International Journal of Difference Equations
ISSN 0973-6069, Volume 12, Number 1, pp. 119–129 (2017)
http://campus.mst.edu/ijde

On the Solvability of Nonlinear Discrete Sturm–Liouville Problems at Resonance

Daniel Maroncelli
College of Charleston
Department of Mathematics
Charleston, SC 29424
maroncellidm@cofc.edu

Jesús Rodríguez
North Carolina State University
Department of Mathematics
Raleigh, NC 27695
rodrigu@ncsu.edu

Abstract

In this work we provide conditions for the existence of solutions to nonlinear, discrete Sturm–Liouville problems of the form

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \lambda x(t) = f(x(t)); \;\; t \in \{a+1, \cdots, b+1\}$$

subject to

$$a_{11}x(a) + a_{12}\Delta x(a) = 0 \text{ and } a_{21}x(b+1) + a_{22}\Delta x(b+1) = 0.$$

The parameter λ will be assumed to be an eigenvalue of the associated linear Sturm–Liouville boundary value problem. Our results generalize those found in [11, 18].

AMS Subject Classifications: 34B15.
Keywords: Sturm–Liouville, resonance, Lyapunov–Schmidt procedure, topological degree theory.

1 Introduction

In this paper we analyze the existence of real-valued solutions to nonlinear, discrete Sturm–Liouville problems of the form

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \lambda x(t) = f(x(t)); \;\; t \in \{a+1, \cdots, b+1\} \tag{1.1}$$

subject to

$$a_{11}x(a) + a_{12}\Delta x(a) = 0 \text{ and } a_{21}x(b+1) + a_{22}\Delta x(b+1) = 0, \tag{1.2}$$

Received May 5, 2016; Accepted September 5, 2016
Communicated by Eugenia Petropoulou

where we assume p and q are defined for all t in $\{a+1, \cdots, b+1\}$, $p(t) > 0$ for all such t, $a_{11}^2 + a_{12}^2 > 0$ and $a_{21}^2 + a_{22}^2 > 0$, $f : \mathbb{R} \to \mathbb{R}$ is continuous, and the parameter λ is assumed to be an eigenvalue of the associated homogeneous linear Sturm–Liouville boundary value problem

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) = 0; \;\; t \in \{a+1, \cdots, b+1\} \tag{1.3}$$

subject to

$$a_{11}x(a) + a_{12}\Delta x(a) = 0 \text{ and } a_{21}x(b+1) + a_{22}\Delta x(b+1) = 0. \tag{1.4}$$

In [18], the author proves the existence of solutions to (1.1) subject to the boundary conditions (1.2) under the following assumptions:

HR1. f is bounded.

HR2. There exists a positive constant, z_0, such that for all x in $\mathbb{R}$ with $|x| > z_0$, $xf(x) > 0$.

In [11], the author extends the work in [18], by proving the existence of solutions under the following assumptions:

HM1. There exists positive constants M_1, M_2, and β, $0 \leq \beta < 1$, such that for every $x \in \mathbb{R}$, $|f(x)| \leq M_1|x|^\beta + M_2$.

HM2. There exists a positive constant, z_0, such that for all x in $\mathbb{R}$ with $|x| > z_0$, $xf(x) > 0$.

In this work, we generalize the results of [11, 18] to allow for more general nonlinearities, f. Our main result is Theorem 4.1, which proves the existence of solutions to (1.1) subject to the boundary conditions (1.2) under rather mild growth conditions placed on the nonlinearity, f, and a condition similar to (HR2.) or (HM2.). The distinguishing feature of Theorem 4.1 is that our conditions are required only on a bounded subset of $\mathbb{R}$, and as such allow for a much wider class of nonlinearities. Theorem 4.1 gives specific, and easily calculated, descriptions of these bounded intervals.

Our approach is topological. In order to fully exploit the structure of the linear homogeneous Sturm–Liouville problem, (1.3)–(1.4), we use a projection scheme often referred to as the Lyapunov–Schmidt reduction. This reduction, in combination with ideas regarding Brouwer degree, is then used to prove the existence of solutions to (1.1)–(1.2) under the assumptions of Theorem 4.1.

The literature regarding nonlinear discrete eigenvalue problems and more general resonant difference equations is vast. The results of this work complement those found in [7, 11, 13, 16–19, 21]. For those interested readers, similar ideas applied to continuous boundary value problems at resonance can be found in [2–5, 8, 9, 12, 14, 15, 20, 22].

2 Preliminaries

We will analyze the nonlinear Sturm–Liouville boundary value problem, (1.1)–(1.2), as an operator equation. In this section, we give a brief description of the discrete Sturm–Liouville theory that will be needed for our analysis. We then introduce the operators and function spaces that will be used to analyze the existence of solutions to (1.1)–(1.2). For a more detailed introduction to the ideas of discrete Sturm–Liouville problems, we suggest [6, 10].

It is well known that every eigenvalue of the linear Sturm–Liouville problem, (1.3)–(1.4), is real and simple; that is, the eigenspace corresponding to each eigenvalue is one-dimensional. Furthermore, if $h : \{a+1, \cdots, b+1\} \to \mathbb{R}$, then the linear nonhomogeneous boundary value problem

$$\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \lambda x(t) = h(t) \quad t \in \{a+1, \cdots, b+1\} \tag{2.1}$$

subject to

$$a_{11}x(a) + a_{12}\Delta x(a) = 0 \text{ and } a_{21}x(b+1) + a_{22}\Delta x(b+1) = 0. \tag{2.2}$$

is solvable if and only if $\sum_{s=a+1}^{b+1} u(s)h(s) = 0$, where u is any (real) eigenfunction corresponding to the eigenvalue λ. This characterization of the image of the linear "Sturm–Liouville" operator will play an important role in our analysis of the nonlinear problem (1.1)–(1.2).

The underlying function spaces for our operator problem are as follows:

$$X = \{\phi : \{a, \cdots, b+2\} \to \mathbb{R} \mid \phi \text{ satisfies (1.2)}\}$$

and

$$Z = \{\phi : \{a+1, \cdots, b+1\} \to \mathbb{R}\}.$$

The topologies used on X and Z will be that of the supremum norm. We use $\|\cdot\|$ to denote these norms and we will use $|\cdot|$ to denote the absolute value on $\mathbb{R}$.

From this point forward, we will assume that the eigenfunction, u, corresponding to λ has been chosen such that $\sum_{s=a+1}^{b+1} u^2(s) = 1$.

Remark 2.1. We would like to point out that saying $\sum_{s=a+1}^{b+1} u^2(s) = 1$ is not equivalent to saying $\|u\|_2 = 1$, where $\|\cdot\|_2$ represents the standard norm on $\ell^2(X)$, since u takes on, possibly nonzero, values at a and $b+2$.

We now define operators as follows: $\mathcal{L}\colon X \to Z$ by

$$\mathcal{L}x(t) = \Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) + \lambda x(t)$$

and $\mathcal{F}\colon X \to Z$ by

$$\mathcal{F}(x)(t) = f(x(t)).$$

With these definitions, it is now clear that solving the nonlinear boundary value problem (1.1)–(1.2) is equivalent to solving the operator equation $\mathcal{L}x = \mathcal{F}(x)$. We would also like to point out that under this notation, $\mathrm{Ker}(\mathcal{L}) = \mathrm{span}\{u\}$ and $\mathrm{Im}(\mathcal{L}) = \left\{ h : \{a+1, \cdots, b+1\} \to \mathbb{R} \mid \sum_{s=a+1}^{b+1} u(s)h(s) = 0 \right\}$.

3 Alternative Method

With the characterizations of $\mathrm{Ker}(\mathcal{L})$ and $\mathrm{Im}(\mathcal{L})$ generated by the Sturm–Liouville theory of the previous section, we are now in a position to discuss the alternative method we will use to analyze the operator equation $\mathcal{L}x = \mathcal{F}(x)$. In this regard, we choose to follow [18].

Proposition 3.1 (See [18]). *If we define $P : X \to X$ by*

$$(Px)(t) = \begin{cases} x(t) & \text{if } t = a, b+2 \\ u(t) \displaystyle\sum_{s=a+1}^{b+1} x(s)u(s) & \text{if } t \in \{a+1, \cdots, b+1\}, \end{cases}$$

then P is a projection onto the $\mathrm{Ker}(\mathcal{L})$.

Proposition 3.2 (See [18]). *If we define $E : Z \to Z$ by*

$$(Eh)(t) = h(t) - u(t) \sum_{s=a+1}^{b+1} x(s)u(s),$$

then E is a projection with $\mathrm{Im}(E) = \mathrm{Im}(\mathcal{L})$.

Remark 3.3. Note that u in the definition of E is actually $u_{|\{a+1,\cdots,b+1\}}$.

The following is a formulation of the projection scheme, often referred to as the Lyapunov–Schmidt procedure, which we will use to analyze the nonlinear problem, (1.1)–(1.2). Interested readers may consult [1] for further explanation of these ideas. We include the proof for the benefit of the reader.

Proposition 3.4. *Solving $\mathcal{L}x = \mathcal{F}(x)$ is equivalent to solving the system*

$$\begin{cases} (I - E)\mathcal{F}(\alpha u + v) = 0 \\ \quad\quad \textit{and} \\ v - M_p E \mathcal{F}(\alpha u + v) = 0 \end{cases}$$

where M_p is $(\mathcal{L}_{|\,\mathrm{Ker}(P)})^{-1}$, α is a real number, and v is in $\mathrm{Im}(I - P)$.

Proof. We have

$$\mathcal{L}x = \mathcal{F}(x) \Longleftrightarrow \begin{cases} (I-E)(\mathcal{L}x - \mathcal{F}(x)) = 0 \\ \text{and} \\ E(\mathcal{L}x - \mathcal{F}(x)) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (I-E)\mathcal{F}(x) = 0 \\ \text{and} \\ \mathcal{L}x - E\mathcal{F}(x) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (I-E)\mathcal{F}(x) = 0 \\ \text{and} \\ M_p\mathcal{L}x - M_pE\mathcal{F}(x) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (I-E)\mathcal{F}(x) = 0 \\ \text{and} \\ (I-P)x - M_pE\mathcal{F}(x) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} (I-E)\mathcal{F}(\alpha u + v) = 0 \\ \text{and} \\ v - M_pE\mathcal{F}(\alpha u + v) = 0. \end{cases}$$

This completes the proof. □

4 Main Results

We now come to our main existence theorem. We start by introducing some notation that will simplify the statement of our main result and its proof. We define $A_\lambda := \|M_pE\|$ (Operator norm), $A_+ := \{t \in \{a+1, \cdots, b+1\} \mid u(t) > 0\}$, $A_- := \{t \in \{a+1, \cdots, b+1\} \mid u(t) < 0\}$, $u_{\max} := \max_{t\in\{a,\cdots,b+2\}} |u(t)|$, $u_{\min} := \min_{t\in\{a+1,\cdots,b+1\}, u(t)\neq 0} |u(t)|$, $\|f\|_k = \sup_{x\in[-k,k]} |f(x)|$, and $g : \mathbb{R} \times \operatorname{Im}(I-P) \to \operatorname{Im}(I-P)$ by $g(\alpha, v) = M_pE\mathcal{F}(\alpha u + v)$.

Theorem 4.1. *Suppose the following conditions hold.*

C1. There exist positive constants c *and* d*,* $c < d$*, such that* $f(x) > 0$ *for each* x *in* $[c, d]$ *and* $f(x) < 0$ *for each* x *in* $[-d, -c]$*.*

C2. $d > \dfrac{cu_{\max} + A_\lambda \|f\|_d (u_{\max} + u_{\min})}{u_{\min}}$.

Then the nonlinear Sturm–Liouville problem, (1.1)–(1.2)*, has at least one solution.*

Proof. Define $H : \mathbb{R} \times \mathrm{Im}(I-P) \to \mathbb{R} \times \mathrm{Im}(I-P)$ by

$$H(\alpha, x) = \begin{pmatrix} \displaystyle\sum_{s=a+1}^{b+1} u(s) f(\alpha u(s) + g(\alpha, v)(s)) \\ v - g(\alpha, v) \end{pmatrix}. \tag{4.1}$$

From Proposition 3.4, the zeros of H are precisely the solutions of (1.1)–(1.2). We will show the existence of a solution to the nonlinear Sturm–Liouville boundary value problem, (1.1)–(1.2), by showing that the Brouwer degree of H, $\deg(H, \Omega, 0)$, is nonzero for some appropriately chosen set Ω. To this end, endow $\mathbb{R} \times \mathrm{Im}(I-P)$ with the product topology and define

$$\Omega = \{(\alpha, v) \mid |\alpha| \le \alpha^* \text{ and } \|v\| \le r^*\},$$

where $\alpha^* = \dfrac{c + A_\lambda \|f\|_d}{u_{\min}}$ and $r^* = A_\lambda \|f\|_d$. Define $Q : [0,1] \times \overline{\Omega} \to \mathbb{R} \times \mathrm{Im}(I-P)$ by

$$Q(\gamma, (\alpha, v)) = \begin{pmatrix} (1-\gamma)\alpha + \gamma \displaystyle\sum_{s=a+1}^{b+1} u(s) f(\alpha u(s) + g(\alpha, v)(s)) \\ v - \gamma g(\alpha, v) \end{pmatrix}.$$

It is evident that Q is a homotopy between the identity mapping and H. In what follows, we will show that $Q(\gamma, (\alpha, v))$ is nonzero for each $\gamma \in (0,1)$ and every (α, v) in $\partial(\Omega) = \{(\alpha, v) \mid |\alpha| = \alpha^* \text{ and } \|v\| \le r^* \text{ or } |\alpha| \le \alpha^* \text{ and } \|v\| = r^*\}$, so that, by the invariance of the Brouwer degree under homotopy, $\deg(H, \Omega, 0) \ne 0$. In what follows, it will be useful to note that

$$\begin{aligned} \alpha^* u_{\max} + r^* &= u_{\max} \left(\frac{c + A_\lambda \|f\|_d}{u_{\min}} \right) + A_\lambda \|f\|_d \\ &= \frac{c u_{\max} + A_\lambda \|f\|_d u_{\max} + A_\lambda \|f\|_d u_{\min}}{u_{\min}} \\ &< d. \end{aligned} \tag{4.2}$$

We now turn our attention to showing that $Q(\gamma, (\alpha, v)) \ne 0$ for each $\gamma \in (0,1)$ and every $(\alpha, v) \in \partial(\Omega)$. We start by assuming $(\alpha, v) \in \partial(\Omega)$, with $|\alpha| \le \alpha^*$ and $\|v\| = r^*$. Since for every s, $|\alpha u(s) + v(s)| \le \alpha^* u_{\max} + r^*$, we have, using (4.2), that $\alpha u(s) + v(s) \in [-d, d]$. It follows that

$$\begin{aligned} \|g(\alpha, v)\| &= \|M_p E \mathcal{F}(\alpha u + v)\| \\ &\le \|M_p E\| \, \|\mathcal{F}(\alpha u + v)\| \\ &= A_\lambda \max_s |f(\alpha u(s) + v(s))| \\ &\le A_\lambda \|f\|_d \\ &= r^*. \end{aligned}$$

Thus, $\|g(\alpha,v)\| \leq r^* = \|v\|$ and it becomes clear that $Q(\gamma,(\alpha,v)) \neq 0$ for every γ in (0,1), since $v-\gamma g(\alpha,v) \neq 0$. We finish the proof by looking at the case when $(\alpha,v) \in \partial(\Omega)$ with $|\alpha| = \alpha^*$ and $\|v\| \leq r^*$. Using the fact that $\|g(\alpha,v)\| \leq r^*$, the triangle inequality, and (4.2), we conclude that for each s, $|\alpha u(s)+g(\alpha,v)(s)| \leq d$ for each $(\alpha,v) \in \partial(\Omega)$. Further, if $|\alpha| = \alpha^*$, then for all $s \in \{a+1,\cdots,b+1\}$ with $u(s) \neq 0$, we have

$$\begin{aligned}|\alpha u(s)+g(\alpha,v)(s)| &\geq \alpha^* u_{\min} - \|g(\alpha,v)\| \\ &\geq \alpha^* u_{\min} - A_\lambda \|f\|_d \\ &= \left(\frac{c+A_\lambda \|f\|_d}{u_{\min}}\right) u_{\min} - A_\lambda \|f\|_d \\ &= c.\end{aligned}$$

Thus, we have shown that when $(\alpha,v) \in \partial(\Omega)$ with $|\alpha| = \alpha^*$ and $\|v\| \leq r^*$, then for all $s \in \{a+1,\cdots,b+1\}$ with $u(s) \neq 0$, $|\alpha u(s)+g(\alpha,v)(s)| \in [c,d]$. In fact, we have shown, that if $\alpha = \alpha^*$ and $s \in A_+$, then $\alpha u(s)+g(\alpha,v)(s) \in [c,d]$ and if $s \in A_-$, then $\alpha u(s)+g(\alpha,v)(s) \in [-d,-c]$. Similarly, if $\alpha = -\alpha^*$ and $s \in A_+$, then $\alpha u(s) + g(\alpha,v)(s) \in [-d,-c]$ and if $s \in A_-$, then $\alpha u(s)+g(\alpha,v)(s) \in [c,d]$. Recalling that f is positive on the interval $[c,d]$ and negative on the interval $[-d,-c]$, it follows, that when $\alpha = \alpha^*$, $u(s)f(\alpha u(s)+g(\alpha,v)(s)) > 0$ for every s in which $u(s) \neq 0$. Arguing in the same fashion, we get that when $\alpha = -\alpha^*$, then $u(s)f(\alpha u(s)+g(\alpha,v)(s)) < 0$ for every s in which $u(s) \neq 0$. Since $(1-\gamma)\alpha + \gamma \sum_{s=a+1}^{b+1} u(s)f(\alpha u(s)+g(\alpha,v)(s))$ would be 0 for some γ in (0,1) if and only if α and $\sum_{s=a+1}^{b+1} u(s)f(\alpha u(s)+g(\alpha,v)(s))$ have opposite sign, we conclude, after noting that $\sum_{s=a+1}^{b+1} u(s)f(\alpha u(s)+g(\alpha,v)(s)) = \sum_{s,u(s)\neq 0} u(s)f(\alpha u(s)+g(\alpha,v)(s))$, that $Q(\gamma,(\alpha,v)) \neq 0$ for each $(\alpha,v) \in \partial(\Omega)$ with $|\alpha| = \alpha^*$ and $\|v\| \leq r^*$, since its first component is nonzero. We now conclude, by the homotopy invariance of the Brouwer degree, that

$$\deg(H,\Omega,0) = \deg(I,\Omega,0) = 1.$$

The result now follows. □

The following corollary is a concrete application of Theorem 4.1 to the cases of sublinear and linear growth. Note that for the case in which $\beta = 0$, easily verifiable conditions are given by direct application of Theorem 4.1.

Corollary 4.2. *Suppose the following conditions hold.*

C1. There exist positive constants* c *and* d*,* $c < d$*, such that* $f(x) > 0$ *for each* x *in* $[c, d]$ *and* $f(x) < 0$ *for each* x *in* $[-d, -c]$*.*

C2. There exist positive constants* M_1 *and* M_2 *such that* $|f(x)| \le M_1|x|^\beta + M_2$ *for every* x *in* $[-d, d]$*, where* $0 < \beta \le 1$*.*

C3.* $d > \dfrac{cu_{\max} + (K_1(1-\beta) + K_2)(u_{\max} + u_{\min})}{u_{\min} - K_1\beta(u_{\max} + u_{\min})}$*, where*

$$K_1 = A_\lambda M_1, \quad K_2 = A_\lambda M_2,$$

and we are assuming $u_{\min} - K_1\beta(u_{\max} + u_{\min}) > 0$*; i.e.,* $K_1 < \dfrac{u_{\min}}{\beta(u_{\min} + u_{\max})}$*.*

Then the nonlinear Sturm–Liouville problem, (1.1)–(1.2)*, has at least one solution.*

Proof. From (C2*), we get $\|f\|_d \le M_1 d^\beta + M_2$. Thus,

$$\begin{aligned}\frac{cu_{\max} + A_\lambda \|f\|_d (u_{\max} + u_{\min})}{u_{\min}} &\le \frac{cu_{\max} + A_\lambda(M_1 d^\beta + M_2)(u_{\max} + u_{\min})}{u_{\min}} \\ &= \frac{cu_{\max} + (K_1 d^\beta + K_2)(u_{\max} + u_{\min})}{u_{\min}}.\end{aligned}$$

Using (C3*), we have $\dfrac{cu_{\max} + (K_1(1-\beta) + K_2)(u_{\max} + u_{\min})}{u_{\min} - K_1\beta(u_{\max} + u_{\min})} < d$, from which we conclude

$$\begin{aligned}cu_{\max} + K_2(u_{\max} + u_{\min}) &< du_{\min} - dK_1\beta(u_{\max} + u_{\min}) - K_1(1-\beta)(u_{\max} + u_{\min}) \\ &= du_{\min} - K_1(1 + \beta(d-1))(u_{\max} + u_{\min}) \\ &\le du_{\min} - K_1(1 + (d-1))^\beta(u_{\max} + u_{\min}) \\ &= du_{\min} - K_1 d^\beta(u_{\max} + u_{\min}).\end{aligned}$$

Rearranging, it follows that

$$\frac{cu_{\max} + A_\lambda \|f\|_d (u_{\max} + u_{\min})}{u_{\min}} \le \frac{cu_{\max} + (K_1 d^\beta + K_2)(u_{\max} + u_{\min})}{u_{\min}} < d.$$

This completes the proof. □

Remark 4.3. We would like to point out that if f is sublinear on all of $\mathbb{R}$; that is, there exist positive numbers M_1, M_2 and a constant β, $0 \le \beta < 1$, such that $|f(x)| \le M_1|x|^\beta + M_2$ for every $x \in \mathbb{R}$, and there is a $z_0 > 0$ such that if $x \in \mathbb{R}$, with $|x| > z_0$, $xf(x) > 0$, then hypotheses of Theorem 4.1 hold. This can easily be seen, since $\lim_{r\to\infty} \dfrac{\frac{cu_{\max} + A_\lambda \|f\|_r (u_{\max} + u_{\min})}{u_{\min}}}{r} = 0 < 1$. In fact, if f has "small" linear growth; that is , $|f(x)| \le M_1|x| + M_2$ for all x with

$$A_\lambda M_1 \left(\frac{u_{\max} + u_{\min}}{u_{\min}}\right) < 1,$$

and there is a $z_0 > 0$ such that if $x \in \mathbb{R}$, with $|x| > z_0$, $xf(x) > 0$, then the hypotheses of Theorem 4.1 again hold, since in this case $\lim_{r\to\infty} \frac{\frac{cu_{\max} + A_\lambda \|f\|_r (u_{\max}+u_{\min})}{u_{\min}}}{r} \leq A_\lambda M_1 \left(\frac{u_{\max}+u_{\min}}{u_{\min}}\right) < 1$. This shows that the results of [11, 18] are in fact a very special case of Theorem 4.1 in which we may view d as ∞.

5 Example

In this section we give a concrete example of the application of Theorem 4.1. Consider

$$\Delta(\Delta x(t-1)) + \lambda_k x(t) = f(x(t));\;\; t \in \{1, \cdots, N-1\}$$

subject to

$$x(0) = 0 \text{ and } x(N) = 0,$$

where for a given k, $k = 1, \cdots, N-1$, λ_k is an eigenvalue of the associated linear Sturm–Liouville boundary value problem, $f : \mathbb{R} \to \mathbb{R}$ is continuous, $|f|$ is bounded by 1 on $[-1, 1]$,

$$f(x) = \begin{cases} \sin(x^{1/m}) & \text{if } 1 \leq x \leq 2^m\pi^m \\ -\sin((-x)^{1/m}) & \text{if } -2^m\pi^m \leq x \leq -1, \end{cases}$$

and m is a positive integer satisfying

$$\frac{\pi^m + A_{\lambda_k}(1 + \sin(\frac{\gcd(k,N)\pi}{N}))}{\sin(\frac{\gcd(k,N)\pi}{N})} < 2^m\pi^m, \tag{5.1}$$

where $\gcd$ denotes greatest common divisor.

It is well known, see [10], that the eigenpairs of the associated linear homogeneous Sturm–Liouville problem are given by $\left(\lambda_k, \sin\left(\frac{k\pi(\cdot)}{N}\right)\right)$, $k = 1, \cdots, N-1$, where $\lambda_k = 2 + 2\cos\left(\frac{k\pi}{N}\right)$.

For $k = 1, \cdots, N-1$, as in the notation of Theorem 4.1, let

$$u_{k_{\max}} = \max_{t\in\{0,\cdots,N\}} \left|\sin\left(\frac{k\pi t}{N}\right)\right|, \quad u_{k_{\min}} = \min_{t\in\{1,\cdots,N-1\}, \sin\left(\frac{k\pi t}{N}\right)\neq 0} \left|\sin\left(\frac{k\pi t}{N}\right)\right|.$$

In this case, we clearly have $u_{k_{\max}} \leq 1$. Further, it can be shown that

$$\sin\left(\frac{\gcd(k,N)\pi}{N}\right) = u_{k_{\min}} \quad \text{for every} \quad k.$$

Now, $f(x) \geq 0$ on $[\pi^m, 2^m\pi^m]$ and $f(x) \leq 0$ on $[-2^m\pi^m, -\pi^m]$, with the only zeros in these intervals occurring at the endpoints. Thus, using (5.1), there exists c, d, with $\pi^m < c < d < 2^m\pi^m$ such that $xf(x) > 0$ when $|x| \in [c, d]$, and

$$\begin{aligned}\frac{cu_{k_{\max}} + A_{\lambda_k} \|f\|_d (u_{k_{\max}} + u_{k_{\min}})}{u_{k_{\min}}} &\leq \frac{c + A_{\lambda_k}(1 + \sin(\frac{\gcd(k,N)\pi}{N}))}{\sin(\frac{\gcd(k,N)\pi}{N})} \\ &< d.\end{aligned}$$

It follows from Theorem 4.1 that the nonlinear Sturm–Liouville boundary value problem has at least one solution. Note that if $2^m\pi^m > \dfrac{\pi^m + 2\max_k A_{\lambda_k}}{\sin(\frac{\pi}{N})}$, then the nonlinear Sturm–Liouville problem has a solution for all eigenvalues, λ_k.

It is important to note that the behavior of $f(x)$ for $|x| > 2^m\pi^m$ is absolutely inconsequential in this example. Thus, on $\mathbb{R} \setminus [-2^m\pi^m, 2^m\pi^m]$, f may have arbitrary growth and there is no restriction on the number of zeros which occur in this region.

References

[1] S. Chow and J. K. Hale. *Methods of Bifurcation Theory*. Springer, Berlin, 1982.

[2] P. Drábek. On the resonance problem with nonlinearity which has arbitrary linear growth. *J. Math. Anal. Appl.*, 127:435–442, 1987.

[3] P. Drábek. Landesman-Lazer type condition and nonlinearities with linear growth. *Czechoslovak Math. J.*, 40:70–86, 1990.

[4] P. Drábek and S. B. Robinson. Resonance problems for the p-laplacian. *J. Funct. Anal.*, 169:189–200, 1999.

[5] Z. Dua, X. Lin, and W. Ge. Some higher-order multi-point boundary value problems at resonance. *J. Comput. Appl. Math.*, 177:55–65, 2005.

[6] S. Elaydi. *An introduction to difference equations*. Springer, 2005.

[7] D. L. Etheridge and J. Rodríguez. Periodic solutions of nonlinear discrete-time systems. *Appl. Anal.*, 62:119–137, 1996.

[8] R. Iannacci and M.N. Nkashama. Nonlinear two point boundary value problems without Landesman-Lazer condition. *Proc. Amer. Math. Soc.*, 106:943–952, 1989.

[9] W. Jiang, B. Wang, and Z. Wang. Solvability of a second-order multi-point boundary-value problems at resonance on a half-line with $dim\, ker\, L = 2$. *Electron. J. Differential Equations*, 2011(120):1–11, 2011.

[10] W. G. Kelley and A.C. Peterson. *Difference Equations*. Academic Press, New York, 1978.

[11] R. Ma. Nonlinear discrete Sturm-Liouville problems at resonance. *Nonlinear Anal.*, 67:3050–3057, 2007.

[12] D. Maroncelli. Scalar multi-point boundary value problems at resonance. *Differ. Equ. Appl.*, 7(4):449–468, 2015.

[13] D. Maroncelli and J. Rodríguez. On the solvability of multipoint boundary value problems for discrete systems at resonance. *J. Difference Equ. Appl.*, 20:24–35, 2013.

[14] D. Maroncelli and J. Rodríguez. Existence theory for nonlinear Sturm-Liouville problems with unbounded nonlinearities. *Differ. Eqn. Appl.*, 6(4):455–466, 2014.

[15] D. Maroncelli and J. Rodríguez. On the solvability of nonlinear impulsive boundary value problems. *Topol. Methods Nonlinear Anal.*, 44(2):381–398, 2014.

[16] D. Maroncelli and J. Rodríguez. Periodic behaviour of nonlinear, second-order, discrete dynamical systems. *J. Difference Equ. Appl.*, 22(2):280–294, 2015.

[17] D. Pollack and P.Taylor. Multipoint boundary value problems for discrete nonlinear systems at resonance. *Int. J. Pure Appl. Math.*, 63:311–326, 2010.

[18] J. Rodríguez. Nonlinear discrete Sturm-Liouville problems. *J. Math. Anal. Appl.*, 308, Issue 1:380–391, 2005.

[19] J. Rodríguez and Z. Abernathy. Nonlinear discrete Sturm-Liouville problems with global boundary conditions. *J. Difference Equ. Appl.*, 18:431–445, 2012.

[20] J. Rodríguez and Z. Abernathy. On the solvability of nonlinear Sturm-Liouville problems. *J. Math. Anal. Appl.*, 387:310–319, 2012.

[21] J. Rodríguez and D. L. Etheridge. Periodic solutions of nonlinear second-order difference equations. *Adv. Difference Eqn.*, 2005(2):173–192, 2005.

[22] J. F. Rodríguez. Existence theory for nonlinear eigenvalue problems. *Appl. Anal.*, 87(3):293–301, 2008.