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Oscillation Criteria for Second-Order Neutral Difference Equations via Third-Order Difference Equations

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Abstract

In this paper, we are concerned with oscillation of a second-order nonlinear neutral delay difference equation of the form

$$\Delta\left(a_n\Delta\left(x_n - p_n x_{n-1}\right)^{\gamma}\right) + q_n f(x_{n-\tau}) = 0.$$

We consider the two cases when $\gamma \geq 1$ and $0 < \gamma < 1$. The main oscillation results will be of Kamenev's type, Philos'type as well as Hille and Nehari types. The results will be established by using the oscillation criteria of the corresponding third order nonlinear difference equation.

AMS Subject Classifications: 34C10, 34K11, 34B05.
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1 Introduction

In this paper, we are concerned with oscillation of solutions of the second-order nonlinear neutral delay difference equation

$$\Delta\left(a_n\Delta\left(x_n - p_n x_{n-1}\right)^{\gamma}\right) + q_n f(x_{n-\tau}) = 0, \text{ for } n \in \mathbb{N}_{n_0}, \tag{1.1}$$

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where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\mathbb{N}_i = \{i+1, i+2, \ldots\}$. Throughout this paper, we will assume that the real sequences a_n, p_n, q_n are nonnegative, γ is a quotient of odd positive integers, τ is a nonnegative integer and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(u) > 0$ for $u \neq 0$. As usual, a nontrivial solution of (1.1) is called nonoscillatory if it eventually positive or eventually negative. Otherwise it is called oscillatory.

In recent years, there has been much research activity concerning oscillation of second-order neutral delay difference equations, we refer the reader to the papers [1, 2, 4, 5, 7, 9, 12, 16, 19, 22–24] and the references cited therein. Most of the results in the above references are obtained when $p_n \leq 0$. Here, we recall some of the results that motivate our study.

In [17], Thandapani and Mahalingam studied the oscillatory and asymptotic behavior of the equation

$$\Delta^2 \left(x_{n-1} - p_n x_{n-k-1}\right) + q_n f\left(x_{n-\tau}\right) = 0, \ \ n = 1, 2, 3, \ldots \tag{1.2}$$

with the condition $0 \leq p_n < 1$, where f satisfies the superlinear condition

$$0 < \int_n^{\infty} \frac{du}{f(u)}, \int_{-\infty}^{-n} \frac{du}{f(u)} < \infty, \text{ for all } n > 0, \tag{1.3}$$

and also, when f satisfies the sublinear condition

$$0 < \int_0^{n} \frac{du}{f(u)}, \int_{-n}^{0} \frac{du}{f(u)} < \infty, \text{ for all } n > 0. \tag{1.4}$$

Thandapani and Mohan [18] considered the equation

$$\Delta^2 \left(x_n - p_n x_{n-k}\right) + q_n f\left(x_{n-\tau}\right) = 0, \ n \geq n_0, \tag{1.5}$$

with the condition $0 < p_n \leq 1$ and f satisfies the conditions (1.3) and (1.4). Thandapani et al. [15] considered the generalized equation

$$\Delta \left(a_n \Delta \left(x_n - p_n x_{n-k}\right)^{\alpha}\right) + q_n f\left(x_{n-\tau}\right) = 0, \tag{1.6}$$

under the conditions

$$0 \leq p_n < p < 1, \ |f(z)| \geq k\,|z|^{\alpha} \text{ and } \sum_{n=n_0}^{\infty} \left(\frac{1}{a_n}\right)^{\frac{1}{\alpha}} = \infty.$$

Thandapani et al. [14] considered the equation

$$\Delta \left(a_n \Delta \left(x_n - p x_{n-k}\right)^{\alpha}\right) - q_n f\left(x_{\sigma(n)}\right) = 0, \tag{1.7}$$

where p is a real number and $\sigma(n)$ is a sequence of integers such that $\sigma(n) \to \infty$ as $n \to \infty$, and

$$\sum_{n=n_0}^{\infty} (a_n)^{\frac{-1}{\alpha}} = \infty.$$

Thandapani et al. [21] also considered the equation

$$\Delta\left(a_n \Delta\left(x_n - p x_{n-k}^{\alpha}\right)\right) + q_n f\left(x_{n+1-l}\right) = 0,\ n \geq n_0 \geq 0,$$

where p is a real number and a_n is a positive sequence with

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty.$$

Recently Thandapani et al. [20] considered the generalized equation (1.6) when $\alpha = 1$, and

$$0 \leq p_n \leq p < 1 \text{ and } \sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty.$$

In this paper in Section 2, we will consider the case when $0 < p_n \leq p \leq 1$ and establish some sufficient conditions for oscillation. The main results will be proved by employing a different technique which depends on an application of an invariant substitution that transforms the second nonlinear neutral difference equation to a third nonlinear difference equation. The results will be of Kamenev's type, Philos' type and Hille and Nehari types.

2 Main Results

In this section we state and prove the main results and is organized as follows. First, we use the invariant substitution to obtain the corresponding third order difference equation. Next, we present some basic lemmas will be used in the proof of the main results and then state and prove the main results by considering two different cases when $\gamma \geq 1$ and when $0 < \gamma < 1$. Now we apply an invariant substitution which transforms the neutral equation to a non-neutral third order difference equations. This substitution is given by (see [3])

$$y_{n+1} = x_n \prod_{i=1}^{n} \frac{1}{p_i}, \quad \text{where} \quad \prod_{i=1}^{n} p_i = O\left(n\right), \tag{2.1}$$

This gives us that

$$x_n = y_{n+1} \prod_{i=1}^{n} p_i, \quad x_{n-1} = y_n \prod{}_{i=1}^{n-1} p_i, \quad \text{and } x_{n-\tau} = y_{n-\tau+1} \prod_{i=1}^{n-\tau} p_i. \tag{2.2}$$

From (2.2), we have

$$x_n - p_n x_{n-1} = \Delta y_n \prod_{i=1}^{n} p_i. \tag{2.3}$$

Substituting (2.3) into (1.1), we obtain

$$\Delta \left(a_n \Delta \left(\Delta y_n \prod_{i=1}^{n} d_i \right)^{\gamma} \right) + q_n f \left(\prod_{i=1}^{n-\tau} p_i y_{n-\tau+1} \right) = 0. \tag{2.4}$$

Setting $d_n = \prod_{i=1}^{n} p_i$, (2.4) becomes

$$\Delta \left(a_n \Delta \left(d_n \Delta y_n \right)^{\gamma} \right) + q_n f(d_{n-\tau} y_{n-\tau+1}) = 0. \tag{2.5}$$

Throughout the paper, we will assume that

(H_1) $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty,$

(H_2) there exists a constant $k > 0$ such that $|f(u)| \geq k |u|^{\gamma}$, for all $u \neq 0$,

(H_3) p_n is a real sequence with $0 < p_n \leq p < 1$ for all $n \in \mathbb{N}_{\max\{1,\tau\}}$,

(H_4) $\sum_{n=1}^{\infty} \frac{1}{d_n} = \infty$ (see [3, Lemma 2]),

(H_5) $\sum_{i=n_0}^{\infty} Q_n = \infty,$

(H_6) $\lim_{n\to\infty} \sum_{s=n_0}^{n-1} \frac{1}{d_s} \left[\sum_{t=n_0}^{s-1} \left(\frac{1}{c_t} \sum_{i=n_0}^{t-1} Q_i \right) \right]^{\frac{1}{\gamma}} = \infty.$

Now, are ready to state some fundamental lemmas for third order difference equations that will be used in the proofs of the main results. By using the condition (H_2), (2.5) becomes

$$\Delta \left(a_n \Delta \left(d_n \Delta y_n \right)^{\gamma} \right) + Q_n y_{n-\tau+1}^{\gamma} \leq 0, \quad \text{for } n \geq n_0, \tag{2.6}$$

where $Q_n = k q_n d_{n-\tau}^{\gamma}$. The following lemmas are adapted from [10].

Lemma 2.1. *Let* y_n *be an eventually positive solution of* (2.6) *and suppose that Case* (i) *of Lemma 2.2 hold. Then there exists* $n_1 \geq n_0$ *such that*

$$\left(\Delta y_{n-\tau+1} \right)^{\gamma} \geq a_n \frac{\varphi_{n-\tau+1}}{\left(d_{n-\tau+1} \right)^{\gamma}} \Delta \left(d_n \Delta y_n \right)^{\gamma}, \quad n \geq n_1, \tag{2.7}$$

where $\varphi(n) = \sum_{i=n_0}^{n-1} (1/a_i)$.

Lemma 2.2. *Assume that* (H_1)–(H_6) *hold. Suppose that* y_n *is an eventually positive solution of* (2.6). *Then there is* $n_1 \geq n_0$ *sufficiently large such that for* $n \geq n_1$ *we have only the following two cases:*

(i) $y_n > 0,\ \Delta y_n > 0,\ \Delta (d_n \Delta y_n)^\gamma > 0,$
(ii) $y_n > 0,\ \Delta y_n < 0,\ \Delta (d_n \Delta y_n)^\gamma > 0.$

Using the Case (ii) in Lemma 2.2, we can prove that (see [8, Lemma 2.1]) $\lim_{n\to\infty} y_n = 0$. Using this we can prove the following lemma which can be considered as an extension of the proof of [3, Lemma 4].

Lemma 2.3. *Assume that* $(H_1) - (H_6)$ *hold. If* y_n *be a nonoscillatory solution of* (2.6) *which satisfies Case (ii) of Lemma 2.2, then* $\lim_{n\to\infty} y_n = 0$ *and hence*

$$\lim_{n\to\infty} \frac{x_n}{d_n} = 0. \tag{2.8}$$

Now, we are ready to state and prove the sufficient conditions which ensure that each solution of equation (1.1) is oscillatory or satisfies (2.8)). We begin with the first case when $\gamma \geq 1$.

Theorem 2.4. *Assume that* (H_1)–(H_6) *hold and let* x_n *be solution of* (1.1). *Furthermore, assume that there exists a positive sequence* ρ_n *such that*

$$\lim_{n\to\infty} \sum_{i=n_0}^{n} \left[\rho_i Q_i - \frac{(d_{i-\tau+1})^\gamma (\Delta \rho_i)^2}{2^{3-\gamma} \varphi_{i-\tau+1}} \right] = \infty. \tag{2.9}$$

Then x_n *oscillates or satisfies* (2.8).

Proof. Let x_n be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x_n > 0$ and $x_{n-1} > 0$ for $n > n_1$ sufficiently large. From (2.1), we see that y_n is a positive solution of (2.6) and there are only the following two cases for $n \geq n_1$:

Case (i) $y_n > 0,\ \Delta y_n > 0,\ \Delta (d_n \Delta y_n)^\gamma > 0,$
Case (ii) $y_n > 0,\ \Delta y_n < 0,\ \Delta (d_n \Delta y_n)^\gamma > 0.$

We consider the Case (i). Defining the sequence ω_n by the Riccati substitution

$$\omega_n = \rho_n \frac{a_n \Delta (d_n \Delta y_n)^\gamma}{y_{n-\tau+1}^\gamma} \quad \text{for} n > n_1. \tag{2.10}$$

This implies $\omega_n > 0$ and

$$\Delta \omega_n = a_{n+1} \Delta (d_{n+1} \Delta y_{n+1})^\gamma \, \Delta \left[\frac{\rho_n}{y_{n-\tau+1}^\gamma} \right] + \rho_n \frac{\Delta a_n \Delta (d_n \Delta y_n)^\gamma}{y_{n-\tau+1}^\gamma}.$$

Hence

$$\begin{aligned}\Delta\omega_n &= a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\left[\frac{\Delta\rho_n\left(y^{\gamma}_{n-\tau+1}\right)-\rho_n\left(\Delta y^{\gamma}_{n-\tau+1}\right)}{y^{\gamma}_{n-\tau+1}y^{\gamma}_{n-\tau+2}}\right]\\ &\quad+\rho_n\frac{\Delta a_n\Delta\left(d_n\Delta y_n\right)^{\gamma}}{y^{\gamma}_{n-\tau+1}}.\end{aligned}$$

From this, (2.6) and (2.10), we see that

$$\Delta\omega_n \leq \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}-\left[\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\rho_n\Delta y^{\gamma}_{n-\tau+1}}{y^{\gamma}_{n-\tau+2}y^{\gamma}_{n-\tau+1}}\right]-\rho_nQ_n. \qquad (2.11)$$

From Lemma 2.2, since $\Delta y_n \geq 0$, we have that $y_{n-\tau+2} \geq y_{n-\tau+1}$. Then from (2.11), we obtain

$$\Delta\omega_n \leq -\rho_nQ_n+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}-\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\rho_n\Delta y^{\gamma}_{n-\tau+1}}{y^{2\gamma}_{n-\tau+2}}. \qquad (2.12)$$

Using the inequality

$$x^{\gamma}-y^{\gamma} \geq 2^{1-\gamma}\left(x-y\right)^{\gamma},\ \text{for all } x \geq y > 0 \text{ where } \gamma \geq 1, \qquad (2.13)$$

we have

$$\begin{aligned}\Delta y^{\gamma}_{n-\tau+1} &= \left(y^{\gamma}_{n-\tau+2}-y^{\gamma}_{n-\tau+1}\right) \geq 2^{1-\gamma}\left(y_{n-\tau+2}-y_{n-\tau+1}\right)^{\gamma} \qquad (2.14)\\ &= 2^{1-\gamma}(\Delta y_{n-\tau+1})^{\gamma}.\end{aligned}$$

Substituting (2.14) into (2.12), we obtain

$$\Delta\omega_n \leq -\rho_nQ_n+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}-\rho_n2^{1-\gamma}\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\left(\Delta y_{n-\tau+1}\right)^{\gamma}}{y^{2\gamma}_{n-\tau+2}}. \qquad (2.15)$$

$$\Delta\omega_n \leq -\rho_nQ_n+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}-\rho_n2^{1-\gamma}\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\left(\Delta y_{n-\tau+1}\right)^{\gamma}}{y^{2\gamma}_{n-\tau+2}}.$$

From Lemma 2.1, substituting (2.7) into (2.15),

$$\begin{aligned}\Delta\omega_n &\leq -\rho_nQ_n+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}\\ &\quad-2^{1-\gamma}\rho_n\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\varphi_{n-\tau+1}a_n\Delta\left(d_n\Delta y_n\right)^{\gamma}}{\left(d_{n-\tau+1}\right)^{\gamma}y^{2\gamma}_{n-\tau+2}}.\end{aligned}$$

From (2.6), since $\Delta\left(a_n\Delta\left(d_n\left(\Delta y_n\right)\right)^{\gamma}\right) \leq 0$, we have

$$a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma} \leq a_n\Delta\left(d_n\Delta y_n\right)^{\gamma}, \qquad (2.16)$$

$$\begin{aligned}\Delta\omega_n &\leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} \\ &\quad -2^{1-\gamma}\rho_n \frac{\left(a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^\gamma\right)^2 \varphi_{n-\tau+1}}{\left(d_{n-\tau+1}\right)^\gamma y_{n-\tau+2}^{2\gamma}}.\end{aligned} \tag{2.17}$$

From (2.10) and (2.17), we have

$$\Delta\omega_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - 2^{1-\gamma}\frac{\rho_n\varphi_{n-\tau+1}}{\rho_{n+1}^2\left(d_{n-\tau+1}\right)^\gamma}\left(\omega_{n+1}\right)^2. \tag{2.18}$$

The rest of the proof of this case is similar to the proof of [10, Theorem 6.4.1]. The proof of the case when Case (ii) holds is similar to the proof of [11, Lemma 2.4]. The proof is complete. □

From Theorem 2.4, we can obtain different condition for oscillation of all solutions of (1.1) by different choices of ρ_n. For example, if we take $\rho_n = n^\lambda$, $n \geq n_0$ and $\lambda > 1$ is a constant, then we have the following result.

Corollary 2.5. *Assume that all the assumptions of Theorem 2.4 hold, except that* (2.9) *is replaced by*

$$\lim_{n\to\infty}\sum_{s=n_0}^{n}\left[s^\lambda Q_i - \frac{\left(d_{s-\tau+1}\right)^\gamma\left(\left(s+1\right)^\lambda - s^\lambda\right)^2}{2^{3-\gamma}\varphi_{s-\tau+1}}\right] = \infty.$$

Then x_n oscillates or satisfies (2.8).

The following theorem gives new oscillation criteria for (1.1) of Philos-type. The proof is based on the technique investigated in [11] by using the Riccati inequality (2.18).

Theorem 2.6. *Assume that (H_1)–(H_6) hold and x_n be solution of* (1.1). *Let ρ_n be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H_{m,n} : m \geq n \geq 0\}$ such that*

(i) $H_{m,m} = 0$ *for* $m \geq 0$.

(ii) $H_{m,n} \geq 0$, *for* $m \geq n \geq 0$.

(iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *for* $m \geq n \geq 0$.

If

$$\lim_{n\to\infty}\frac{1}{H_{m,n_0}}\sum_{i=n_0}^{m-1}\left[H_{m,i}\rho_i Q_i - \frac{\left(\rho_{i+1}\right)^2}{4\rho_i}A_{m,i}^2\right] = \infty, \tag{2.19}$$

where

$$\overline{\rho_n} \; : \; = 2^{1-\gamma} \frac{\rho_n \varphi_{n-\tau+1}}{(d_{n-\tau+1})^{\gamma}}, \; A_{m,n} := \left(h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right),$$

$$h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}},$$

then x_n *oscillates or satisfies* (2.8).

By choosing the sequence $H_{m,n} = (m-n)^{\lambda}$ in appropriate manners, we can derive an oscillation criteria for (1.1) of Kamenev's type. The technique of proof based on the results investigated in [25].

Theorem 2.7. *Assume that all the assumptions of Theorem 2.6 hold. Furthermore, assume that there exists a positive sequence* ρ_n *such that for every positive number* $\lambda \geq 1$*,*

$$\lim_{n\to\infty} \frac{1}{m^{\lambda}} \sum_{i=n_0}^{n-1} (m-i)^{\lambda} \left[\rho_i Q_i - \frac{(\rho_{i+1})^2}{4\overline{\rho_n}} A_{m,i}^2 \right] = \infty, \tag{2.20}$$

where

$$\overline{\rho_n} := 2^{1-\gamma} \frac{\rho_n \varphi_{n-\tau+1}}{(d_{n-\tau+1})^{\gamma}}, \; A_{m,n} := \left(\frac{\Delta \rho_n}{\rho_{n+1}} - \frac{\lambda (m-n-1)}{(m-n)^{\lambda}} \right).$$

Then x_n *oscillates or satisfies* (2.8).

In the following, we establish new oscillation criteria for (1.1) of Hille and Nehari types. Now, from the proof of Theorem 2.4, we see, by choosing $\rho_n = 1$, that the corresponding Riccati inequality when Case (i) of Lemma 2.2 hold is given by,

$$\Delta \omega_n \leq -Q_n - \frac{1}{r_n} \omega_{n+1}^2, \text{ for } \; n \geq n_1, \tag{2.21}$$

where $r_n = 2^{\gamma-1} (d_{n-\tau+1})^{\gamma} / \varphi_{n-\tau+1}$. Using this inequality and proceeding as in the proof of [10, Theorem 6.4.5], we obtain the following theorem.

Theorem 2.8. *Assume that* (H_1)*–*(H_6) *hold and* x_n *be solution of* (1.1)*. Furthermore assume that* $\Delta r_n \geq 0$*. If*

$$\lim_{n\to\infty} \inf \frac{n}{r_n} \sum_{s=n+1}^{\infty} Q_s > \frac{1}{4}, \tag{2.22}$$

or

$$\lim_{n\to\infty} \inf \frac{n}{r_n} \sum_{s=n+1}^{\infty} Q_s + \lim_{n\to\infty} \inf \frac{1}{n} \sum_{s=n+1}^{\infty} \frac{s^2}{r_n} Q_s > 4,$$

then x_n *oscillates or satisfies* (2.8).

Next, in the following we consider the second case when $0 < \gamma < 1$.

Theorem 2.9. *Assume that all the assumptions of Theorem 2.4 hold, except that* (2.19) *is replaced by*

$$\lim_{n\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{i=n_0}^{n-1} \left(H_{m,.i}\psi_i - \phi_{m,i}\right) = \infty, \tag{2.23}$$

where

$$\psi_n = \rho_n Q_n,\ \phi_{m,n} = \frac{\rho_{n+1}^{\gamma+1}}{(1+\gamma)^{1+\gamma}\rho_n^{\gamma}\delta_{n-\tau}d_{n-\tau+1}^{-\gamma}H_{m,n}^{\gamma}}\left(\frac{\psi_n H_{m.n}}{\rho_{n+1}} - h_{m,n}H_{m,n}^{\frac{1}{1+\gamma}}\right)^{1+\gamma}.$$

Then x_n oscillates or satisfies (2.8).

Proof. Let x_n be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x_n > 0$ and $x_{n-1} > 0$ for $n > n_1$ sufficiently large. From (2.1), we see that y_n is a positive solution of (2.6) and there are only the following two cases for $n \geq n_1$:

Case (i) $y_n > 0,\ \Delta y_n > 0,\ \Delta\left(d_n \Delta y_n\right)^{\gamma} > 0,$

Case (ii) $y_n > 0,\ \Delta y_n < 0,\ \Delta\left(d_n \Delta y_n\right)^{\gamma} > 0.$

We consider the Case (i). Defining the sequence ω_n by the Riccati substitution (2.10) and proceeding as in the proof Theorem 2.4 (2.12) to get

$$\Delta\omega_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\rho_n \Delta y_{n-\tau+1}^{\gamma}}{y_{n-\tau+2}^{2\gamma}}. \tag{2.24}$$

By using the inequality [6, p. 39], we get

$$x^{\gamma} - y^{\gamma} \geq \gamma x^{\gamma-1}(x-y),\ \text{for all } x \neq y > 0 \text{ and } 0 < \gamma < 1. \tag{2.25}$$

We may write

$$\Delta y_{n-\tau+1}^{\gamma} = \left(y_{n-\tau+2}^{\gamma} - y_{n-\tau+1}^{\gamma}\right) \geq \gamma y_{n-\tau+2}^{\gamma-1}\Delta y_{n-\tau+1}. \tag{2.26}$$

Substituting (2.26) into (2.25), we obtain

$$\Delta\omega_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\rho_n \gamma\Delta\left(y_{n-\tau+1}\right)}{y_{n-\tau+2}^{\gamma+1}}. \tag{2.27}$$

From Lemma 2.1, and (2.27), we find

$$\begin{aligned}\Delta\omega_n &\leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} \\ &\quad -\rho_n\gamma\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}}{y_{n-\tau+2}^{\gamma+1}}\frac{\varphi_{n-\tau+1}^{\frac{1}{\gamma}}}{d_{n-\tau+1}}\left(a_n\Delta\left(d_n\Delta y_n\right)^{\gamma}\right)^{\frac{1}{\gamma}}.\end{aligned} \tag{2.28}$$

Since $\Delta a_n \Delta\left(d_n\left(\Delta y_n\right)\right)^{\gamma} \leq 0$, it follows that

$$a_{n+1}\Delta\left(d_{n+1}\left(\Delta y_{n+1}\right)\right)^{\gamma} \leq a_n\Delta\left(d_n\Delta y_n\right)^{\gamma}$$

and thus

$$\left(a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\right)^{\frac{1}{\gamma}} \leq \left(a_n\Delta\left(d_n\Delta y_n\right)^{\gamma}\right)^{\frac{1}{\gamma}}. \tag{2.29}$$

Substituting (2.29) into (2.28) as

$$\begin{aligned}\Delta\omega_n &\leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} \\ &\quad -\rho_n\gamma\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}}{y_{n-\tau+2}^{\gamma+1}}\frac{\varphi_{n-\tau+1}^{\frac{1}{\gamma}}}{d_{n-\tau+1}}\left(a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}\right)^{\frac{1}{\gamma}},\end{aligned}$$

$$\begin{aligned}\Delta\omega_n &\leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} \\ &\quad -\rho_n\gamma\frac{\varphi_{n-\tau+1}^{\frac{1}{\gamma}}}{d_{n-\tau+1}}\left(\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}}{y_{n-\tau+2}^{\gamma}}\right)^{\frac{\gamma+1}{\gamma}}.\end{aligned}$$

This yields that

$$\begin{aligned}\Delta\omega_n &\leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} \\ &\quad -\gamma\frac{\varphi_{n-\tau+1}^{\frac{1}{\gamma}}}{d_{n-\tau+1}}\frac{\rho_n}{\rho_{n+1}^{\frac{\gamma+1}{\gamma}}}\left(\rho_{n+1}\frac{a_{n+1}\Delta\left(d_{n+1}\Delta y_{n+1}\right)^{\gamma}}{y_{n-\tau+2}^{\gamma}}\right)^{\frac{\gamma+1}{\gamma}},\end{aligned}$$

$$\rho_n Q_n \leq -\Delta\omega_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \gamma\frac{\varphi_{n-\tau+1}^{\frac{1}{\gamma}}}{d_{n-\tau+1}}\frac{\rho_n}{\left(\rho_{n+1}\right)^{\frac{\gamma+1}{\gamma}}}\omega_{n+1}^{1+\frac{1}{\gamma}}. \tag{2.30}$$

Multiplying (2.30) by $H_{m,n}$ and summation, we obtain

$$\begin{aligned}\sum_{i=n_1}^{n-1} H_{m,i}\,\psi_i &\leq -\sum_{i=n_1}^{n-1} H_{m,i}\Delta\omega_i + \sum_{i=n_1}^{n-1} H_{m,i}\frac{\Delta\rho_i}{\rho_{i+1}}\omega_{i+1} \\ &\quad -\sum_{i=n_1}^{n-1} H_{m,i}\Phi_i\frac{1}{\left(\rho_{i+1}\right)^{\frac{1+\gamma}{\gamma}}}\omega_{i+1}^{1+\frac{1}{\gamma}},\end{aligned}$$

where $\Phi_n = \gamma\rho_n\varphi_{n-\tau+1}^{\frac{1}{\gamma}}/d_{n-\tau+1}$, which yields after summing by parts on the term $\sum_{i=n_1}^{n-1} H_{m,i}\Delta\omega_i$,

$$\begin{aligned}\sum_{i=n_1}^{n-1} H_{m,i}\psi_i &\leq H_{m,n_1}\omega_{n_1} - \sum_{i=n_1}^{n-1}\Delta_2 H_{m,i}\omega_{i+1} + \sum_{i=n_1}^{n-1} H_{m,i}\frac{\Delta\rho_i}{\rho_{i+1}}\omega_{i+1} \\ &\quad -\sum_{i=n_1}^{n-1}\left(\frac{H_{m,i}\Phi_i}{\left(\rho_{i+1}\right)^{\frac{1+\gamma}{\gamma}}}\right)\omega_{i+1}^{1+\frac{1}{\gamma}}.\end{aligned}$$

$$\sum_{i=n_1}^{n-1} H_{m,i}\psi_i \;\leq\; H_{m,n_1}\omega_{n_1} + \sum_{i=n_1}^{n-1}\left(H_{m,i}\frac{\Delta\rho_i}{\rho_{i+1}} - \Delta_2 H_{m,i}\right)\omega_{i+1} \tag{2.31}$$
$$-\sum_{i=n_1}^{n-1}\left(\frac{H_{m,i}\Phi_i}{(\rho_{i+1})^{\frac{1+\gamma}{\gamma}}}\right)\omega_{i+1}^{1+\frac{1}{\gamma}}.$$

Using the inequality

$$Bu - Au^{1+\frac{1}{\gamma}} \leq \frac{B^B}{(1+B)^{B+1}}\frac{B^{B+1}}{A^B} \tag{2.32}$$

for

$$A = \frac{H_{m,n}\Phi_n}{(\rho_{n+1})^{\frac{1+\gamma}{\gamma}}},\; B = H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} - \Delta_2 H_{m,n},$$

(2.31) become

$$\sum_{i=n_0}^{n-1}(H_{m,i}\psi_i - \phi_{m,n}) \leq H_{m,n_0}\left(\omega_{n_1} + \sum_{i=n_0}^{n_1-1}\psi_i\right) < \infty.$$

Then

$$\lim_{n\to\infty}\sup\frac{1}{H_{m,n_0}}\sum_{i=n_0}^{n-1}(H_{m,i}\psi_i - \phi_{m,n}) < \infty.$$

which contradicts with (2.23). The proof of the case when Case (ii) holds is similar to the proof of Lemma 2.4 [11]. The proof is complete. □

The following result is an immediate consequence of Theorem 2.9.

Corollary 2.10. *Let x_n be a solution of* (1.1) *and assume that all the assumptions of Theorem 2.9 hold, except that* (2.23) *is replaced by*

$$\lim_{n\to\infty}\sup\frac{1}{H_{m,n_0}}\sum_{i=n_0}^{n-1}H_{m,.i}\psi_i = \infty \text{ and } \lim_{n\to\infty}\sup\frac{1}{H_{m,n_0}}\sum_{i=n_0}^{n-1}\phi_{m,n} < \infty.$$

Then x_n oscillates or satisfies (2.8).

The following is an oscillation result for (1.1) of Kamenev's type.

Theorem 2.11. *Let x_n be a solution of* (1.1) *and assume that all the assumptions of Theorem 2.9 hold, except that* (2.23) *is replaced by*

$$\lim_{n\to\infty}\sup\frac{1}{m^\lambda}\sum_{i=n_0}^{n-1}(m-i)^\lambda\left(\psi_i - \frac{\rho_{n+1}^{\gamma+1}\left(\frac{\psi_n}{\rho_{n+1}} - \lambda(m-i)^{-1}\right)^{1+\gamma}}{(1+\gamma)^{1+\gamma}\rho_n^\gamma\delta_{n-\tau}d_{n-\tau+1}^{-\gamma}(m-i)^{\lambda\gamma}}\right) = \infty.$$

Then x_n oscillates or satisfies (2.8).

References

[1] R. P. Agarwal, M. M. S. Manuel and E. Thandapani, Oscillatory and nonoscillatory behavior of second-order neutral delay difference equations, Math. Comput. Modelling 24 (1996), 5–11.

[2] R. P. Agarwal, M. M. S. Manuel, and E. Thandapani, Oscillatory and nonoscillatory behavior of second-order neutral delay difference equations II, Appl. Math. Lett. 10 (1997), 103–109.

[3] A. Bezubik, M. Migda, M. Nockowska and E. Schmeidel, Trichotomy of nonoscillatory solutions to second-order neutral difference equation with quasi-difference, . Advances in Difference Equations (2015).

[4] M. Budericenic, Oscillation of a second-order neutral difference equation, Bull Cl. Sci, Math. Nat. Sci. Math. 22 (1994), 1–8.

[5] S. R. Grace and B. S. Lalli, Oscillation theorem for second order delay and neutral difference equations, Utilitaes Math. 45 (1994), 197–211.

[6] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, 2nd Ed. Cambridge Univ. Press 1952.

[7] H. J. Li and C. C. Yeh, Oscillation criteria for second-order neutral delay difference equations, Comput. Math. Appl. 36 (1998), 123–132.

[8] S. H. Saker, Oscillation and asymptotic behavior of third-order nonlinear neutral difference equations, Dynam. Systems Appl. 15 (2006), 549–568.

[9] S. H. Saker, New oscillation criteria for second-order nonlinear neutral delay difference equations, Appl. Math. Comput. 142 (2003), 99–111.

[10] S. H. Saker, *Oscillation Theory of Delay Differential and Difference Equations Second and Third Orders,* VDM Verlag Dr. Muller Aktiengesellschaft & Co. KG (2010).

[11] S. H. Saker, and S. S. Cheng, Kamenev Type Oscillation Criteria for Nonlinear Difference Equations, Czechoslovak Math. J. 54 (2004) 955–967.

[12] Y. G. Sun and S. H. Saker, Oscillation for second-order nonlinear neutral delay difference equations, Appl. Math. Comput. 163 (2005), 909–918.

[13] A. Sternal and B. Szmanda, Asymptotic and oscillatory behavior of certain difference equations, Le Mat. 51 (1996), 77–86.

[14] E. Thandapani, S. Pandian and K. Balasubramanlan, Oscillatory Behavior of Secaned Order Unstable Type Neutral Diffrence Equations, Tamkang, J. of Math. 36 (2005) 57–68.

[15] E. Thandapani, P. Sundaram, J. R. Graef and P. W. Spikes, Asymptotic properties of solutions of nonlinear second-order neutral delay difference equations, Dymanic Sys. Appl. 4 (1995), 125–136.

[16] E. Thandapani and R. Arul, Oscillation properties of second order nonlinear neutral delay difference equations, Indian J. Pure Appl. Math. 28 (1997), 1567–1571.

[17] E. Thandapani and K. Mahalingam, Necessary and sufficient conditions for oscillation of second-order neutral difference equations, Tamkang J. Math. 34 (2003), 137–145.

[18] E. Thandapani and P. Mohan Kumar, Oscillation and Nonoscillation of nonlinear neutral delay difference equations, Tamkang J. Math. 38 (2007), 323–333.

[19] E. Thandapani, V. Balasubramanian and J. R. Graef, Oscillation criteria for second-order difference equations with neqative neutral term, Inter. J. Pure. Appl. 87 (2013), 283–292.

[20] E. Thandapani, D. Segher and S. Pinelas, Oscillation theorems for second-order difference equations with neqative neutral term, submitted.

[21] E. Thandapani, Z. Liue, R. Arul, P. S. Raja, Oscillation and Asymptotic Behavior of Secaned Order Difference Equation with Nonlinear Neutral Terms, Applied Math, E-Notes, 4 (2004) 59–67.

[22] X. H. Tang and Y. J. Liu, Oscillations for nonlinear delay difference equations, Tamkang. J. Math. 32 (2001), 275–280.

[23] G. Zhang, Oscillation for nonlinear neutral difference equations, Applied Math. E-Notes 2 (2002), 22–24.

[24] B. G. Zhang and S. H. Saker, Kamenev-type oscillation criteria for nonlinear neutral delay difference equations, Indian J. Pure Appl. Math. 34 (2003), 1571–1584.

[25] B. G. Zhang and S. H. Saker, Kamenev-type oscillation criteria for nonlinear neutral delay difference equations, Indian J. Pure Appl. Math. 34 (2003), 1571–1584.