subseteq \mathcal{M}_2 \Rightarrow N(\mathcal{M}_3) \subseteq \mathcal{M}_3.$$

Proceeding by induction, we get

$$N(\mathcal{M}_k) \subseteq \mathcal{M}_k \text{ for every } k \in \mathbb{N} \Rightarrow N(C) \subseteq C.$$

Hence by Theorem 2.6, N has at least one fixed point. □

As a consequence of Theorem 3.4, we present versions of Schaefer's fixed point theorem and the nonlinear alternative Leray–Schauder type theorem for β-condensing operators in a generalized Banach space.

Theorem 3.5. *Let E be a generalized Banach space and $N : E \to E$ is a continuous and β-condensing operator. Moreover assume that the set*

$$A = \{x \in E : x = \lambda N(x) \textit{ for some } \lambda \in (0, 1)\}$$

is bounded. Then N has a fixed point.

Theorem 3.6. *Let E be a generalized Banach space, $U \subset E$ be a bounded, convex open neighborhood of zero and let $G : \overline{U} \to E$ be a continuous and β-condensing mapping. If G satisfies the boundary condition*

$$x \neq \lambda G(x)$$

for all $x \in \partial U$ and $0 \leq \lambda \leq 1$, then the set $\mathrm{Fix}(G) = \{x \in U : x = G(x)\}$ is nonempty.

4 Existence and Uniqueness Results

Consider the equation

$$x(k+1) = A(k)x(k), \quad k \in \mathbb{N}_0. \tag{4.1}$$

With a variable linear operator $A(k)$ on a Banach space X. Then the linear operator

$$U(k,s) : X \to X, \quad k,s \in \mathbb{N}_0,$$

defined by the equalities

$$U(k,j) = A(k-1)\cdots A(j),\ k = j+1, j+2, \dots \text{ and } U(j,j) = I,\ j \in \mathbb{N}_0, \tag{4.2}$$

will be called be the evolution operator of (4.1). Recall that I is the unit operator on X.

It is simple to check that the evolution operator has the following properties:

$$U(k,j) = U(k,t)U(t,j), \quad k \geq t \geq j,\ j \in \mathbb{N}_0$$

and

$$U(k+1,j) = A(k)U(k,j), \quad k \geq j,\ j \in \mathbb{N}_0.$$

Lemma 4.1. *Let $x(k)$ be a solution of* (4.1). *Then*

$$x(k) = U(k,s)x(s), \quad k \geq s. \tag{4.3}$$

The operator $U(k) = U(k,0)$ will be called the Cauchy operator of (4.1). *If $A(k) = A$ is a constant operator, then*

$$U(k) = A^k \text{ and } U(k,s) = U(k-s) = A^{k-s}, \quad k \geq s.$$

A solution of the nonhomogeneous equation

$$x(k+1) = A(k)x(k) + f(k,x(k)), \quad k \in \mathbb{N}_0,$$

with a given sequence $\{f(k,x(k)) \in X\}_{k=0}^{\infty}$, can be represented in the form

$$x(k) = U(k,0)x_0 + \sum_{t=0}^{k-1} U(k,t+1)f(k,x(k)), \quad k \in \mathbb{N}. \tag{4.4}$$

Let us introduce the following hypothesis, for $f_1, f_2 : \mathbb{N}(a,b) \times X \times X \to X$:

(H_1) There exist nonnegative numbers a_i and b_i for each $i \in \{1,2\}$

$$|f_1(k,x,y) - f_1(k,\overline{x},\overline{y})| \leq a_1|x-\overline{x}| + b_1|y-\overline{y}|$$

and

$$|f_2(k,x,y) - f_2(k,\overline{x},\overline{y})| \leq a_2|x-\overline{x}| + b_2|y-\overline{y}|,$$

for all $x, y, \overline{x}, \overline{y} \in X$.

(H_2) there exists a positive number Q such that

$$\|U(k,s)\| \le Q, \text{ for all } k,s \in \mathbb{N}(a,b).$$

(H_3) The evolution semigroup $\{U(k,s)\}_{k-s>0}$ is compact in X.

(H_4) There exist $p_1, p_2, \bar{p}_1, \bar{p}_2, \in \mathrm{C}(\mathbb{N}(a,b), \mathbb{R}_+)$ such that

$$|f_1(k,x,y)| \le p_1(k)(|x|+|y|) + \bar{p}_1(k), \ k \in \mathbb{N}(a,b), \ (x,y) \in X \times X,$$

and

$$|f_2(k,x,y)| \le p_2(k)(|x|+|y|) + \bar{p}_2(k), \ k \in \mathbb{N}(a,b), \ (x,y) \in X \times X.$$

Theorem 4.2. *Assume that* (H_1)–(H_2) *are satisfied and that the matrix*

$$M = Qb \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}_+).$$

If M converges to zero, then (1.1) *has a unique solution.*

Proof. Let $N : \mathrm{C}(\mathbb{N}(a,b), X) \times \mathrm{C}(\mathbb{N}(a,b), X) \to C(\mathbb{N}(a,b), X) \times \mathrm{C}(\mathbb{N}(a,b), X)$ be the operator defined by

$$N(x,y) = (N_1(x,y), N_2(x,y)), \ (x,y) \in \mathrm{C}(\mathbb{N}(a,b), X) \times \mathrm{C}(\mathbb{N}(a,b), X),$$

where

$$N_1(x(k), y(k)) = U(k,a)x_0 + \sum_{l=a}^{k-1} U(k,l+1) f_1(l, x(l), y(l)), \quad k \in \mathbb{N}(a,b)$$

and

$$N_2(x(k), y(k)) = U(k,a)y_0 + \sum_{l=a}^{k-1} U(k,l+1) f_2(l, x(l), y(l)), \quad k \in \mathbb{N}(a,b).$$

We shall use Theorem 2.5 to prove that N has a unique fixed point. Indeed, let

$$(x,y), (\overline{x}, \overline{y}) \in \mathrm{C}(\mathbb{N}(a,b), X) \times \mathrm{C}(\mathbb{N}(a;b), X).$$

Then we have for each $k \in \mathbb{N}(a,b)$

$$\begin{aligned} &|N_1(x(k), y(k)) - N_1(\overline{x}(k), \overline{y}(k))| \\ &\qquad = \left| \sum_{l=s}^{k-1} U(k,l+1)[f_1(l, x(l), y(l)) - f_1(l, \overline{x}(l), \overline{y}(l))] \right|. \end{aligned}$$

Then

$$\|N_1(x,y) - N_1(\overline{x},\overline{y})\|_\infty \le Qba_1\|x-\overline{x}\|_\infty + Qbb_1\|y-\overline{y}\|_\infty.$$

Similarly we have

$$\|N_2(x,y) - N_2(\overline{x},\overline{y})\|_\infty \le Qba_2\|x-\overline{x}\|_\infty + Qbb_2\|y-\overline{y}\|_\infty.$$

Hence

$$\begin{aligned} \|N(x,y) - N(\overline{x},\overline{y})\|_\infty &= \begin{pmatrix} \|N_1(x,y) - N_1(\overline{x},\overline{y})\|_\infty \\ \|N_2(x,y) - N_2(\overline{x},\overline{y})\|_\infty \end{pmatrix} \\ &\le Qb \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \|x-\overline{x}\|_\infty \\ \|y-\overline{y}\|_\infty \end{pmatrix}. \end{aligned}$$

Therefore,

$$\|N(x,y) - N(\overline{x},\overline{y})\|_\infty \le M \begin{pmatrix} \|x-\overline{x}\|_\infty \\ \|y-\overline{y}\|_\infty \end{pmatrix},$$

for all $(x,y), (\overline{x},\overline{y}) \in \mathrm{C}(\mathbb{N}(a,b),X) \times \mathrm{C}(\mathbb{N}(a,b),X)$. From the fixed point theorem, Theorem 2.5, the mapping N has a unique fixed $(x,y) \in \mathrm{C}(\mathbb{N}(a,b),X) \times \mathrm{C}(\mathbb{N}(a,b),X)$ which is the unique solution of (1.1). $\square$

Now we give an existence result based on Schaefer's fixed point theorem type in a vector-valued Banach space.

Theorem 4.3. *Let* $f_1, f_2 : \mathbb{N}(a,b) \times X \times X \to X$ *be continuous functions for which* (H_2)–(H_4) *hold. Then* (1.1) *has at least one solution.*

Proof. Let N be the operator defined in Theorem 4.2.

Step 1: N is continuous: $N = (N_1, N_2)$. Let (x_m, y_m) be a sequence such that

$$(x_m, y_m) \to (x,y) \in \mathrm{C}(\mathbb{N}(a,b),X) \times \mathrm{C}(\mathbb{N}(a,b),X) \text{ as } m \to \infty.$$

Then

$$\begin{aligned} &|N_1(x_m(k), y_m(k)) - N_1(x(k), y(k))| \\ &= \left| \sum_{l=a}^{k-1} [f_1(l, x_m(l), y_m(l)) - f_1(l, x(l), y(l))] \right| \\ &\le \sum_{l=a}^{b} |f_1(l, x_m(l), y_m(l)) - f_1(l, x(l), y(l))| \end{aligned}$$

and

$$|N_2(x_m(k), y_m(k)) - N_2(x(k), y(k))| \leq \sum_{l=a}^{b} |f_2(l, x_m(l), y_m(l)) - f_2(l, x(l), y(l))|.$$

Since f_1 and f_2 are continuous functions, we get,

$$\|N_1(x_m, y_m) - N_1(x, y)\|_\infty \to 0, \text{ as } m \to \infty$$

and

$$\|N_2(x_m, y_m) - N_2(x, y)\|_\infty \to 0, \text{ as } m \to \infty.$$

Then N is continuous.

Step 2: N maps bounded sets into bounded sets in $\mathrm{C}(\mathbb{N}(a, b), X) \times \mathrm{C}(\mathbb{N}(a, b), X)$. Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $(x, y) \in B_q$, where

$$B_q = \{(x, y) \in \mathrm{C}(\mathbb{N}(a, b), X) \times \mathrm{C}(\mathbb{N}(a, b), X) : \|x\|_\infty \leq q, \|y\|_\infty \leq q\},$$

we have

$$\|N(x, y)\|_\infty \leq l := (l_1, l_2).$$

Then for each $k \in \mathbb{N}(a, b)$, we get

$$\|N_1(x, y)\|_\infty \leq |x_0| + 2q \sum_{k=a}^{b} p_1(k) := l_1.$$

Similarly, we have

$$\|N_2(x, y)\|_\infty \leq |y_0| + 2q \sum_{k=a}^{b} p_2(k) := l_2.$$

Observe that from (H_4), there exist $Q_1, Q_2 > 0$ such that

$$|f_1(k, x, y)| \leq 2q\|p_1\|_\infty + \|\bar{p}_1\|_\infty, \text{for all } k \in \mathbb{N}(a, b),\ x, y \in \mathrm{C}(\mathbb{N}(a, b), X),$$

and

$$|f_2(k, x, y)| \leq 2q\|p_2\|_\infty + \|\bar{p}_2\|_\infty, \text{ for all } k \in \mathbb{N}(a, b),\ x, y \in \mathrm{C}(\mathbb{N}(a, b), X).$$

Since $\{U(k, s)\}_{k-s>0}$ is compact then, for each $k, s \in \mathbb{N}(a, b)$, we have

$$\{N_1(x(k), y(k)) : (x, y) \in B_q\} \text{ and } \{N_2(x(k), y(k)) : (x, y) \in B_q\}$$

are relatively compact in X. In the case when $k = s$, we have

$$\{N_1(x(k), y(k)) : (x, y) \in B_q\} = \{x_0\} \text{ and } \{N_2(x(k), y(k)) : (x, y) \in B_q\} = \{y_0\}.$$

Then, from as a consequence of Theorem 2.10, we conclude that $\overline{N(B_q \times B_q)}$ is compact. As consequence of Steps 1 to 2, N is completely continuous.

Step 3: It remains to show that

$$\mathcal{A} = \{(x, y) \in \mathrm{C}(\mathbb{N}(a, b), X) \times \mathrm{C}(\mathbb{N}(a, b), X) : \ (x, y) = \lambda N(x, y), \lambda \in (0, 1)\}$$

is bounded. Let $(x, y) \in \mathcal{A}$. Then $x = \lambda N_1(x, y)$ and $y = \lambda N_2(x, y)$ for some $0 < \lambda < 1$. Thus, for $k \in \mathbb{N}(a, b)$, we have

$$\begin{aligned} |x(k)| &\leq |x_0| + \sum_{l=a}^{l=k-1} |f_1(l, x(l), y(l))| \\ &\leq |x_0| + \sum_{l=a}^{b} \bar{p}_1(l) + \sum_{l=a}^{k-1} p_1(l)(|x(l)| + |y(l)|). \end{aligned}$$

and

$$|y(k)| \leq |y_0| + \sum_{l=a}^{b} \bar{p}_2(l) + \sum_{l=a}^{k-1} p_2(l)(|x(l)| + |y(l)|).$$

Therefore

$$|x(k)| + |y(k)| \leq |x_0| + |y_0| + \sum_{l=a}^{b} \bar{p}_1(l) + \sum_{l=a}^{b} \bar{p}_2(l) + \sum_{l=a}^{k-1} p(l)(|x(l)| + |y(l)|),$$

where

$$p(k) = p_1(k) + p_2(k), \ k \in \mathbb{N}(a, b).$$

By Theorem 2.9, we have

$$\begin{aligned} &|x(k)| + |y(k)| \\ &\qquad \leq \left(|x_0| + |y_0| + \sum_{l=a}^{b} \bar{p}_1(l) + \sum_{l=a}^{b} \bar{p}_2(l)\right) \left(1 + \sum_{l=a}^{k-1} p(l) \prod_{l+1}^{k-1} (1 + p(\tau))\right). \end{aligned}$$

Hence

$$\begin{aligned} &\|x\|_\infty + \|y\|_\infty \\ &\qquad \leq \left(|x_0| + |y_0| + \sum_{l=a}^{b} \bar{p}_1(l) + \sum_{l=a}^{b} \bar{p}_2(l)\right) \left(1 + \sum_{l=a}^{b} p(l) \prod_{l+1}^{b} (1 + p(\tau))\right). \end{aligned}$$

This shows that $\mathcal{A}$ is bounded. As a consequence of Theorem 2.7, we deduce that N has a fixed point (x, y) which is a solution to (1.1). $\square$

5 Boundary Value Problems

The following condition will be needed in the sequel.

($\mathcal{L}$) The operators $\widetilde{L}_1, \widetilde{L}_2 : X \to X$ defined by

$$\widetilde{L}_1(x) = L_1(U(\cdot, 0)x), \quad \widetilde{L}_2(x) = L_2(U(\cdot, 0)x), \quad x \in X,$$

have a bounded inverses $\widetilde{L}_1^{-1}, \widetilde{L}_2^{-1} : X \to X$.

Lemma 5.1. *Under the condition ($\mathcal{L}$), the mild solution $(x, y) \in \mathrm{C}(\mathbb{N}(0, b), X) \times \mathrm{C}(\mathbb{N}(0, b), X)$ of (1.2) can be written as*

$$x(k) = \widetilde{L}_1^{-1} \left(l_1 - L_1 \sum_{i=0}^{k-1} U(k, i+1) f_1(i, x(i), y(i)) \right) + \sum_{i=0}^{k-1} U(k, i+1) f_1(i, x(i), y(i))$$

and

$$y(k) = \widetilde{L}_2^{-1} \left(l_2 - L_2 \sum_{i=0}^{k-1} U(k, i+1) f_2(i, x(i), y(i)) \right) + \sum_{i=0}^{k-1} U(k, i+1) f_2(i, x(i), y(i)).$$

Proof. The function $(x, y) \in \mathrm{C}(\mathbb{N}(0, b), X) \times \mathrm{C}(\mathbb{N}(0, b), X)$ is a mild solution of (1.2) if and only if

$$x(k) = U(k, 0)x(0) + \sum_{i=0}^{k-1} U(k, i+1) f_1(i, x(i), y(i)),$$

$$L_1(U(k, 0)x(0)) + L \left(\sum_{t=0}^{k-1} U(k, t+1) f_1(t, x(t), y(t)) \right) = l_1,$$

and

$$y(k) = U(k, 0)y(0) + \sum_{i=0}^{k-1} U(k, i+1) f_2(i, x(i), y(i)),$$

$$L_2(U(k, 0)y(0)) + L_2 \left(\sum_{t=0}^{k-1} U(k, t+1) f_2(t, x(t), y(t)) \right) = l_2.$$

Hence

$$\widetilde{L}_1(U(k, 0)x(0)) = l_1 - L_1 \left(\sum_{t=0}^{k-1} U(k, t+1) f_2(t, x(t), y(t)) \right),$$

and

$$\widetilde{L}_2(U(k, 0)y(0)) = l_2 - L_2 \left(\sum_{t=0}^{k-1} U(k, t+1) f_2(t, x(t), y(t)) \right).$$

Since $\widetilde{L}_1$ and $\widetilde{L}_2$ are invertible operators, we get

$$x(0) = \widetilde{L}_1^{-1} l - U(k,0)\widetilde{L}_1^{-1}\left(L_1 \sum_{i=0}^{k-1} U(k,i+1) f_1(i,x(i))\right)$$

and

$$y(0) = \widetilde{L}_2^{-1} l - U(k,0)\widetilde{L}_2^{-1}\left(L_2 \sum_{i=0}^{k-1} U(k,i+1) f_2(i,x(i))\right).$$

Hence

$$\begin{aligned} x(k) &= U(k,0)\widetilde{L}_1^{-1}\left(l_1 - L_1 \sum_{i=0}^{k-1} U(k,i+1) f_1(i,x(i),y(i))\right) \\ &\quad + \sum_{i=0}^{k-1} U(k,i+1) f_1(i,x(i),y(i)) \end{aligned}$$

and

$$\begin{aligned} y(k) &= U(k,0)\widetilde{L}_2^{-1}\left(l_2 - L_2 \sum_{i=0}^{k-1} U(k,i+1) f_2(i,x(i),y(i))\right) \\ &\quad + \sum_{i=0}^{k-1} U(k,i+1) f_2(i,x(i),y(i)). \end{aligned}$$

The proof is complete. □

Now, we present the first result of this section.

Theorem 5.2. *Let $f_1, f_2 : \mathbb{N}(0,b) \times X \times X \to X$ be continuous functions. Let conditions (H_2), (H_3) and $(\mathcal{L})$ be satisfied. Assume that the condition*

(H_5) *There exist $p_3, p_4, \bar{p}_3, \bar{p}_4 \in \mathrm{C}(\mathbb{N}(0,b), \mathbb{R}_+)$ and $\gamma_1, \gamma_2 \in [0,1)$ such that*

$$|f_1(k,x,y)| \le p_3(k)(|x|+|y|)^{\alpha} + \bar{p}_3(k),\ k \in \mathbb{N}(0,b),\ (x,y) \in X \times X,$$

and

$$|f_2(k,x,y)| \le p_4(k)(|x|+|y|)^{\beta} + \bar{p}_4(k),\ k \in \mathbb{N}(0,b),\ (x,y) \in X \times,$$

holds. Then (1.2) *has at least one solution.*

Proof. Define the operator $\bar{N} : \mathrm{C}(\mathbb{N}(0,b), X) \times \mathrm{C}(\mathbb{N}(0,b), X) \to \mathrm{C}(\mathbb{N}(0,b), X)$ by

$$\bar{N}(x,y) = (\bar{N}_1(x,y), \bar{N}_2(x,y)),\ (x,y) \in \mathrm{C}(\mathbb{N}(0,b), X) \times \mathrm{C}(\mathbb{N}(0,b), X),$$

where

$$\begin{aligned}\bar{N}_1(x(k),y(k)) &= U(k,0)\widetilde{L}_1^{-1}\left(l_1 - L_1\sum_{i=0}^{k-1}U(k,i+1)f_1(i,x(i),y(i))\right)\\ &\quad+\sum_{i=0}^{k-1}U(k,i+1)f_1(i,x(i),y(i)),\end{aligned}$$

and

$$\begin{aligned}\bar{N}_2(x(k),y(k)) &= U(k,0)\widetilde{L}_2^{-1}\left(l_2 - L_2\sum_{i=0}^{k-1}U(k,i+1)f_2(i,x(i),y(i))\right)\\ &\quad+\sum_{i=0}^{k-1}U(k,i+1)f_2(i,x(i),y(i)).\end{aligned}$$

Step 1: $N=(\bar{N}_1,\bar{N}_2)$ is continuous. Let (x_m,y_m) be a sequence such that $(x_m,y_m)\to(x,y)\in \mathrm{C}(\mathbb{N}(0,b),X)\times\mathrm{C}(\mathbb{N}(0,b),X)$ as $m\to\infty$. Then

$$\begin{aligned}|N_1(x_m(k),y_m(k))-N_1(x(k),y(k))| &= |U(k,0)\widetilde{L}_1^{-1}\Big(L_1\sum_{i=0}^{k-1}U(k,i+1)\times\\ &\quad[f_1(i,x_m(i),y_m(i))-f_1(i,x(i),y(i))]\Big)\\ &\quad+\sum_{i=0}^{k-1}U(k,i+1)\times\\ &\quad[f_1(i,x_m(i),y_m(i))-f_1(i,x(i),y(i))]|\end{aligned}$$

and

$$\begin{aligned}|N_2(x_m(k),y_m(k))-N_2(x(k),y(k))| &= |U(k,0)\widetilde{L}_2^{-1}\Big(L_2\sum_{i=0}^{k-1}U(k,i+1)\times\\ &\quad[f_2(i,x_m(i),y_m(i))-f_2(i,x(i),y(i))]\Big)\\ &\quad+\sum_{i=0}^{k-1}U(k,i+1)\times\\ &\quad[f_2(i,x_m(i),y_m(i))-f_2(i,x(i),y(i))]|.\end{aligned}$$

Then

$$\begin{aligned}\|N_1(x_m,y_m)-N_1(x,y)\|_\infty &\leq Q^2\|\widetilde{L}_1^{-1}\|_{B(X)}\|L_1\|_{B(X)}\sum_{i=0}^{b}|f_1(i,x_m(i),y_m(i))\\ &\quad-f_1(i,x(i),y(i))|\\ &\quad+Q\sum_{i=0}^{b}|f_1(i,x_m(i),y_m(i))-f_1(i,x(i),y(i))|\end{aligned}$$

and

$$\begin{aligned} \|N_2(x_m,y_m)-N_2(x,y)\|_\infty &\le Q^2\|\widetilde{L}_2^{-1}\|_{B(X)}\|L_2\|_{B(X)}\sum_{i=0}^{b}|f_2(i,x_m(i),y_m(i)) \\ &\quad -f_2(i,x(i),y(i))| \\ &\quad +Q\sum_{i=0}^{b}|f_2(i,x_m(i),y_m(i))-f_2(i,x(i),y(i))|. \end{aligned}$$

Since f_1 and f_2 are continuous functions, we get

$$\|N_1(x_m,y_m)-N_1(x,y)\|_\infty \to 0, \text{ as } m\to\infty$$

and

$$\|N_2(x_m,y_m)-N_2(x,y)\|_\infty \to 0, \text{ as } m\to\infty.$$

So N is continuous.

Step 2: N maps bounded sets into bounded sets in $\mathrm{C}(\mathbb{N}(0,b),X)\times\mathrm{C}(\mathbb{N}(0,b),X)$. Indeed, it is enough to show that for any $q>0$ there exists a positive constant M such that for each $x\in B_1, y\in B_2$, where

$$B_1=\{x\in \mathrm{C}(\mathbb{N}(0,b),X):\ \|x\|_\infty\le q\}$$

and

$$B_2=\{y\in \mathrm{C}(\mathbb{N}(0,b),X):\ \|y\|_\infty\le q\},$$

we have

$$\|N(x,y)\|_\infty\le M=(M_1,M_2).$$

Then for each $k\in\mathbb{N}(0,b)$, we get

$$\begin{aligned} \|N_1(x,y)\|_\infty &\le Q\left[\|\widetilde{L}_1^{-1}\|_{B(X)}\|l_1\|+2q^{\gamma_1}Q\|L_1\|_{B(X)}\sum_{k=1}^{b}p_3(k)+\sum_{k=1}^{b}\bar{p}_3(k)\right] \\ &\quad +2q^{\gamma_1}Q\sum_{i=0}^{b}\bar{p}_3(k). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|N_2(x,y)\|_\infty &\le Q\left[\|\widetilde{L}_2^{-1}\|_{B(X)}\|l_2\|+2q^{\gamma_2}Q\|L_2\|_{B(X)}\sum_{k=1}^{b}p_4(k)+\sum_{k=1}^{b}\bar{p}_4(k)\right] \\ &\quad +2q^{\gamma_2}Q\sum_{i=0}^{b}\bar{p}_4(k). \end{aligned}$$

By (H_3), we can easily prove that, for each $k \in \mathbb{N}(0,b)$, we have

$$\{N_1(x(k),y(k)) : (x,y) \in B\} \text{ and } \{N_2(x(k),y(k)) : (x,y) \in B\}$$

are relatively compact in X. Then as a consequence of Theorem 2.10, we conclude that $\overline{N(B)}$ is compact. As consequence of Steps 1 to 2, N is completely continuous.

Step 3: It remains to show that $\mathcal{A} = \{(x,y) \in \mathrm{C}(\mathbb{N}(0,b),X) \times \mathrm{C}(\mathbb{N}(0,b),X) : (x,y) = \lambda N(x,y), \lambda \in (0,1)\}$ is bounded. Let $(x,y) \in \mathcal{A}$. Then $x = \lambda N_1(x,y)$ and $y = \lambda N_2(x,y)$ for some $0 < \lambda < 1$. Thus, for $k \in \mathbb{N}(0,b)$, we have

$$\begin{aligned} \|x(k)\| \leq\; & Q\Big[\|\widetilde{L}_1^{-1}\|_{B(X)}\|l_1\| + (Q\|\widetilde{L}_1^{-1}\|_{B(X)}\|L_1\|_{B(X)} + 1) \times \\ & \left(\sum_{i=0}^{b} p_3(i)(|x(i)| + |y(i)|)^{\gamma_1} + \sum_{i=0}^{b} \bar{p}_3(i)\right)\Big] \end{aligned}$$

and

$$\begin{aligned} \|y(k)\| \leq\; & Q\Big[\|\widetilde{L}_2^{-1}\|_{B(X)}\|l_2\| + (Q\|\widetilde{L}_2^{-1}\|_{B(X)}\|L_2\|_{B(X)} + 1) \times \\ & \left(\sum_{i=0}^{k-1} p_4(i)(|x(i)| + |y(i)|)^{\gamma_2} + \sum_{i=0}^{b} \bar{p}_3(i)\right)\Big]. \end{aligned}$$

Therefore

$$\|x\|_\infty + \|y\|_\infty \leq Q_1 + Q_2 \sum_{i=0}^{b} p_*(i)(\|x\|_\infty + \|y\|_\infty)^{\max(\gamma_1,\gamma_2)},$$

where

$$\begin{aligned} Q_1 &= Q(\|\widetilde{L}_1^{-1}\|_{B(X)}\|l_1\| + \|\widetilde{L}_2^{-1}\|_{B(X)}\|l_2\| + Q_2 M), \\ Q_2 &= Q(\|\widetilde{L}_1^{-1}\|_{B(X)}\|L_1\|_{B(X)} + \|\widetilde{L}_2^{-1}\|_{B(X)}\|L_2\|_{B(X)} + 2), \end{aligned}$$

and

$$M = \sum_{i=0}^{b}(\bar{p}_3(i) + p_4(i)),\ \|p_*(k) = p_3(k) + p_4(k),\ k \in \mathbb{N}(0,b).$$

Then

$$\|x\|_\infty + \|y\|_\infty \leq \max(1, C),$$

where

$$C = \left[Q_1 + Q_2 \sum_{i=0}^{b} p_*(i)\right]^{\frac{1}{1-\max(\gamma_1,\gamma_2)}}.$$

This shows that A is bounded. As a consequence of Theorem 2.7, we deduce that N has a fixed point (x,y) which is a solution to (1.2). $\square$

By measure of noncompactness arguments, we have a second result of this part of the paper.

Theorem 5.3. *Let $f_1, f_2 : \mathbb{N}(0,b) \times X \times X \to X$ be continuous functions. Assume that (H_2), (H_5) and the condition*

(H_6) *There exist nonnegative numbers a_i and b_i for each $i \in \{1,2\}$ such that for every bounded set $B_1 \times B_2 \subset X \times X$, we have*

$$\alpha(f_1(k, B_1, B_2)) \leq a_1\alpha(B_1) + a_2\alpha(B_2),\ k \in \mathbb{N}(0,b),\ (x,y) \in X \times X$$

and

$$\alpha(f_2(k, B_1, B_2)) \leq b_1\alpha(B_1) + b_2\alpha(B_2),\ k \in \mathbb{N}(0,b),\ (x,y) \in X \times X,$$

where α is Kuratowski's measure of noncompactness.

If the matrix

$$M = Qb\begin{bmatrix} (Q\|\widetilde{L}_1^{-1}\|_{B(X)}\|L_1\|_{B(X)}+1)a_1 & (Q\|\widetilde{L}_1^{-1}\|_{B(X)}\|L_1\|_{B(X)}+1)a_2 \\ (Q\|\widetilde{L}_2^{-1}\|_{B(X)}\|L_2\|_{B(X)}+1)b_1 & (Q\|\widetilde{L}_2^{-1}\|_{B(X)}\|L_2\|_{B(X)}+1)b_2 \end{bmatrix}$$

converges to zero, then (1.2) *has a solution.*

Proof. Let $\bar{N} : \mathrm{C}(N(0,b),X) \times \mathrm{C}(\mathbb{N}(0,b),X) \to \mathrm{C}(\mathbb{N}(0,b),X)$ be the operator defined in Theorem 5.2. As in Theorem 5.2, we can easily prove that $\bar{N}$ is continuous, transforms bounded sets into bounded sets and

$$\mathcal{A} = \{(x,y) \in \mathrm{C}(\mathbb{N}(a,b),X) \times \mathrm{C}(\mathbb{N}(a,b),X) :\ (x,y) = \lambda\bar{N}(x,y),\ \lambda \in (0,1)\}$$

is bounded. In order to apply Theorem 3.5, we prove that $\bar{N}$ is β-condensing, where

$$\beta(B_1 \times B_2) = (\beta_1(B_1), \beta_2(B_2)),$$

$$\beta_1(B) = \beta_2(B) = \sup\{\alpha(B(k)) :\ k \in \mathbb{N}(0,b)\},\ b \in \mathrm{C}(\mathbb{N}(0,b),X),$$

and

$$\beta_*(B_1 \times B_2) = (\alpha_1(B_1), \alpha_2(B_2)).$$

For that, for each $k \in \mathbb{N}(0,b)$, we have

$$
\begin{aligned}
\beta_*(N(B_1\times B_2)(k)) &= \begin{pmatrix} \alpha_1(\{N_1(x(k),y(k)):\ x\in B_1, y\in B_2\}) \\ \alpha_2(\{N_2(x(k),y(k)):\ x\in B_1, y\in B_2\}) \end{pmatrix} \\
&\le \begin{pmatrix} Q^2\|G^{-1}\|\|L\|\displaystyle\sum_{i=s}^{k-1}\alpha_1(\{f_1(i,x(i),y(i)):\ x\in B_1, y\in B_2\}) \\ Q^2\|G^{-1}\|\|L\|\displaystyle\sum_{i=s}^{k-1}\alpha_2(\{f_2(i,x(i),y(i)):\ x\in B_1, y\in B_2\}) \end{pmatrix} \\
&\quad + \begin{pmatrix} Q\displaystyle\sum_{i=s}^{k-1}\alpha_1(\{f_1(i,x(i),y(i)):\ x\in B_1, y\in B_2\}) \\ Q\displaystyle\sum_{i=s}^{k-1}\alpha_2(\{f_2(i,x(i),y(i)):\ x\in B_1, y\in B_2\}) \end{pmatrix} \\
&\le \begin{pmatrix} Q^2\|G^{-1}\|\|L\|b\alpha_1(f_1(k,B_1,B_2)) + Qb\alpha_1(f_1(k,B_1,B_2)) \\ Q^2\|G^{-1}\|\|L\|b\alpha_2(f_2(k,B_1,B_2)) + Qb\alpha_2(f_2(k,B_1,B_2)) \end{pmatrix} \\
&\le \begin{pmatrix} Q^2\|G^{-1}\|\|L\|b(a_1\beta_1(B_1)+a_2\beta_2(B_2)) \\ +Qb(a_1\beta_1(B_1)+a_2\beta_2(B_2)) \\ \\ Q^2\|G^{-1}\|\|L\|b(b_1\beta_1(B_1)+b_2\beta_2(B_2)) \\ +Qb(b_1\beta_1(B_1)+b_2\beta_2(B_2)) \end{pmatrix} \\
&\le Qb\begin{pmatrix} Q\|G^{-1}\|\|L\|+1 \\ Q\|G^{-1}\|\|L\|+1 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} \beta_1(B_1) \\ \beta_2(B_2) \end{pmatrix}.
\end{aligned}
$$

Hence

$$\beta(N(B)) \le M\beta(B).$$

As a consequence of Theorem 3.5, we deduce that $\bar{N}$ has a fixed point (x,y) which is a solution to (1.2). □

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