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The Systems Complexity Problem for Nonlinear Polynomial Discrete Systems with Many Delays and two Components. An Algebraic Approach.

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Abstract

By the means of special operators and operations, the so-called D-operators and the star-product, a special algebraic description for nonlinear polynomial discrete systems in two dimensions is developed. By using this description, we can check if these nonlinear systems are “similar”or “equivalent” with linear systems, in the sense that the evolution of both systems, under the same initial conditions, are related among each other. Different kind of solutions of the problem, seem to determine different degrees of complexity for the original nonlinear systems. The whole approach has algebraic and algorithmic nature and no analytical tools are used.

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1 Introduction

Difference equations or discrete systems, which are in use for the creation of models in a variety of domains are in principle nonlinear. On the other hand, most of the existing results refer upon linear systems and various linearization processes are in practice, not always successfully. The reason is that the initial systems possess complexities and due to this fact, basic characteristics do not inherit into linearization. It is therefore a necessity to rethink about linearization processes, their tools and degrees of acceptance for the obtained results.

Mathematical control theory provides a unifying framework for posing and studying such problems [5]. In this respect, we treat in this manuscript with systems known as nonlinear discrete systems of polynomial type with two components, which have the general form:

$$\left.\begin{array}{l} x(n)=\sum\sum a_{\mathbf{ij}}x(n-i_1)\cdots x(n-i_\tau)y(n-j_1)\cdots y(n-j_\xi) \\ y(n)=\sum\sum b_{\mathbf{i'j'}}x(n-i'_1)\cdots x(n-i'_{\tau'})y(n-j'_1)\cdots y(n-j'_{\xi'}) \end{array}\right\}. \tag{1.1}$$

The coefficients $a_{\mathbf{i}j}, b_{\mathbf{i'j'}}$ are real numbers and the quantities $x(n), y(n)$ real sequences. These systems, transform a pair of sequences to another pair of sequences in a nonlinear polynomial way, involving a certain number of products among different delays of those sequences. We usually refer to these products as "cross products". These systems appear in many branches of dynamical systems theory, see for instance [2, 14], as well as, in control theory and signal process Analysis, where one of the two sequences is considered as the input. See [4].

The aim of this paper is to deal with the above nonlinearities by using mainly algebraic tools based upon the so called **star product**. The star product corresponds to the composition of polynomial functions, in other words to the substitution of one polynomial into another. This star product allows to describe the evolution of the system along naturally defined operations, the D-operator. This operation is compatible with the cascade connection of one system with another. The problems of evolution and stability of those systems have been studied in a series of papers. See [10–12]. Particularly, in these works, we realized (1.1) as $\vec{x}(n) = \mathbf{A}\vec{x}(n-1)$, where $\vec{x}(n) = [x(n), y(n)]^T$ the state vector and $\mathbf{A}$ a proper D-operator. Then we proved that the dynamic evolution of a such system is given by $\vec{x}(k) = \mathbf{A^k}\vec{x}(n-k)$, where the power $\mathbf{A^k}$ has been calculated with respect to the star product. Both results are analogous to those we have from the classical theory of linear systems. Based on this framework stability theorems were also established.

In the present paper, inspired by similar problems in control theory, see for instance [5, 13], we set down the problem of equivalence of two such systems, and we look for conditions in order to transform one system into a (sometimes given) equivalent one, with the same future evolution. In the linear case the problem is well understood and completely solved. Indeed, if $\vec{x}(n) = \mathbf{A}\vec{x}(n-1)$ is a linear system, $\mathbf{A}$ the coefficient matrix, $\vec{x}(n)$ the state vector, then it is known that we can transform this system to another linear one $\vec{z}(n) = L\vec{z}(n-1)$, by using the transformation $\vec{x}(n) = T\vec{z}(n)$ where T is a matrix satisfying the relation: $AT = TL$. To extend this idea in our case we use the aforementioned D-operators description and the star product. By their means, we reduce the equivalence problem to the solution of the equation $\mathbf{F}{*}T = T{*}\mathbf{G}$ of D-operators with respect to $\mathbf{T}$, when $\mathbf{F}$ and $\mathbf{G}$ are given. For a specific system $\mathbf{F}$ and when the given system $\mathbf{G}$ is a linear one, the equivalence problem is of prime interest. To face it, a recursive procedure is introduced, constructing first the linear part of $\mathbf{T}$, then the quadratic, the cubic and so on. To achieve that, we solve the systems of equations arising, by the application of the star product. The whole approach is algebraic in nature [1, 3] and certain theorems provide us with criteria about the structure of $\mathbf{T}$, if it is a polynomial series (an infinite sum of polynomials) or a series of series (an infinite sum of series).

It turns out that in this case a notion of complexity could be introduced, which realizes the intrinsic nonlinearity of the system. See [6]. As we said, the solution $\mathbf{T}$ may be a polynomial operator, a series of operators, a series of series, to be invertible or not and to converge or not. Each one of these situations determines a type of nonlinearity complexity for the underlying model. A corresponding classification of different degrees of complexity is presented. This is a little bit arbitrary and reflects the authors first attempt to categorize the different issues of the model complexity problem.

To summarize, the advantages of this work are:

1. It is algebraic in nature and no analytical tools are used.
2. The operations can be carried out by using suitable software.
3. It can provide us with a computational tool proper to make nonlinear systems equivalent to linear ones.
4. It can be used as a "measure" of the system complexity.

Here are the contents of this work. In the beginning, we give the preliminary notion of a D-operator and develop the algebraic tools which allow the transformation of the given equation in an algebraic like object. After that, we deal with the main object of study, the Nonlinear Discrete Polynomial Systems. Initially, we define an equivalence relation among D-operations, which turns out to be the appropriate one to characterize the evolution of the underlying systems. This relation is used to define the notion the $\mathbf{T}$-similarity, between two pairs of sequences and reduce this algebraically to a corresponding D-operators. The determination of the operator $\mathbf{T}$ in the equation $\mathbf{F} * \mathbf{T} = \mathbf{T} * \mathbf{G}$, requires a lot of machinery in order to solve the occurring linear like systems. This is achieved in an algorithmic manner. A main theorem ensures that under middle restriction, for a given nonlinear discrete polynomial system the linear T-similarity problem accepts a series solution. Along same considerations, a table for the levels of model complexity is established. All the above situations are illustrated through some indicative arithmetic examples, which conclude this presentation.

2 The Algebraic Framework

In this section, we shall describe the algebraic tools, which shall be used later for the description of nonlinear polynomial discrete systems with two components. The cornerstone of our approach is the so called D-operator. It has been introduced in [9, 11], and transforms a pair of sequences to a pair of sequences. For the sake of completeness of the manuscript, we shall present the basic definitions here, adding some new results. We shall follow a constructive method, starting from simpler operators and proceeding step by step, but let us first recall some basic notions from the indices.

2.1 The Indices

Let

$$\bigcup_{n=1}^{\infty} (\mathbb{N})^n,$$

be the set of all the n-tuples with positive integers as elements. Any subset of it consisting from n-tuples of finite length, is called a set of multiindices. We denote it by $\mathbf{I}$. In other words,

$$\mathbf{I} = \{\mathbf{i} = (i_1, i_2, \ldots, i_n), \ \ n \in \mathbb{N}, \ \ i_k \in \mathbb{N}\}.$$

Usually, the elements of a multiindex $\mathbf{i}$ are ordered in an increasing way, that is,

$$i_1 \leq i_2 \leq \cdots \leq i_n.$$

By $d(\mathbf{i})$, we denote the minimum element of $\mathbf{i}$, that is i_1 and by $\dim(\mathbf{i})$ the number n, called the dimension or the length of $\mathbf{i}$. By $e = ()$, we denote the empty index, that is, the index with no elements. Apparently, $\dim(e) = 0$.

Let $\mathbf{I}$ be a set of multi-indices. We equip it with the following operations:

- **Vector Addition**. Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ and $\mathbf{j} = (j_1, j_2, \ldots, j_n)$ be two multiindices, with $\dim(\mathbf{i}) = \dim(\mathbf{j}) = n$. Their vector addition is a new multiindex, denoted by $\mathbf{i} + \mathbf{j}$ and defined as

$$\mathbf{i} + \mathbf{j} = (i_1 + j_1, i_2 + j_2, \ldots, i_n + j_n).$$

- **Scalar Multiplication**. Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be a multiindex and ρ an integer. The multiplication of $\mathbf{i}$ by ρ is a new multiindex, defined as

$$\rho \cdot \mathbf{i} = (\rho \cdot i_1, \rho \cdot i_2, \ldots, \rho \cdot i_n).$$

- **Cross Addition**. Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ and $\mathbf{j} = (j_1, j_2, \ldots, j_m)$ be two multiindices. Their cross addition is a new multiindex, denoted by $\mathbf{i} \oplus \mathbf{j}$. This is done, by putting together all the elements of $\mathbf{i}$ and $\mathbf{j}$ in an increasing order. Explicitly:

$$\mathbf{i} \oplus \mathbf{j} = (i_1, j_1, i_2, i_3, j_2, \ldots, i_n, j_m),$$

where

$$i_1 \le j_1 \le i_2 \le i_3 \le j_2 \le \cdots \le i_n \le j_m.$$

- **Pointwise Addition**. Let $\mathbf{j} = (j_1, j_2, \dots, j_m)$ be a multiindex and i an integer. The pointwise sum is a new multiindex, denoted by $\mathbf{j}\dot{+}i$, and is defined as

$$\mathbf{j}\dot{+}i = (j_1 + i, j_2 + i, \dots, j_m + i).$$

- **Star product**. Let $\mathbf{i} = (i_1, i_2, \dots, i_n)$ and $\mathbf{j} = (j_1, j_2, \dots, j_n)$ be two multiindices. The star product, is a new multiindex, denoted by $\mathbf{i} * \mathbf{j}$, and is defined as

$$\mathbf{i} * \mathbf{j} = (\mathbf{j}\dot{+}i_1) \oplus (\mathbf{j}\dot{+}i_2) \oplus \cdots \oplus (\mathbf{j}\dot{+}i_n).$$

- **The Multistar Product**. We introduce the multiindex $\mathbf{k}$=$(k_1, k_2, \dots, k_\theta)$, and the set of multiindices $\mathbf{I} = \{\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_\theta\}$. Each of these multiindices has different dimension. The multistar product is a new multiindex, denoted by $\mathbf{k} * \mathbf{I}$, and is defined as

$$\mathbf{k} * \mathbf{I} = \mathbf{k} * \{\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_\theta\} = (\mathbf{i}_1\dot{+}k_1) \oplus (\mathbf{i}_2\dot{+}k_2) \oplus \cdots \oplus (\mathbf{i}_\theta\dot{+}k_\theta).$$

- **The General Star product**. Let $\mathbf{K} = \{\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_f\}$, be the set of multiindices and $\mathcal{J}$ be a collection of multiindices, that is, $\mathcal{J} = \{\mathbf{I}_1, \mathbf{I}_2, \dots \mathbf{I}_f\}$, where $\mathbf{I}_t = (\mathbf{i}_{t,1}, \mathbf{i}_{t,2}, \dots, \mathbf{i}_{t,s_t})$, $t = 1, 2, \dots, f$. The general star product is a new multiindex, denoted by $\mathbf{K} * \mathcal{J}$, and is defined as

$$\mathbf{K} * \mathcal{J} = (\mathbf{k}_1 * \mathbf{I}_1) \oplus (\mathbf{k}_2 * \mathbf{I}_2) \oplus \cdots \oplus (\mathbf{k}_f * \mathbf{I}_f).$$

- **Order.** Let $\mathbf{I}$ be a set of multiindices. It may be ordered in a lexicographical way as follows: The multi-index $\mathbf{i} = (i_1, i_2, \dots, i_n)$ is "less" than the multi-index $\mathbf{j} = (j_1, j_2, \dots, j_m)$, and we write

$$\mathbf{i} \prec \mathbf{j},$$

if

$$n < m \text{ or } n = m,$$

and the right-most nonzero entry of the vector $\mathbf{j} - \mathbf{i}$, is positive.

Example 2.1. To clarify how the above operations work in practise, let us take the indices

$$\mathbf{i} = (-1, 0, 2) \text{ and } \mathbf{j} = (-1, 1, 3).$$

Then

$$\mathbf{i} + \mathbf{j} = (-2, 1, 5), \quad 5\mathbf{i} = (-5, 0, 10), \quad \mathbf{i} \oplus \mathbf{j} = (-1, -1, 0, 1, 2, 3), \quad \mathbf{j}\dot{+}2 = (1, 3, 5),$$

and

$$\mathbf{i} * \mathbf{j} = (\mathbf{j} \dot{+} (-1)) \oplus (\mathbf{j} \dot{+} 0) \oplus (\mathbf{j} \dot{+} 2)$$
$$= (-2, 0, 2) \oplus (-1, 1, 3) \oplus (1, 3, 5) = (-2, -1, 0, 1, 1, 2, 3, 3, 5).$$

Furthermore, if

$$\mathbf{I} = \{(2, 3), (1, 1, 2)\} \text{ and } k = (0, 2, 3),$$

then

$$k * \mathbf{I} = (k * (2, 3)) \oplus (k * (1, 1, 2)).$$

However,

$$k * (2, 3) = (2, 4, 5) \oplus (3, 5, 6) = (2, 3, 4, 5, 5, 6)$$

and

$$k * (1, 1, 2) = (1, 3, 4) \oplus (1, 3, 4) \oplus (2, 4, 5) = (1, 1, 2, 3, 3, 4, 4, 5).$$

Thus,

$$k * \mathbf{I} = (1, 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6).$$

Finally, if

$$\mathbf{K} = \{(0), (0, 3)\} \text{ and } \mathcal{J} = \{\mathbf{I_1}, \mathbf{I_2}\},$$

where

$$\mathbf{I_1} = \{(1, 1), (-1, 1, 2)\} \text{ and } \mathbf{I_2} = \{(0), (-1, -1), (0, 1, 1)\},$$

then

$$\begin{aligned} \mathbf{K} * \mathcal{J} &= ((0) * \mathbf{I_1} \oplus ((0, 3) * \mathbf{I}_2)) \\ &= [(1, 1) \dot{+} 0] \oplus [(-1, 1, 2) \dot{+} 0] \oplus [(-1, -1) \dot{+} 0] \oplus [(0, 1, 1) \dot{+} 3] \\ &= (-1, -1, -1, 1, 1, 1, 2, 3, 4, 4). \end{aligned}$$

Proposition 2.2. *The next relations are valid.*

(A)

$$e \oplus \mathbf{j} = e, \quad e \dot{+} i = e, \quad \mathbf{i} * e = e.$$

(B)

$$\mathbf{i} \oplus \mathbf{j} = \mathbf{j} \oplus \mathbf{i}.$$

(C)

$$(\mathbf{j} \oplus \mathbf{k}) \dot{+} r = (\mathbf{j} \dot{+} r) \oplus (\mathbf{k} \dot{+} r).$$

(D)

$$\mathbf{i} * \mathbf{j} = \mathbf{j} * \mathbf{i}.$$

(E)

$$\mathbf{i} * (\mathbf{j} * \mathbf{h}) = (\mathbf{i} * \mathbf{j}) * \mathbf{h}.$$

(G)

$$\mathbf{i} * (\mathbf{j} \oplus \mathbf{k}) = (\mathbf{i} * \mathbf{j}) \oplus (\mathbf{i} * \mathbf{k}).$$

(H)

$$\mathbf{K} * \mathcal{J} = (\mathbf{i}_{1,1} \dot{+} k_{11}) \oplus (\mathbf{i}_{1,2} \dot{+} k_{12}) \oplus \cdots \oplus (\mathbf{i}_{1,s_1} \dot{+} k_{1s_1}) \oplus (\mathbf{i}_{2,1} \dot{+} k_{21})$$
$$\oplus (\mathbf{i}_{2,2} \dot{+} k_{22}) \oplus \cdots \oplus (\mathbf{i}_{2,s_2} \dot{+} k_{2s_2}) \oplus \cdots \oplus (\mathbf{i}_{t,1} \dot{+} k_{t1}) \oplus (\mathbf{i}_{t,2} \dot{+} k_{t2})$$
$$\oplus \cdots \oplus (\mathbf{i}_{t,s_t} \dot{+} k_{ts_t}) \oplus (\mathbf{i}_{f,1} \dot{+} k_{f1}) \oplus (\mathbf{i}_{f,2} \dot{+} k_{f2}) \oplus \cdots \oplus (\mathbf{i}_{f,s_f} \dot{+} k_{fs_f}).$$

Proof. Relations in (A) are a straightforward result of the emptiness of the index e. Since, $\mathbf{i} \oplus \mathbf{j}$ is an index consisting from all the elements of $\mathbf{i}$ and $\mathbf{j}$, in an increasing order, it is uniquely determined. Whence, we immediately conclude that $\mathbf{i} \oplus \mathbf{j} = \mathbf{j} \oplus \mathbf{i}$. Relation (B) has been proved. For the relation (C), in view of the assumption,

$$j_1 \leq k_1 \leq \cdots \leq k_n \leq j_n,$$

we have

$$(\mathbf{j} \oplus \mathbf{k}) \dot{+} r = (j_1 + r, k_1 + r, \ldots, k_n + r, j_n + r).$$

Since, $j_a \leq k_b$ implies that $j_a + r \leq k_b + r$, we easily conclude that

$$(j_1 + r, k_1 + r, \ldots, k_n + r, j_n + r) = (\mathbf{j} \dot{+} r) \oplus (\mathbf{k} \dot{+} r),$$

which proves (C). For proving relation (D), let us consider two indices

$$\mathbf{i} = (i_1, i_2, \ldots, i_n) \text{ and } \mathbf{j} = (j_1, j_2, \ldots, j_m).$$

Successively, we have

$$\mathbf{i} * \mathbf{j} = (\mathbf{j} \dot{+} i_1) \oplus (\mathbf{j} \dot{+} i_2) \oplus \cdots \oplus (\mathbf{j} \dot{+} i_n) = (j_{\alpha_\phi} + i_{\beta_\phi})_{\phi=1,\ldots,nm},$$

where

$$j_{\alpha_1} + i_{\beta_1} \leq j_{\alpha_2} + i_{\beta_2} \leq \cdots \leq j_{\alpha_\phi} + i_{\beta_\phi}.$$

Similarly,

$$\mathbf{j} * \mathbf{i} = (\mathbf{i} \dot{+} j_1) \oplus (\mathbf{i} \dot{+} j_2) \oplus \cdots \oplus (\mathbf{i} \dot{+} j_n) = (i_{\beta_\phi} + j_{\alpha_\phi})_{\phi=1,\ldots,nm},$$

where

$$i_{\beta_1} + j_{\alpha_1} \leq i_{\beta_2} + j_{\alpha_2} \leq \cdots \leq i_{\beta_\phi} + j_{\alpha_\phi}.$$

In view of the commutative property of the common addition, the proof of (D) follows. For the proof of (G), we have successively

$$\mathbf{i} * (\mathbf{j} \oplus \mathbf{k}) = [(\mathbf{j} \oplus \mathbf{k}) \dot{+} i_1] \oplus [(\mathbf{j} \oplus \mathbf{k}) \dot{+} i_2] \oplus \cdots \oplus [(\mathbf{j} \oplus \mathbf{k}) \dot{+} i_n],$$

which, in view of (C), becomes

$$(\mathbf{j}\dotplus i_1) \oplus (\mathbf{k}\dotplus i_1) \oplus (\mathbf{j}\dotplus i_2) \oplus (\mathbf{k}\dotplus i_2) \oplus \cdots \oplus (\mathbf{j}\dotplus i_n) \oplus (\mathbf{k} + i_n).$$

Using property (B), we may rewrite the last expression as

$$(\mathbf{j}\dotplus i_1) \oplus (\mathbf{j}\dotplus i_2) \oplus \cdots \oplus (\mathbf{j}\dotplus i_n) \oplus (\mathbf{k}\dotplus i_1) \oplus (\mathbf{k}\dotplus i_2) \oplus \cdots \oplus (\mathbf{k}\dotplus i_n),$$

which is equal with,

$$(\mathbf{i} * \mathbf{j}) \oplus (\mathbf{i} * \mathbf{k}).$$

This proves (G). Next, we prove (E). Suppose that a third index $\mathbf{h} = (h_1, h_2, \ldots, h_\nu)$ is available. Using (G), we get

$$(\mathbf{i} * \mathbf{j}) * \mathbf{h} = (h_{\rho_\phi} + j_{\alpha_\phi} + i_{\beta_\phi}), \quad \phi = 1, 2, \ldots, nm\nu,$$

where

$$h_{\rho_1} + j_{\alpha_1} + i_{\beta_1} \leq h_{\rho_2} + j_{\alpha_2} + i_{\beta_2} \leq \cdots \leq h_{\rho_\phi} + j_{\alpha_\phi} + i_{\beta_\phi}.$$

Analogously, we have

$$(\mathbf{j} * \mathbf{h}) = (h_{\rho_\phi} + j_{\alpha_\phi}), \quad \phi = 1, \ldots, m\nu.$$

Thus,

$$\mathbf{i} * (\mathbf{j} * \mathbf{h}) = (h_{\rho_\phi} + j_{\alpha_\phi} + i_{\beta_\phi}), \quad \phi = 1, 2, \ldots, nm\nu,$$

from which the proof of (E) follows. Finally, we establish (H). We have

$$\begin{aligned}
\mathbf{K} * \mathcal{J} &= (\mathbf{k}_1 * \mathbf{I}_1) \oplus (\mathbf{k}_2 * \mathbf{I}_2) \oplus \cdots \oplus (\mathbf{k}_f * \mathbf{I}_f) \\
&= (\mathbf{k}_1 * \{\mathbf{i}_{1,1}, \mathbf{i}_{1,2}, \ldots, \mathbf{i}_{1,s_1}\}) \oplus (\mathbf{k}_2 * \{\mathbf{i}_{2,1}, \mathbf{i}_{2,2}, \ldots, \mathbf{i}_{2,s_2}\}) \\
&\oplus \cdots \oplus (\mathbf{k}_t * \{\mathbf{i}_{t,1}, \mathbf{i}_{t,2}, \ldots, \mathbf{i}_{t,s_t}\}) \oplus (\mathbf{k}_f * \{\mathbf{i}_{f,1}, \mathbf{i}_{f,2}, \ldots, \mathbf{i}_{f,s_f}\})
\end{aligned}$$

$$= (\mathbf{i}_{1,1}\dotplus k_{11}) \oplus (\mathbf{i}_{1,2}\dotplus k_{12}) \oplus \cdots \oplus (\mathbf{i}_{1,s_1}\dotplus k_{1s_1}) \oplus (\mathbf{i}_{2,1}\dotplus k_{21}) \oplus (\mathbf{i}_{2,2}\dotplus k_{22}) \oplus \cdots \oplus (\mathbf{i}_{2,s_2}\dotplus k_{2s_2})$$

$$\oplus \cdots \oplus (\mathbf{i}_{t,1}\dotplus k_{t1}) \oplus (\mathbf{i}_{t,2}\dotplus k_{t2})$$

$$\oplus \cdots \oplus (\mathbf{i}_{t,s_t}\dotplus k_{ts_t}) \oplus (\mathbf{i}_{f,1}\dotplus k_{f1}) \oplus (\mathbf{i}_{f,2}\dotplus k_{f2}) \oplus \cdots \oplus (\mathbf{i}_{f,s_f}\dotplus k_{fs_f}).$$

The proof is complete. □

Proposition 2.3. *Let us suppose that the next indices and sets are as in the aforementioned definitions. Then the following relations hold:*

(i)

$$\dim(\mathbf{i} + \mathbf{j}) = \dim(\mathbf{i}) = \dim(\mathbf{j}).$$

(ii)

$$\dim(\rho \cdot \mathbf{i}) = \dim(\mathbf{i}).$$

(iii)

$$\dim(\mathbf{i} \oplus \mathbf{j}) = \dim(\mathbf{i}) + \dim(\mathbf{j}).$$

(iv)

$$\dim(\mathbf{j} \dot{+} i) = \dim(\mathbf{j}).$$

(v)

$$\dim(\mathbf{i} * \mathbf{j}) = \dim(\mathbf{i}) \cdot dim(\mathbf{j}).$$

(vi)

$$\dim(\mathbf{k} * \mathbf{I}) = \left(\sum_{\mu=1}^{\theta} \dim(\mathbf{i}_\mu) \right).$$

(vii)

$$\dim(\mathbf{K} * \mathcal{J}) = \sum_{\rho=1}^{f} \left[\sum_{\mu=1}^{s_\rho} \dim(\mathbf{i}_{\rho\mu}) \right],$$

where f is the amount of the sets $\mathbf{I}_\rho$, contained into the collection $\mathcal{J}$, which is equal to the amount of multiindices contained into the set $\mathbf{K}$. Also, s_ρ is the amount of multiindices, contained into the sets $\mathbf{I}_\rho$.

Proof. The proof in each case, follows easily by using the definitions. □

2.2 The δ-Operator and the δ-Polynomials

In this section, we correspond to each index $\mathbf{i}$ an operator $\delta_{\mathbf{i}}$ which acts, as we shall see in the next section, on sequences. We define the set of symbols

$$\Delta = \left\{ \delta_{\mathbf{i}}, \mathbf{i} \in \bigcup_{n=1}^{\infty} (\mathbb{N})^n \right\}.$$

The members of this set are called δ-operators. We correspond to the empty set $\mathbf{I} = \{\}$, the operator δ_e. If $\mathbf{I} = \mathbb{N}$, the operators δ_i, $i \in \mathbf{I}$ are called linear. A special case is the identity operator δ_0.

Among the δ-operators, we define two internal operations: The dot product and the star product. The dot product is defined as

$$\delta_{\mathbf{i}} \cdot \delta_{\mathbf{j}} = \delta_{\mathbf{i} \oplus \mathbf{j}}$$

and the star product as

$$\delta_{\mathbf{i}} * \delta_{\mathbf{j}} = \delta_{\mathbf{i} * \mathbf{j}}.$$

Proposition 2.4. *The following properties are valid:*

(i)
$$\delta_{\mathbf{i}} \cdot \delta_{\mathbf{j}} = \delta_{\mathbf{j}} \cdot \delta_{\mathbf{i}}.$$

(ii)
$$\delta_e * \delta_{\mathbf{i}} = \delta_e.$$

(iii)
$$\delta_{\mathbf{i}} * (\delta_{\mathbf{j}} \cdot \delta_{\mathbf{k}}) = (\delta_{\mathbf{i}} * \delta_{\mathbf{j}}) \cdot (\delta_{\mathbf{i}} * \delta_{\mathbf{k}}).$$

Proof. The proofs are straightforward applications of the relations A, B and E of the Proposition 2.2. □

Expressions of the form
$$A = \sum_{n=0}^{w} \sum_{\mathbf{i} \,\in\, \mathbf{I_n}} a_{\mathbf{i}} \delta_{\mathbf{i}} = \sum_{\mathbf{i} \,\in\, \bigcup_n \mathbf{I}_n} a_{\mathbf{i}} \delta_{\mathbf{i}}, \quad a_{\mathbf{i}} \in \mathbf{R},$$

are called δ-polynomials, where by $\mathbf{I_n}$, we denote the set of multiindices with dimension n. By convention,
$$\mathbf{I_0} = \{e\}.$$

For each polynomial A, we define $d(A)$ as
$$d(A) = \min\{d(\mathbf{i}), a_{\mathbf{i}} \neq 0\}.$$

We define as degree of A, and we denote it by $\deg(A)$, the maximum $\dim(\mathbf{i})$ with $\delta_{\mathbf{i}}$ appeared in A. The maximum term of a nonlinear polynomial A, denoted by $\max(A)$, is the quantity $a_{\mathbf{i}}\delta_{\mathbf{i}}$, where $\mathbf{i}$ is the largest index according to the lexicographical order defined in the previous section. An expression of the form $\sum_{i \in \mathbf{Z}} a_i \delta_i$ is called a linear polynomial.

Definition 2.5. Two δ-polynomials,
$$A = \sum_{\mathbf{i} \,\in\, \bigcup_n \mathbf{I}_n} a_{\mathbf{i}} \delta_{\mathbf{i}} \ \text{ and } \ B = \sum_{\mathbf{j} \,\in\, \bigcup_m \mathbf{J}_m} b_{\mathbf{j}} \delta_{\mathbf{j}},$$

are equal, if
$$\mathbf{I}_n = \mathbf{J}_m, \quad n, m = 0, 1, \ldots, \nu \ \text{ and } \ a_{\mathbf{i}} = b_{\mathbf{j}}.$$

Definition 2.6. Let
$$A = \sum_{\mathbf{i} \,\in\, \bigcup_n \mathbf{I}_n} a_{\mathbf{i}} \delta_{\mathbf{i}} \ \text{ and } \ B = \sum_{\mathbf{j} \in \bigcup_m \mathbf{J}_m} b_{\mathbf{j}} \delta_{\mathbf{j}},$$

be two δ-polynomials. Their dot product is defined as
$$A \cdot B = \sum_{\mathbf{i} \,\in\, \bigcup_n \mathbf{I}_n} \sum_{\mathbf{j} \,\in\, \bigcup_m \mathbf{J}_m} a_{\mathbf{i}} b_{\mathbf{j}} \delta_{\mathbf{i} \oplus \mathbf{j}}.$$

Definition 2.7. Let us suppose that we have the δ-polynomials

$$A = \sum_{\mathbf{i} \in \bigcup_n \mathbf{I}_n} a_{\mathbf{i}} \delta_{\mathbf{i}} \text{ and } B = \sum_{\mathbf{j} \in \bigcup_m \mathbf{J}_m} b_{\mathbf{j}} \delta_{\mathbf{j}}.$$

Their star product is defined by the relation

$$A * B = \sum_{\mathbf{i} \in \bigcup_n \mathbf{I}_n} \sum_{J=(\mathbf{j}_1,\mathbf{j}_2,\ldots,\mathbf{j}_n) \in (\bigcup_m \mathbf{J}_m)^n} a_{\mathbf{i}} b_{\mathbf{j}_1} b_{\mathbf{j}_2} \cdots b_{\mathbf{j}_n} \delta_{\mathbf{i}*J}.$$

Many times, for the calculation of simple star-products, we use the properties presented in the next proposition.

Proposition 2.8. *Let A and B be δ-polynomials, defined as before. Then the following are valid:*

(1)

$$\delta_i * B = \sum_{\mathbf{j} \in \bigcup_m \mathbf{J}_m} b_{\mathbf{j}} \delta_{\mathbf{j} \dot{+} i}, \quad i \in \mathbf{Z}.$$

(2)

$$\delta_{\mathbf{i}} * B = (\delta_{i_1} * B) \cdot (\delta_{i_2} * B) \cdots (\delta_{i_n} * B), \quad \mathbf{i} = (i_1, i_2, \ldots, i_n).$$

(3)

$$\delta_{\mathbf{i}} * (A \cdot B) = (\delta_{\mathbf{i}} * A) \cdot (\delta_{\mathbf{i}} * B).$$

Proof. (1) In this case, the polynomial A has the trivial form $A = \delta_i$, that is $\mathbf{I}_1 = \{i\}$ and $a_i = 1$. Applying the formula of Definition 2.7, we get

$$\delta_i * B = \sum_{\mathbf{i} \in \mathbf{I_1}} \sum_{J \in (\in (\cup_m \mathbf{J}_m)^1} a_{\mathbf{i}} b_{\mathbf{j}_1} b_{\mathbf{j}_2} \cdots b_{\mathbf{j}_n} \delta_{\mathbf{i}*J}.$$

Using the previous data this expression becomes

$$\sum_{\mathbf{j} \in (\cup_m \mathbf{J}_m)} b_{\mathbf{j}} \delta_{i*\mathbf{j}}.$$

In view of,

$$i * \mathbf{j} = \mathbf{j} \dot{+} i,$$

the result follows.

(2) For the sake of presentation, we shall prove the relation for $n = 2$, that is, we shall prove

$$\delta_{\mathbf{i}} * B = (\delta_{i_1} * B) \cdot (\delta_{i_2} * B), \quad \mathbf{i} = (i_1, i_2).$$

The generalization for arbitrary n, comes straightforward. By applying the formula of Definition 2.7, with $A = \sum_{\mathbf{i}\in\mathbf{I_2}} \delta_{\mathbf{i}}$, $\mathbf{I}_2 = \{(i_1, i_2)\}$, and $a_{\mathbf{i}} = 1$, we get

$$\delta_{\mathbf{i}} * B = \sum_{\mathbf{J}=(\mathbf{j}_1,\mathbf{j}_2)\in(\cup_m \mathbf{J}_m)^2} b_{\mathbf{j_1}} b_{\mathbf{j_2}} \delta_{\mathbf{i}*\mathbf{J}}.$$

But,

$$\mathbf{i} * \mathbf{J} = (\mathbf{i} * \mathbf{j}_1) \oplus (\mathbf{i} * \mathbf{j}_2)$$

and

$$\mathbf{i} * \mathbf{j}_1 = (\mathbf{j}_1 \dot{+} i_1) \oplus (\mathbf{j}_1 \dot{+} i_2), \mathbf{i} * \mathbf{j}_2 = (\mathbf{j}_2 \dot{+} i_1) \oplus (\mathbf{j}_2 \dot{+} i_2),$$

and so,

$$\delta_{\mathbf{i}} * B = \sum_{\mathbf{J}=(\mathbf{j}_1,\mathbf{j}_2)\in(\cup_m \mathbf{J}_m)^2} b_{\mathbf{j_1}} b_{\mathbf{j_2}} \delta_{(\mathbf{j}_1 \dot{+} i_1)\oplus(\mathbf{j}_1 \dot{+} i_2)\oplus(\mathbf{j}_2 \dot{+} i_1)\oplus(\mathbf{j}_2 \dot{+} i_2)}.$$

In view of the fact that

$$\delta_{(\mathbf{j}_1 \dot{+} i_1)\oplus(\mathbf{j}_1 \dot{+} i_2)\oplus(\mathbf{j}_2 \dot{+} i_1)\oplus(\mathbf{j}_2 \dot{+} i_2)} = \delta_{(\mathbf{j}_1 \dot{+} i_1)\oplus(\mathbf{j}_2 \dot{+} i_1)\oplus(\mathbf{j}_1 \dot{+} i_2)\oplus(\mathbf{j}_2 \dot{+} i_2)},$$

the relations

$$\delta_{i_1} * B = \sum_{\mathbf{j}\in\cup_\mathbf{m}\mathbf{J_m}} b_{\mathbf{j}} \delta_{\mathbf{j}\dot{+} i_1} \quad , \quad \delta_{i_2} * B = \sum_{\mathbf{j}\in\cup_\mathbf{m}\mathbf{J_m}} b_{\mathbf{j}} \delta_{\mathbf{j}\dot{+} i_2},$$

and the formula of Definition 2.6, we may easily see that

$$\delta_{\mathbf{i}} * B = (\delta_{i_1} * B) \cdot (\delta_{i_2} * B).$$

(3) First, we show that

$$\delta_\rho * (A \cdot B) = (\delta_\rho * A) \cdot (\delta_\rho * B), \quad \rho \in \mathbf{Z}.$$

Indeed,

$$\begin{aligned} \delta_\rho * (A \cdot B) &= \delta_\rho * \left(\sum_{\mathbf{i}\in\cup_n \mathbf{I}_n} \sum_{\mathbf{j}\in\cup_m \mathbf{J}_m} a_{\mathbf{i}} b_{\mathbf{j}} \delta_{\mathbf{i}\oplus\mathbf{j}} \right) \\ &= \sum_{\mathbf{i}\in\cup_n \mathbf{I}_n} \sum_{\mathbf{j}\in\cup_m \mathbf{J}_m} a_{\mathbf{i}} b_{\mathbf{j}} \delta_{(\mathbf{i}\oplus\mathbf{j})\dot{+}\rho}. \end{aligned}$$

Using Proposition 2.2 (C), the latter becomes

$$\begin{aligned} \sum_{\mathbf{i}\in\cup_n \mathbf{I}_n} \sum_{\mathbf{j}\in\cup_m \mathbf{J}_m} a_{\mathbf{i}} b_{\mathbf{j}} \delta_{(\mathbf{i}\dot{+}\rho)\oplus(\mathbf{j}\dot{+}\rho)} &= \left(\sum_{\mathbf{i}\in\cup_n \mathbf{I}_n} a_{\mathbf{i}} \delta_{\mathbf{i}\dot{+}\rho} \right) \cdot \left(\sum_{\mathbf{j}\in\cup_m \mathbf{J}_m} b_{\mathbf{j}} \delta_{\mathbf{j}\dot{+}\rho} \right) \\ &= (\delta_\rho * A) \cdot (\delta_\rho * B). \end{aligned}$$

By using Part (2) of Proposition 2.8 twice, we get

$$\begin{aligned} \delta_{\mathbf{i}} * (A \cdot B) &= [\delta_{i_1} * (A \cdot B)] \cdot [\delta_{i_2} * (A \cdot B)] \cdots [\delta_{i_n} * (A \cdot B)] \\ &= (\delta_{i_1} * A) \cdot (\delta_{i_1} * B) \cdot (\delta_{i_2} * A) \cdot (\delta_{i_2} * B) \cdots (\delta_{i_n} * A) \cdot (\delta_{i_n} * B) \\ &= (\delta_{i_1} * A) \cdot (\delta_{i_2} * A) \cdots (\delta_{i_n} * A) \cdot (\delta_{i_1} * B) \cdot (\delta_{i_2} * B) \cdots (\delta_{i_n} * B) \\ &= (\delta_{\mathbf{i}} * A) \cdot (\delta_{\mathbf{i}} * B). \end{aligned}$$

The proof is complete. □

The following proposition is stated without a proof. For the proof, see [7, 8, 11].

Proposition 2.9. *The following statements hold.*

(1)
$$A \cdot B = B \cdot A.$$

(2)
$$A * B \neq B * A.$$

(3)
$$(A * B) * \Gamma = A * (B * \Gamma).$$

(4)
$$d(A * B) = d(A) + d(B)$$

(5)
$$\deg(A * B) = \deg(A) \cdot \deg(B).$$

(6)
$$(A \cdot B) * C = (A * C) \cdot (B * C).$$

Example 2.10. Let

$$A = \delta_{\mathbf{i}} = \delta_1 \delta_2^2 = \delta_{(1,2,2)}$$

and

$$B = 2\delta_1 + 3\delta_2\delta_3 = 2\delta_1 + 3\delta_{(2,3)}.$$

Then

$$\begin{aligned} A * B &= \delta_{(1,2,2)} * B = (\delta_1 * B) \cdot (\delta_2 * B) \cdot (\delta_2 * B) \\ &= (2\delta_2 + 3\delta_{(3,4)}) \cdot (2\delta_3 + 3\delta_{(4,5)}) \cdot (2\delta_3 + 3\delta_{(4,5)}) \\ &= 12\delta_{(3,3)}\delta_{(3,4)} + 24\delta_{(2,3,4,5)} + 36\delta_{(3,3,4,4,5)} + 18\delta_{(2,4,4,5,5)} + 27\delta_{(3,4,4,4,5,5)} + 8\delta_{(2,3,3)} \\ &= 12\delta_3^3\delta_4 + 24\delta_2\delta_3\delta_4\delta_5 + 36\delta_3^2\delta_4^2\delta_5 + 18\delta_2\delta_4^2\delta_5^2 + 27\delta_3\delta_4^3\delta_5^2 + 8\delta_2\delta_3^2. \end{aligned}$$

From this and

$$\begin{aligned} B * A &= (2\delta_1 + 3\delta_{(2,3)}) * \delta_{(1,2,2)} = 2\delta_1 * \delta_{(1,2,2)} + 3\delta_{(2,3)} * \delta_{(1,2,2)} \\ &= 2\delta_2\delta_3^2 + 3\delta_3\delta_4^3\delta_5^2, \end{aligned}$$

we see that $A * B \neq B * A$.

2.3 The $\delta\epsilon$-Operators and the $\delta\epsilon$-Polynomials

The δ-operator will be used later in order to cope with the delays of one sequence and hence with one dimension dynamical systems. Nevertheless, when we focus our attention to two dimensional dynamical systems or to input-output control systems, see [11], we have to work with two sequences. Therefore, we need one operator devoted to the first sequence and another one, devoted to the second sequence. To achieve that, we extend the above introduced notion in a direct way, by considering the set

$$\Delta \times \Delta.$$

A member of the set $\mathcal{D} = \Delta \times \Delta$, is called $\delta\epsilon$-operator and it is denoted by

$$\delta_{\mathbf{i}} \times \delta_{\mathbf{j}}.$$

Sometimes, for the sake of appearance, it is more convenient to use the following notation:

$$\delta_{\mathbf{i}} \times \delta_{\mathbf{j}} = \delta_{\mathbf{i}}\epsilon_{\mathbf{j}}.$$

Therefore, the ϵ-operator is just a second operator with properties identical similar to the properties of δ-operator. Obviously,

$$\delta_{\mathbf{i}} \times \delta_e = \delta_{\mathbf{i}} \;\text{ and }\; \delta_e \times \delta_{\mathbf{j}} = \epsilon_{\mathbf{j}}.$$

A $\delta\epsilon$-operator,

$$\delta_{\mathbf{i}}\epsilon_{\mathbf{j}},$$

with the property that

$$d(\mathbf{i}) = d(\mathbf{j}) = 0,$$

is called a *zero* $\delta\epsilon$-operator. A special case of a zero $\delta\epsilon$-operator is the operator

$$\delta_0\epsilon_0.$$

If

$$\mathbf{i} = (i_1, i_2, \ldots, i_a) \;\text{ and }\; \mathbf{j} = (j_1, j_2, \ldots, j_b),$$

the number $a + b$ is called degree of $\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}$ and it is denoted by $\deg(\delta_{\mathbf{i}}\epsilon_{\mathbf{j}})$. The pair

$$(a, b) = (\dim(\mathbf{i}), \dim(\mathbf{j})),$$

is called multidegree of $\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}$ and it is denoted by $\deg_m(\delta_{\mathbf{i}}\epsilon_{\mathbf{j}})$. We say that

$$\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} \preceq \delta_{\mathbf{i}'}\epsilon_{\mathbf{j}'},$$

if

$$\text{either } \; \mathbf{i} < \mathbf{i}' \; \text{ or } \; (\mathbf{i} = \mathbf{i}' \; \text{ and } \; \mathbf{j} < \mathbf{j}'),$$

with respect to the lexicographical order, defined previously, among multiindices. Now we are ready to extend the two operations defined previously, in the case of $\delta\epsilon$-operators.

Definition 2.11. Let
$$\delta_{\mathbf{i}} \times \delta_{\mathbf{j}} = \delta_{\mathbf{i}}\epsilon_{\mathbf{j}} \text{ and } \delta_{\mathbf{i}'} \times \delta_{\mathbf{j}'} = \delta_{\mathbf{i}'}\epsilon_{\mathbf{j}'}$$
be two $\delta\epsilon$-operators. As their dot product, denoted by $\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} \cdot \delta_{\mathbf{i}'}\epsilon_{\mathbf{j}'}$, we define the operator

$$\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} \cdot \delta_{\mathbf{i}'}\epsilon_{\mathbf{j}'} = \delta_{\mathbf{i}\oplus\mathbf{i}'}\epsilon_{\mathbf{j}\oplus\mathbf{j}'}.$$

The star product cannot be determined between two $\delta\epsilon$-operators, but between a $\delta\epsilon$-operator and a pair of $\delta\epsilon$-operators. The reason for that will be clear later, when we study the action of these operators to a pair of sequences.

Definition 2.12. The star product of the operator $\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}$ and the pair of operators $[\delta_{\mathbf{i}_1}\epsilon_{\mathbf{j}_1}, \delta_{\mathbf{i}_2}\epsilon_{\mathbf{j}_2}]$ is defined as
$$\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} * [\delta_{\mathbf{i}_1}\epsilon_{\mathbf{j}_1}, \delta_{\mathbf{i}_2}\epsilon_{\mathbf{j}_2}] = \delta_{\mathbf{i}*\{\mathbf{i}_1,\mathbf{i}_2\}}\epsilon_{\mathbf{j}*\{\mathbf{j}_1,\mathbf{j}_2\}}.$$

By using the null operators δ_e, ϵ_e, we can include the star product among simple δ-operators and $\delta\epsilon$-operators into the above definition. For instance,

$$\delta_{\mathbf{i}} * \delta_{\mathbf{i}_1}\epsilon_{\mathbf{j}_1} = \delta_{\mathbf{i}}\epsilon_e * [\delta_{\mathbf{i}_1}\epsilon_{\mathbf{j}_1}, \delta_e\epsilon_e] = \delta_{\mathbf{i}*\mathbf{i}_1}\epsilon_{\mathbf{i}*\mathbf{j}_1}.$$

Definition 2.13. Let $\mathbf{I}_n$, $\mathbf{J}_m$ be sets of multiindices with dimensions n and m respectively. An expression of the form

$$A = \sum_{n=0}^{\nu}\sum_{m=0}^{\mu}\sum_{(\mathbf{i},\mathbf{j})\in\mathbf{I}_n\times\mathbf{J}_m} c_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}},$$

where $c_{\mathbf{ij}}$ are real numbers. By convention, $\mathbf{I}_0 = \mathbf{J}_0 = \{e\}$, is called a $\delta\epsilon$-polynomial.

Sometimes, for the sake of abbreviation, we write

$$A = \sum_{(\mathbf{i},\mathbf{j})\in\mathbf{I}\times\mathbf{J}} c_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}},$$

where $\mathbf{I}$ and $\mathbf{J}$ are sets containing multiindices of any dimension, not of a constant one. Actually, they contain multiindices of dimensions from 0 up to ν and μ, respectively.

As in the case of δ-polynomials, we can extend this notion to $\delta\epsilon$-series, in a natural manner. Indeed, if both ν and μ tend to infinity or if $\mathbf{I}_n, \mathbf{J}_n$ are infinite sets of multiindices, or if both are true, then we have a $\delta\epsilon$-series. Specifically,

$$A = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{(\mathbf{i},\mathbf{j})\in\mathbf{I}_n\times\mathbf{J}_m} c_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}.$$

By means of the null index e, we can decompose a $\delta\epsilon$-polynomial into its pure δ-part, ϵ-part and $\delta\epsilon$-part. Indeed,
$$A = A_\delta + A_\epsilon + A_{\delta\epsilon},$$

where

$$A_{\delta} = \sum_{n=0}^{\nu} \sum_{(\mathbf{i},\mathbf{j})\in \mathbf{I}_n\times \mathbf{J}_0} c_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} = \sum_{n=0}^{\nu}\sum_{\mathbf{i}\in \mathbf{I}_n} c_{\mathbf{i}e}\delta_{\mathbf{i}}\epsilon_e = \sum_{n=0}^{\nu}\sum_{\mathbf{i}\in \mathbf{I}_n} c_{\mathbf{i}}\delta_{\mathbf{i}}$$

is the pure δ-part of A,

$$A_{\epsilon} = \sum_{m=0}^{\mu} \sum_{(\mathbf{i},\mathbf{j})\in \mathbf{I}_0\times \mathbf{J}_m} c_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} = \sum_{m=0}^{\mu}\sum_{\mathbf{j}\in \mathbf{J}_m} c_{e\mathbf{j}}\delta_e\epsilon_{\mathbf{j}} = \sum_{m=0}^{\mu}\sum_{\mathbf{j}\in \mathbf{J}_m} c_{\mathbf{j}}\epsilon_{\mathbf{j}}$$

is the pure ϵ-part, and $A_{\delta\epsilon}$ is the pure $\delta\epsilon$-part. Expressions of the form

$$\sum_{(i,j)\in(\mathbf{I}_1\times\mathbf{J}_0)\cup(\mathbf{I}_0\times\mathbf{J}_1)} c_{ij}\delta_i\epsilon_j = \sum_{i\in\mathbf{I}_1} c_i\delta_i + \sum_{j\in\mathbf{J}_1} c_j\epsilon_j \tag{2.1}$$

$$\sum_{(i,j)\in(\mathbf{I}_1\times\mathbf{J}_0)} c_{ie}\delta_i\epsilon_e = \sum_{i\in\mathbf{I}_1} c_i\delta_i,$$

and

$$\sum_{(i,j)\in(\mathbf{I}_0\times\mathbf{J}_1)} c_{ej}\delta_e\epsilon_j = \sum_{j\in\mathbf{J}_1} c_j\epsilon_j$$

are called linear $\delta\epsilon$, δ, and ϵ-polynomials, respectively. The term, which according to the order defined previously, is ordered highly among the terms of A, is called the *maximum* term of A. By $d(A)$, we denote the minimum delay of A. In other words,

$$d(A) = \min(\min(\mathbf{i}), \min(\mathbf{j})), \quad (\mathbf{i},\mathbf{j}) \in \mathbf{I}_n \times \mathbf{J}_m, \quad n = 0,\ldots,\nu, \quad m = 0,\ldots,\mu.$$

The highest degree of the terms of A is called *degree* of A, denoted by $\deg(A)$. If all the terms of a $\delta\epsilon$-polynomial or a $\delta\epsilon$-series A have the same multidegree (a, b), then we call it homogeneous, and we usually write $A^{(a,b)}$. Obviously $A^{(1,0)}$ is a δ-polynomial of linear δ-terms only and $A^{(0,1)}$ is an ϵ-polynomial of linear ϵ-terms only. Any $\delta\epsilon$-polynomial or $\delta\epsilon$-series can be considered as sum of homogeneous $\delta\epsilon$-polynomials (series) of an increasing degree. That is,

$$A = \underbrace{A^{(1,0)} + A^{(0,1)}}_{linear-part} + \underbrace{A^{(2,0)} + A^{(1,1)} + A^{(0,2)}}_{quadratic-part}$$

$$+ \underbrace{A^{(3,0)} + A^{(2,1)} + A^{(1,2)} + A^{(0,3)}}_{cubic-part} + \cdots = \sum_{n=1}^{\infty}\sum_{a+b=n} A^{(a,b)}.$$

The equality of two $\delta\epsilon$-polynomials is defined in the same manner as it is defined for the δ-polynomials. Now, working analogously with the case of δ-polynomials, we are going to define the two internal operations, the dot product and the star product for $\delta\epsilon$-polynomials.

Definition 2.14. Let

$$A = \sum_{(\mathbf{i},\mathbf{j})\in \mathbf{I}\times\mathbf{J}} a_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} \quad \text{and} \quad B = \sum_{(\mathbf{s},\mathbf{r})\in \mathbf{S}\times\mathbf{R}} b_{\mathbf{sr}}\delta_{\mathbf{s}}\epsilon_{\mathbf{r}},$$

be two $\delta\epsilon$-polynomials, where $\mathbf{I}, \mathbf{J}, \mathbf{S}, \mathbf{R}$ are sets of multiindices of various dimensions. Their dot product is a new $\delta\epsilon$-polynomial, denoted by $A \cdot B$, and defined as

$$A \cdot B = \sum_{(\mathbf{i},\mathbf{j})\in \mathbf{I}\times\mathbf{J}} \sum_{(\mathbf{s},\mathbf{r})\in \mathbf{S}\times\mathbf{R}} a_{\mathbf{ij}} b_{\mathbf{sr}} \delta_{\mathbf{i}\oplus\mathbf{s}}\epsilon_{\mathbf{j}\oplus\mathbf{s}}.$$

This dot product corresponds to the usual product among polynomials of many variables.

Definition 2.15. We introduce the $\delta\epsilon$-polynomials

$$A = \sum_{(\mathbf{i},\mathbf{j})\in \mathbf{I}\times\mathbf{J}} a_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}, \quad B = \sum_{(\mathbf{s},\mathbf{r})\in \mathbf{S}\times\mathbf{R}} b_{\mathbf{sr}}\delta_{\mathbf{s}}\epsilon_{\mathbf{r}},$$

and

$$C = \sum_{(\mathbf{h},\mathbf{w})\in \mathbf{H}\times\mathbf{W}} c_{\mathbf{hw}}\delta_{\mathbf{h}}\epsilon_{\mathbf{w}},$$

where $\mathbf{I}, \mathbf{J}, \mathbf{S}, \mathbf{R}, \mathbf{H}, \mathbf{W}$ are sets of multiindices of various dimensions. The star product operation $A * [B, C]$ produces a $\delta\epsilon$-polynomial, which is defined as

$$A * [B, C]$$

$$= \sum_{\substack{K=(\mathbf{i},\mathbf{j})\in\mathbf{I}\times\mathbf{J} \\ \mathbf{i}=(i_1,i_2,\ldots,i_n) \\ \mathbf{j}=(j_1,j_2,\ldots,j_m)}} \sum_{\substack{S=(\mathbf{s_1},\mathbf{s_2},\ldots,\mathbf{s_n})\in\mathbf{S}^n \\ R=(\mathbf{r_1},\mathbf{r_2},\ldots,\mathbf{r_m})\in\mathbf{R}^n}} \sum_{\substack{H=(\mathbf{h_1},\mathbf{h_2},\ldots,\mathbf{h_m})\in\mathbf{H}^m \\ W=(\mathbf{w_1},\mathbf{w_2},\ldots,\mathbf{w_m})\in\mathbf{W}^m}}$$

$$a_{\mathbf{ij}} b_{\mathbf{s_1 r_1}} b_{\mathbf{s_2 r_2}} \cdots b_{\mathbf{s_n r_n}} c_{\mathbf{h_1 w_1}} c_{\mathbf{h_2 w_2}} \cdots c_{\mathbf{h_m w_m}} \delta_{K*\{S,H\}}\epsilon_{K*\{R,W\}},$$

where by $K * \{S, H\}$, we mean the general star product among the set of multiindices $K = \{\mathbf{i}, \mathbf{j}\}$ and the collection $\{S, H\}$.

The next propositions are devoted to how the multidegree is handled through the star product. Their proofs arise easily from the definitions of the star product and the multidegree.

Proposition 2.16. *We introduce the following homogeneous $\delta\epsilon$-polynomials T, G_1, G_2 with multidegrees*

$$\deg_m(T) = (\kappa, \lambda), \quad \deg_m(G_1) = (\nu, \mu), \quad \deg_m(G_2) = (\varphi, \omega).$$

Then the following relation holds:

$$\deg_m(T * [G_1, G_2]) = (\kappa, \lambda) \begin{pmatrix} \nu & \mu \\ \varphi & \omega \end{pmatrix}.$$

Proof. We suppose that the polynomials G_1, G_2 and T have the next construction:

$$T = \sum_{(\mathbf{i},\mathbf{j})\in \mathbf{I}_\kappa\times\mathbf{J}_\lambda} t_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}, \quad G_1 = \sum_{(\mathbf{s},\mathbf{r})\in\mathbf{S}_\nu\times\mathbf{R}_\mu} b_{\mathbf{sr}}\delta_{\mathbf{s}}\epsilon_{\mathbf{r}},$$

and

$$G_2 = \sum_{(\mathbf{h},\mathbf{w})\in\mathbf{H}_\varphi\times\mathbf{W}_\omega} c_{\mathbf{hw}}\delta_{\mathbf{h}}\epsilon_{\mathbf{w}}.$$

Using Definition 2.15, we get

$$T*[G_1,G_2] = \sum_{\substack{\mathbf{k}=(\mathbf{i},\mathbf{j})\in\mathbf{I}_\kappa\times\mathbf{J}_\lambda\\ \mathbf{i}=(i_1,i_2,\dots,i_\kappa)\\ \mathbf{j}=(j_1,j_2,\dots,j_\lambda)}} \sum_{\substack{S=(\mathbf{s}_1,\mathbf{s}_2,\dots,\mathbf{s}_\kappa)\in\mathbf{S}_\nu^\kappa\\ R=(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_\kappa)\in\mathbf{R}_\mu^\kappa}} \sum_{\substack{H=(\mathbf{h}_1,\mathbf{h}_2,\dots,\mathbf{h}_\lambda)\in\mathbf{H}_\varphi^\lambda\\ W=(\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_\lambda)\in\mathbf{W}_\omega^\lambda}}$$

$$b_{\mathbf{s}_1\mathbf{r}_1}\cdots b_{\mathbf{s}_\kappa\mathbf{r}_\kappa}c_{\mathbf{h}_1\mathbf{w}_1}\cdots c_{\mathbf{h}_\lambda\mathbf{w}_\lambda}\delta_{K*\{S,H\}}\epsilon_{K*\{R,W\}}.$$

Clearly, all terms of $T*[G_1,G_2]$ have the same multidegree (a,b). We know that in this case

$$(a,b) = (\dim(K*\{S,H\}, \dim(K*\{R,W\})).$$

Let us calculate the first dimension. Using Proposition 2.3 successively, we get

$$\dim(K*\{S,H\}) = \dim(\mathbf{i}*S\oplus\mathbf{j}*H) = \dim(\mathbf{i}*S)+\dim(\mathbf{j}*\mathbf{H})$$

$$= \sum_{\mu=1}^{\kappa}\dim(\mathbf{s}_\mu) + \sum_{\mu=1}^{\lambda}\dim(\mathbf{h}_\mu).$$

But we know from the homogeneity of the series that

$$\dim(\mathbf{s}_\mu) = \nu \text{ and } \dim(\mathbf{h}_\mu) = \varphi,$$

for all values of μ, and so,

$$\dim(K*\{S,H\}) = \kappa\nu+\lambda\varphi.$$

Similarly, we prove the corresponding relation for the second dimension. Using matrix formation, we can rewrite the above equalities in the requested expression. The proof is complete. □

Proposition 2.17. *We introduce the $\delta\epsilon$-series T, G_1, G_2. Also, let W be the star product, that is*

$$W = T*[G_1,G_2].$$

Note that, $W^{(a,b)}$, which is the homogeneous part of W, of multidegree (a,b), satisfies the relation

$$W^{(a,b)} = \sum_{\substack{\kappa=0\\ \lambda=0}}^{\infty} \sum_{(\nu,\mu,\varphi,\omega)\in S} T^{(\kappa,\lambda)}*[G_1^{(\nu,\mu)}, G_2^{(\varphi,\omega)}],$$

where S is the solution set of the equations

$$\kappa\nu + \lambda\varphi = a, \quad \kappa\mu + \lambda\omega = b, \quad \nu, \mu, \varphi, \omega \in \mathbf{N} \cup \{0\},$$

and

$$T^{(\kappa,\lambda)}, \quad G_1^{(\nu,\mu)}, \quad G_2^{(\varphi,\omega)},$$

are the homogeneous parts of the $\delta\epsilon$-series T, G_1, G_2 of multidegrees (κ, λ),(ν, μ), and (φ, ω) respectively.

Proof. It comes straightforward from Proposition 2.16 together with the fact that the homogeneous part $W^{(a,b)}$ of W is consisting from all the parts of $T^{(\kappa,\lambda)} * [G_1^{(\nu,\mu)}, G_2^{(\varphi,\omega)}]$ with multidegree (a, b). The proof is complete. □

Example 2.18. Let us give now a more complicated example, describing the general case. Let

$$A = 2\delta_1\delta_2\epsilon_4^3, \quad B = \delta_2^2\epsilon_1 + \delta_1^2\epsilon_2, \quad \text{and} \quad C = 4\delta_1\epsilon_2^2 + \delta_3\epsilon_3^2.$$

We want to calculate the product $A * [B, C]$. We shall work by using the formula of the Definition 2.15. First we have to determine the indices sets of the above polynomials. We see that

$$\mathbf{I} = \{(1, 2)\}, \quad \mathbf{J} = \{(4, 4, 4)\}, \quad \mathbf{S} = \{(2, 2), (1, 1)\},$$
$$\mathbf{R} = \{(1), (2)\}, \quad \mathbf{H} = \{(1), (3)\}, \text{and} \quad \mathbf{W} = \{(2, 2), (3, 3)\}.$$

For the coefficients, we have

$$a_{(1,2),(4,4,4)} = 2, \quad b_{(2,2),(1)} = 1, \quad b_{(1,1),(2)} = 1,$$
$$c_{(1),(2,2)} = 4, \quad \text{and} \quad c_{(3),(3,3)} = 1.$$

Proceeding with the formula, first we calculate the quantity

$$\mathbf{I} \times \mathbf{J} = \mathbf{I} \times \mathbf{J} = \{((1, 2), (4, 4, 4))\}.$$

This set contains only one element, a pair with first element the index $\mathbf{i} = (1, 2)$ and second element the index $\mathbf{j} = (4, 4, 4)$. Thus $n = 2$ and $m = 3$. Then,

$$\mathbf{S}^2 = \mathbf{S} \times \mathbf{S} = \{((2, 2), (2, 2)), ((2, 2), (1, 1)), ((1, 1), (2, 2)), ((1, 1), (1, 1))\},$$

$$\mathbf{H}^3 = \mathbf{H} \times \mathbf{H} \times \mathbf{H} = \{((1), (1), (1)), ((1), (1), (3)), ((1), (3), (1)), ((1), (3), (3)),$$
$$((3), (1), (1)), ((3), (1), (3)), ((3), (3), (1)), ((3), (3), (3))\},$$
$$\mathbf{R^2} = \{(1), (2)\} \times \{(1), (2)\} = \{((1), (1)), \ldots\},$$

and

$$\begin{aligned} \mathbf{W}^3 &= \{(2, 2), (3, 3)\} \times \{(2, 2), (3, 3)\} \times \{(2, 2), (3, 3)\} \\ &= \{((2, 2), (2, 2), (2, 2)), ((2, 2), (2, 2), (3, 3)), \ldots\}. \end{aligned}$$

We shall calculate the first term of the star-product. It has a δ part and an ϵ one. For the δ part, we have

$$S = ((2,2),(2,2)) \text{ and } H = ((1),(1),(1)).$$

Let $K = ((1,2),(4,4,4))$. Then,

$$K * \{S, H\} = (\mathbf{i} * S) \oplus (\mathbf{j} * H) = (1,2) * ((2,2),(2,2)) \oplus (4,4,4) * ((1),(1),(1))$$
$$= ((2,2)\dot{+}1) \oplus ((2,2)\dot{+}2) \oplus ((1)\dot{+}4) \oplus ((1)\dot{+}4) \oplus ((1)\dot{+}4) = (3,3,4,4,5,5,5),$$

corresponds to the part $\delta_3^2\delta_4^2\delta_5^3$. Moreover,

$$K * \{R, W\} = K * \{((1),(1)),((2,2),(2,2),(2,2))\}$$
$$= (1)\dot{+}1 \oplus (1)\dot{+}2 \oplus (2,2)\dot{+}4 \oplus (2,2)\dot{+}4 \oplus (2,2)\dot{+}4$$
$$= (2,3,6,6,6,6,6,6).$$

This corresponds to the ϵ part $\epsilon_2\epsilon_3\epsilon_6^6$. Let us see now the coefficient

$$a_{(1,2),(4,4,4)} \cdot b_{(2,2),(1)} \cdot c_{(1),(2,2)} \cdot c_{(1),(2,2)} \cdot c_{(1),(2,2)} = 2 \cdot 1 \cdot 4 \cdot 4 \cdot 4 = 128.$$

Thus, the first term is $128\delta_3^2\delta_4^2\delta_5^3\epsilon_2\epsilon_3\epsilon_6^6$. Similarly, we calculate the next terms, and finally, we get

$$\begin{aligned} A * [B, C] &= 128\delta_3^2\delta_4^2\delta_5^3\epsilon_2\epsilon_3\epsilon_6^6 + 128\delta_2^2\delta_4^2\delta_5^3\epsilon_3^2\epsilon_6^+ 128\delta_3^4\delta_5^3\epsilon_2\epsilon_4\epsilon_6^6 + \cdots \\ &= 2(\delta_3^2\epsilon_2 + \delta_2^2\epsilon_3)(\delta_4^2\epsilon_3 + \delta_3^2\epsilon_4)(4\delta_5\epsilon_6^2 + \delta_7\epsilon_7^2)^3. \end{aligned}$$

2.4 The D-Operators

The scope of this section is to extend the previous notions in the case of two $\delta\epsilon$-polynomials. Actually, as we showed in the last section, a $\delta\epsilon$-polynomial can act, by means of the star product, on a pair of $\delta\epsilon$-polynomials, producing one $\delta\epsilon$-polynomial. Now, we want to act on a pair of $\delta\epsilon$-polynomials, getting a pair of $\delta\epsilon$-polynomials. We present now the D-operators. This is achieved by the so called D-operator This is the main tool we shall use later in order to describe nonlinear transformations of pair of sequences. The D-operator is nothing else, than a pair of $\delta\epsilon$-polynomials. In other words:

$$D = \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \sum\limits_{(\mathbf{i},\mathbf{j}) \in \mathbf{I_a} \times \mathbf{J_a}} a_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} \\ \sum\limits_{(\mathbf{i},\mathbf{j}) \in \mathbf{I_b} \times \mathbf{J_b}} b_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}} \end{bmatrix}.$$

If the above $\delta\epsilon$-polynomials are linear, then we speak about a linear D-operator. It has the form

$$L = \begin{bmatrix} \sum\limits_{i=0}^{k_1}(a_{1i}\delta_i + b_{1i}\epsilon_i) \\ \sum\limits_{i=0}^{k_2}(a_{2i}\delta_i + b_{2i}\epsilon_i) \end{bmatrix},$$

where some of the coefficients a_{si}, b_{si}, $s = 1, 2$, may be equal to zero. If instead of the $\delta\epsilon$-polynomials A and B we have the $\delta\epsilon$-series A and B, then the D-operator is called a D-series. The notion of D-operators has been studied exhaustively in the past, see [11], where a matrix like description has been used. We repeat here the main terminology.

Two D-operators are equal if and only if, their corresponding components are equal as $\delta\epsilon$-polynomials. In other words, they have the same sets of multiindices and the same coefficients.

We introduce two D-operators

$$D_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}.$$

Their dot product and star product are defined as

$$D_1 \cdot D_2 = \begin{bmatrix} A_1 \cdot A_2 \\ B_1 \cdot B_2 \end{bmatrix} \text{ and } D_1 * D_2 = \begin{bmatrix} A_1 * [A_2, B_2] \\ B_1 * [A_2, B_2] \end{bmatrix}.$$

We can extend all of the above to the case of $\delta\epsilon$-series, in a similar way.

2.5 Operators and Sequences

As we mentioned in the introduction, the D-operators are used for the description of polynomial discrete systems in a compact way. Before we present that explicitly, we show how the δ and $\delta\epsilon$-operators and the δ and $\delta\epsilon$-polynomials act on sequences producing new sequences. We shall proceed gradually, starting from δ-operators and arriving at the D-operators.

Consider the set of sequences

$$F = \{x(t) : \mathbb{N} \to \mathbb{R}, \text{ where } x(t) = 0, \text{ for } t < 0\} \subset \mathbb{R}^{\mathbb{N}},$$

a set arising from the sampling of continuous functions. Let us further consider a $\delta_{\mathbf{i}}$ operator, $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be a given multiindex. This operator defines a function $\Phi : F \to F$ as

$$\Phi(x(t)) = \delta_{\mathbf{i}} x(t) = x(t - i_1)x(t - i_2) \cdots x(t - i_n).$$

Many times, we shall use for this function the same symbol, as the one we used for the operator, that is $\delta_{\mathbf{i}}$. If $\mathbf{i} = i$, is just a positive integer, then $\delta_i x(t) = x(t-i)$, which means that δ_i coincides with the well-known shift operator. A special case is the operator δ_0, which leaves a sequence unchanged, that is, $\delta_0 x(t) = x(t)$. It is called the identity operator. For the sake of completeness, we define by convention that, $\delta_e x(t) = 1$. Using this type of action of the δ-operators upon sequences, we can define an external operation among δ-operators, namely addition, as

$$(\delta_{\mathbf{i}} + \delta_{\mathbf{j}})x(t) = \delta_{\mathbf{i}} x(t) + \delta_{\mathbf{j}} x(t).$$

The next proposition, which we state without a proof, will be useful in the sequel.

Proposition 2.19. *The following are true:*

a)
$$(\delta_{\mathbf{i}} + \delta_{\mathbf{j}}) \cdot \delta_{\mathbf{k}} = \delta_{\mathbf{i}} \cdot \delta_{\mathbf{k}} + \delta_{\mathbf{j}} \cdot \delta_{\mathbf{k}}.$$

b)
$$(\delta_{\mathbf{i}} + \delta_{\mathbf{j}}) * \delta_{\mathbf{k}} = \delta_{\mathbf{i}} * \delta_{\mathbf{k}} + \delta_{\mathbf{j}} * \delta_{\mathbf{k}}.$$

c)
$$\delta_{\mathbf{k}} * (\delta_{\mathbf{i}} + \delta_{\mathbf{j}}) \neq \delta_{\mathbf{k}} * \delta_{\mathbf{i}} + \delta_{\mathbf{k}} * \delta_{\mathbf{i}}.$$

The latter relation indicates that the set $(\Delta, +, *)$ of the δ-operators, equipped with the operations of addition and the star-product, is not a ring. The next theorem is a crucial one, since it reveals the role of the star product. Actually, it makes clear that it corresponds to the operation of composition among sequences.

Theorem 2.20. *Let $\delta_{\mathbf{i}}$ and $\delta_{\mathbf{j}}$ be δ-operators and $\mathbf{i} = (i_1, \ldots, i_n)$ and $\mathbf{j} = (j_1, \ldots, j_m)$ multi-indices. Consider the functions*

$$\Phi_1 : F \to F, \quad \Phi_1(x(t)) = \delta_{\mathbf{i}} x(t),$$

$$\Phi_2 : F \to F, \quad \Phi_2(x(t)) = \delta_{\mathbf{j}} x(t),$$

and
$$\Phi_3 : F \to F, \quad \Phi_3(x(t)) = \delta_{\mathbf{i}} * \delta_{\mathbf{j}} x(t).$$

Then, $\Phi_3 = \Phi_1 \circ \Phi_2$.

Proof. Let
$$\mathbf{i} = (i_1, i_2, \ldots, i_n) \text{ and } \mathbf{j} = (j_1, j_2, \ldots, j_m).$$

Set
$$w(t) = \Phi_2(x(t)) = \delta_{\mathbf{j}} x(t) = x(t - j_1) x(t - j_2) \cdots x(t - j_m).$$

Then,
$$\Phi_1 \circ \Phi_2(x(t)) = \Phi_1(w(t)) = w(t - i_1) w(t - i_2) \cdots w(t - i_n).$$

Substituting $w(t)$ by its equal, we take

$$\Phi_1(w(t)) = x(t - j_1 - i_1) x(t - j_2 - i_1) \cdots x(t - j_m - i_1) \cdots x(t - j_1 - i_1) x(t - j_2 - i_n)$$
$$\cdots x(t - j_m - i_n) = \delta_{\mathbf{j} \dot{+} i_1} x(t) \delta_{\mathbf{j} \dot{+} i_2} x(t) \cdots \delta_{\mathbf{j} \dot{+} i_n} x(t). \tag{2.2}$$

Taking into account that some of the delays may be equal, we finally get that (2.2) becomes
$$\delta_{(\mathbf{j} \dot{+} i_1) \oplus (\mathbf{j} \dot{+} i_2) \oplus \cdots \oplus (\mathbf{j} \dot{+} i_n)} x(t) = \delta_{\mathbf{i} * \mathbf{j}} x(t).$$

The proof is complete. □

The δ-polynomials work also as functions transforming sequences to sequences as follows. Let

$$A = \sum_{n=0}^{w} \sum_{\mathbf{i} \in \mathbf{I}_n} a_{\mathbf{i}} \delta_{\mathbf{i}},$$

be a δ-polynomial and $x(t)$ a sequence. Then

$$Ax(t) = \sum_{n=0}^{w} \sum_{\mathbf{i}=(i_1,\ldots,i_n) \in \mathbf{I}_n} a_{\mathbf{i}} x(t-i_1)x(t-i_2)\cdots x(t-i_n).$$

The star product corresponds to the composition as before. In other words, to the substitution of one polynomial into another. Indeed, if A,B are two δ-polynomials, defining the maps

$$A : F \to F, \text{ with } w(t) \to Aw(t)$$

and

$$B : F \to F, \text{ with } y(t) \to By(t).$$

Then, the polynomial which corresponds to the map $A \circ B : F \to F$, $A \circ By(t) = A(B(y(t))$ is the $A * B$. An addition of δ-polynomials is defined as $(A + B)x(t) = Ax(t) + Bx(t)$.

Proposition 2.21. *The following hold.*

(1)

$$[A + B] * C = A * C + B * C.$$

(2)

$$C * [A + B] \neq C * A + C * B.$$

Remark 2.22. The latter property means that the set of δ-polynomials equipped with the operation of the addition, is not a ring.

All the above are applied straightforward in the case of δ-series, too. We can extend the methodology so that to act not to a single sequence but to a pair of sequences. We can achieve that by means of the $\delta\epsilon$-operator. Indeed, let $\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}$ be a $\delta\epsilon$-operator, $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, $\mathbf{j} = (j_1, j_2, \ldots, j_m)$ two multi-indices. This operator defines a function $\Phi : F \times F \to F$ as

$$\Phi[x(t), y(t)] = \delta_{\mathbf{i}}\epsilon_{\mathbf{j}}[x(t), y(t)] = x(t-i_1)\cdots x(t-i_n)y(t-j_1)\cdots y(t-j_m).$$

Therefore, the δ-part of the $\delta\epsilon$-operator acts exclusively on the first sequence and the ϵ-part on the second. If

$$\text{either } \mathbf{j} = \{e\} \text{ or } \mathbf{i} = \{e\},$$

then

$$\delta_{\mathbf{i}}\epsilon_e[x(t), y(t)] = \delta_{\mathbf{i}}x(t) \text{ and } \delta_e\epsilon_{\mathbf{j}}[x(t), y(t)] = \epsilon_{\mathbf{j}}x(t).$$

We can define the addition as

$$(\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}+\delta_{\mathbf{i}'}\epsilon_{\mathbf{j}'})[x(t),y(t)]=\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}[x(t),y(t)]+\delta_{\mathbf{i}'}\epsilon_{\mathbf{j}'}[x(t),y(t)].$$

As in the case of simple δ-operators, we can prove here as well, that the distributive property does not hold. That is,

$$\delta_{\mathbf{k}}\epsilon_{\mathbf{h}}*(\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}+\delta_{\mathbf{i}'}\epsilon_{\mathbf{j}'})\neq\delta_{\mathbf{k}}\epsilon_{\mathbf{h}}*\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}+\delta_{\mathbf{k}}\epsilon_{\mathbf{h}}*\delta_{\mathbf{i}'}\epsilon_{\mathbf{j}'}.$$

Let $A=\sum_{n=0}^{\nu}\sum_{m=0}^{\mu}\sum_{(\mathbf{i},\mathbf{j})\in\mathbf{I}_n\times\mathbf{J}_m}c_{\mathbf{ij}}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}$ be a $\delta\epsilon$-polynomial. This polynomial acts on a pair of sequences as

$$A[x(t),y(t)]=\sum_{n=0}^{\nu}\sum_{m=0}^{\mu}\sum_{(\mathbf{i},\mathbf{j})\in\mathbf{I}_n\times\mathbf{J}_m}c_{\mathbf{ij}}x(t-i_1)\cdots x(t-i_n)y(t-j_1)\cdots y(t-j_m).$$

If A is a $\delta\epsilon$-series, then $A[x(t),y(t)]$ is a Volterra series, containing products among delays of $x(t)$ and $y(t)$. In the case of linear polynomials (or linear series) $A[x(t),y(t)]$ is a linear polynomial (or a linear series) of delays of $x(t)$ and $y(t)$. The star product among $\delta\epsilon$-operators (or $\delta\epsilon$-polynomials or $\delta\epsilon$-series) corresponds to the composition among maps. Indeed, let B,C,A be $\delta\epsilon$-polynomials. We define the maps

$$B:F\times F\to F,\ \ [w(t),v(t)]\to y(t)=B[w(t),v(t)],$$

$$C:F\times F\to F,\ \ [w(t),v(t)]\to u(t)=C[w(t),v(t)],$$

$$A:F\times F\to F,[y(t),u(t)]\to z(t)=A[y(t),u(t)].$$

The $\delta\epsilon$-polynomial, which corresponds to the composition

$$A\circ[B,C]:F\times F\to F,\ \ [w(t),v(t)]\to z(t),$$

is called the star product of the polynomials A,B,C and it is denoted by $A*[B,C]$.

Now, let

$$D=\begin{bmatrix}A\\B\end{bmatrix},$$

be a D-operator and A, B, $\delta\epsilon$-polynomials. This operator defines a function

$$Z:F\times F\to F\times F,$$

acting on a pair of sequences and producing pair of sequences as

$$Z[x(t),y(t)]=[A[x(t),y(t)],B[x(t),y(t)].$$

The star-product, between two D-operators corresponds, as before, to the composition of maps. Indeed, let us have two D-operators, D_1, D_2. We define the maps

$$Z_1 : F \times F \to F \times F, \ \ [w(t), r(t)] \to [x(t), y(t)], \ \text{with } [x(t), y(t)] = D_1[w(t), r(t)]$$

and

$$Z_2 : F \times F \to F \times F, \ \ [u(t), v(t)] \to [w(t), r(t)], \text{with } [w(t), r(t)] = D_2[w(t), r(t)].$$

Then, the D-operator which corresponds to the map

$$Z_1 \circ Z_2 : F \times F \to F \times F, \ \ [u(t), v(t)] \to [x(t), y(t)],$$

is the $D_1 * D_2$. The proof of this fact is a direct result of the definition of the star product.

Example 2.23. We introduce the discrete polynomial relations

$$x(t) = u^2(t-1) + v(t-1)u(t-2), \ \ y(t) = u(t-2) + 2v^2(t-1),$$

and

$$w(t) = x(t-1) - 5x(t-1)y(t-2).$$

Using the $\delta\epsilon$-polynomials notation, we rewrite them as

$$x(t) = A[u, v], \ \text{ where } \ A = \delta_1^2 + \delta_2\epsilon_1,$$

$$y(t) = B[u, v], \ \text{ where } \ B = \delta_2 + 2\epsilon_1^2,$$

and

$$w(t) = C[x, y], \ \text{ where } \ C = \delta_1 - 5\delta_1\epsilon_2.$$

Then,

$$\begin{aligned} w(t) &= C * [A, B][u, v] = [\delta_1 * A - 5(\delta_1 * A)(\epsilon_2 * B)][u, v] \\ &= [\delta_2^2 + \delta_3\epsilon_2 - 5(\delta_2^2 + \delta_3\epsilon_2)(\delta_4 + 2\epsilon_3^2)][u, v] \\ &= (\delta_2^2 + \delta_3\epsilon_2 - 5\delta_2^2\delta_4 - 10\delta_2^2\epsilon_3^2 - 5\delta_3\delta_4\epsilon_2 - 10\delta_3\epsilon_2\epsilon_3^2)[u, v] \\ &= u^2(t-2) + u(t-3)v(t-2) - 5u^2(t-2)u(t-4) - 10u^2(t-2)v^2(t-3) \\ &\quad -5u(t-3)u(t-4)v(t-2) - 10u(t-3)v(t-2)v^2(t-3). \end{aligned}$$

3 Nonlinear Discrete Polynomial Systems

In this section, we present how we can use the operators, developed previously, in order to describe nonlinear polynomial discrete systems. Let us start with polynomial discrete systems involving only one sequence. They have the form

$$x(n) = \sum_{k=1}^{\theta} \sum_{\substack{\mathbf{i} \in \mathbf{I}_k \\ \mathbf{i}=(i_1, i_2, \ldots, i_k)}} c_{\mathbf{i}} x(n-1-i_1)x(n-1-i_2)\cdots x(n-1-i_k), \tag{3.1}$$

with $c_{\mathbf{i}} \in \mathbf{R}$ and $\mathbf{I}_k$, a finite set of multiindices of dimension k. To this system, we assign a set of initial conditions, $C = \{\gamma_0, \gamma_1, \ldots, \gamma_{s-1}\} \subset \mathbf{R}$, if and only if,

$$x(0) = \gamma_0, \quad x(1) = \gamma_1, \ldots, x(s-1) = \gamma_{s-1},$$

where s is the maximum delay appeared in (3.1). Starting from these initial conditions and by using (3.1), we can calculate all the future evolution of the system, that is the quantities

$$x(s), x(s+1), x(s+2), \ldots .$$

Now, by using the δ-polynomial

$$A = \sum_{k=1}^{\theta} \sum_{\substack{\mathbf{i} \in \mathbf{I}_k \\ \mathbf{i}=(i_1,i_2,\ldots,i_k)}} c_{\mathbf{i}} \delta_{\mathbf{i}},$$

we can rewrite the above system, shortly as $x(n) = Ax(n-1)$. By means of this notation the evolution of the system is described through the star product. Indeed, it can be proved, as the next theorem states. For the proof see [11].

Theorem 3.1. *The quantity $x(t+n)$, $n \geq t$, is given by the relation*

$$x(t+n) = \underbrace{A * A * \cdots * A}_{n-times} x(t-1) = A^n x(t-1)$$

or equivalently $x(t) = \underbrace{A * A * \cdots * A}_{n-times} x(t-n)$.

Note that the same set of initial conditions C, has been used.

To this end, let us come to polynomial discrete systems with two components, that is, systems transforming a pair of sequences to a pair of sequences in a nonlinear polynomial way. We introduce the sequences $x_1(n), x_2(n)$ and the system

$$\left.\begin{array}{l} x_1(n) = \displaystyle\sum_{\alpha=1}^{\alpha'} \sum_{\beta=1}^{\beta'} \sum_{\substack{(\mathbf{i},\mathbf{j}) \in \mathbf{I}_\alpha \times \mathbf{J}_\beta \\ \mathbf{i}=(i_1,\ldots,i_\tau) \\ \mathbf{j}=(j_1,\ldots,j_\xi)}} c^{(1)}_{\mathbf{ij}} x_1(n-i_1) \cdots x_1(n-i_\tau) x_2(n-j_1) \cdots x_2(n-j_\xi) \\ x_2(n) = \displaystyle\sum_{\alpha=1}^{\alpha''} \sum_{\beta=1}^{\beta''} \sum_{\substack{(\mathbf{i}',\mathbf{j}') \in \mathbf{I}'_\alpha \times \mathbf{J}'_\beta \\ \mathbf{i}'=(i'_1,\ldots,i'_\tau) \\ \mathbf{j}'=(j'_1,\ldots,j'_\xi)}} c^{(2)}_{\mathbf{i}'\mathbf{j}'} x_1(n-i'_1) \cdots x_1(n-i'_{\tau'}) x_2(n-j'_1) \cdots x_2(n-j'_{\xi'}) \end{array}\right\} \quad (3.2)$$

where $\mathbf{I}_a, \mathbf{I}_\beta, \mathbf{J}'_a, \mathbf{J}'_\beta$ sets of multiindices of dimensions α and β respectively. To this system, we assign the following sets of initial values

$$C_1 = \{a_0, a_1, \ldots, a_{\rho-1}\} \text{ and } C_2 = \{b_0, b_1, \ldots, b_{\sigma-1}\}.$$

In fact, we have

$$x_1(0) = a_0, x_1(1) = a_1, \ldots, x_1(\rho - 1) = a_{\rho-1}$$

and

$$x_2(0) = b_0, x_2(1) = b_1, \ldots, x_2(\sigma - 1) = b_{\sigma-1},$$

where ρ and σ are the maximum delays of $\{x_1(n)\}$ and $\{x_2(n)\}$, respectively. If the quantities $\alpha', \beta', \alpha'', \beta''$ are not concrete numbers but equal to infinity or the sets $I_\alpha, J_\beta, I'_\alpha, J'_\beta$ have an infinite number of elements, then (3.2) becomes a Volterra series.

By means of the D-operators, we can rewrite (3.2) as

$$\mathbf{x}(n) = \mathbf{G}\mathbf{x}(n-1), \quad \mathbf{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

where G_1, G_2 are proper $\delta\epsilon$-polynomials and $\mathbf{G}$ the corresponding D-operator.

The next theorem, which lies in the same path with Theorem 3.1, describes the dynamic evolution of the afore mentioned polynomial systems.

Theorem 3.2. *The output,* $\mathbf{x}(n)$*, of a system* $\mathbf{x}(n) = \mathbf{G}\mathbf{x}(n-1)$*, at any time instant* $t+n$*,* $n \geq t$*, is given by the relation*

$$\mathbf{x}(t+n) = \underbrace{\mathbf{G} * \mathbf{G} * \cdots * \mathbf{G}}_{n-times}\mathbf{x}(t-1) = \mathbf{G}^n\mathbf{x}(t-1)$$

Proof. Straightforward from the definitions. □

The issue of the same dynamic behavior must be now under consideration. We have two nonlinear discrete systems

$$\mathbf{x}(n) = \mathbf{G}\mathbf{x}(n-1) \text{ and } \mathbf{z}(n) = \mathbf{F}\mathbf{z}(n-1),$$

where $\mathbf{G}, \mathbf{F}$ are D-operators,

$$\mathbf{x}(n) = (x_1(n), x_2(n)), \quad \mathbf{z}(n) = (z_1(n), z_2(n)),$$

and

$$I_{g,1} = \{a_0, a_1, \ldots, a_{\rho-1}\}, I_{g,2} = \{b_0, b_1, \ldots, b_{\sigma-1}\},$$

two sets of initial conditions. Let

$$m(\mathbf{G}) = \max\{m(G_1), m(G_2)\}, m(\mathbf{F}) = \max\{m(F_1), m(F_2)\},$$

and

$$\xi = \min\{m(\mathbf{G}), m(\mathbf{F})\}.$$

We say that the two systems operate under identical initial conditions if

$$x_1(i) = y_1(i) = a_i, x_2(i) = y_2(i) = b_i, \quad i = 0, \ldots, \xi,$$

provided that ξ appears in the first system (correspondingly to the second one). We use this system to produce the quantities

$$\mathbf{x}(j), \quad j = \xi + 1, \ldots, \max\{m(\mathbf{G}), \mathbf{m}(\mathbf{F})\}.$$

We set $\mathbf{z}(j) = \mathbf{x}(j)$ and then we use these values as initial conditions for the second system (correspondingly for the first one).

Definition 3.3. We say that two systems $\mathbf{x}(n) = \mathbf{G}\mathbf{x}(n-1)$ and $\mathbf{z}(n) = \mathbf{F}\mathbf{z}(n-1)$, $\mathbf{F}, \mathbf{G}$, D-operators, are equivalent, if $\mathbf{x}(n) = \mathbf{z}(n)$, $n = 1, 2, \ldots$, whenever they operate under identical initial conditions.

In this case, we write $\mathbf{G} \sim \mathbf{F}$. It is trivial to be seen that this notion is an equivalence relation. The next theorem combines equivalence of dynamical systems with equality of D-operators.

Theorem 3.4. *We introduce the systems* $\mathbf{x}(n) = \mathbf{G}\mathbf{x}(n-1)$ *and* $\mathbf{y}(n) = \mathbf{F}\mathbf{y}(n-1)$. *These systems are equivalent if and only if* $\mathbf{G} = \mathbf{F}$.

Proof. If $\mathbf{G} = \mathbf{F}$, then the systems are equivalent in a trivial way. The converse now. Let us suppose that the systems are equivalent. This means that $\mathbf{x}(n) = \mathbf{y}(n)$, whenever they operate under identical initial conditions. Let us suppose that

$$\mathbf{I}_{f,1} \neq \mathbf{I}_{g,1},$$

where $\mathbf{I}_{f,1}, \mathbf{I}_{g,1}$ are the multiindices sets. This implies that the $\delta\epsilon$-polynomial

$$F_1 = \sum_{(\mathbf{i},\mathbf{j}) \in \mathbf{I}_{f,1} \times \mathbf{J}_{f,1}} f^{(1)}_{(\mathbf{i},\mathbf{j})} \delta_{\mathbf{i}} \epsilon_{\mathbf{j}},$$

has at least one term namely, $f^{(1)}_{(a,b)} \delta_a \epsilon_b$, $f^{(1)}_{(a,b)} \neq 0$, which does not exist in G_1. The equality $\mathbf{x}(n) = \mathbf{y}(n)$ gives $x_1(n) = y_1(n)$ or $G_1 x(n-1) = F_1 y(n-1)$, $G_1 x(n-1) - F_1 y(n-1) = 0$, $(G_1 - F_1)\mathbf{x}(n-1) = 0$. Since this equality is valid for any sequence $\mathbf{x}(n)$, we conclude that the coefficients of the polynomial $G_1 - F_1$ must be equal to zero, and thus $f^{(1)}_{(a,b)} = 0$, a contradiction. Thus, the polynomials G_1, F_1 do not contain isolated terms. Arguing then in a similar way, we conclude that all the coefficients $f^{(1)}_{(\mathbf{i},\mathbf{j})}$ and $g^{(1)}_{(\mathbf{i},\mathbf{j})}$ are equal. Repeating this procedure for the second pair of polynomials, we finally get that $\mathbf{G} = \mathbf{F}$. □

3.1 Linear Systems

A special class of δ or ϵ-polynomials are the so-called linear polynomials. We present them here briefly, just to indicate that in the linear case the algebraic tools, presented in this manuscript, coincide with the classical ones. To avoid strong complexity of the presentation, we shall be restricted here only to simple δ-operators. All the results can be extended to all the other cases (ϵ and D operators) in a straightforward way. Besides, we notice that a pure $\delta\epsilon$-polynomial, that is a $\delta\epsilon$-polynomial without δ or ϵ terms is nonlinear in nature.

Let A be a linear δ-polynomial, that is

$$A = \sum_{i=0}^{\theta} a_i \delta_i, \quad a_i \in \mathbf{R}.$$

The expression

$$\begin{aligned} x(t) &= Ax(t-1) = a_0 x(t-1) + a_1 x(t-2) + \cdots + a_\theta x(t-1-\theta) \\ &= \sum_{i=0}^{\theta} a_i x(t-i-1) \end{aligned}$$

accompanied by a set of initial conditions, is a linear discrete dynamical system with maximum delay equal to θ.

The following property is very useful. It says that working with linear δ-polynomials and the star product is like working with polynomials of a single variable and the classical product among them.

Proposition 3.5. *Let* $\mathbf{F}$ *be the set of linear* δ*-polynomials. Then the set* $(\mathbf{F}, *, +)$ *is a commutative ring, and it is isomorphic to the ring* $(\mathbf{R}[x], \cdot, +)$*, where* $\mathbf{R}[x]$ *is the set of real polynomials of a single variable and* $\cdot$ *the operation of the polynomial product.*

Proof. That $(\mathbf{F}, *, +)$ is a commutative ring comes as a straightforward result from Proposition 2.9. To any linear δ-polynomial $M = \sum_{i=0}^{k} m_i \delta_i$, we correspond a real polynomial

$$\bar{M} = \sum_{i=0}^{k} m_i x^i,$$

defining hence the map

$$\varphi : (\mathbf{F}, *, +) \to (\mathbf{R}[x], \cdot, +), \varphi(M) = \bar{M}.$$

It can be easily proved that

$$\varphi(M+N) = \varphi(M) + \varphi(N),$$

where N is another linear δ-polynomial and that φ is one-to-one and onto. Now, let

$$N = \sum_{j=0}^{h} n_j \delta_j.$$

Then

$$M * N = \sum_{i=0}^{k} \sum_{j=0}^{h} m_i n_j \delta_{i+j}.$$

This means that

$$\begin{aligned}\varphi(M * N) &= \sum_{i=0}^{k} \sum_{j=0}^{h} m_i n_j x^{i+j} = \sum_{\theta=0}^{k+h} \left(\sum_{s=0}^{\theta} m_s n_{\theta-s} \right) x^{\theta} \\ &= \left(\sum_{i=0}^{k} m_i x^i \right) \cdot \left(\sum_{j=0}^{h} n_j x^j \right) = \varphi(M) \cdot \varphi(N).\end{aligned}$$

This relation ensures that ϕ is an isomorphism. The proof is complete. □

The next lemma ensures that in the case of simple linear systems the description of the dynamic evolution of the system by means of δ-polynomials and the star product, coincides with the well-known state space description of the literature. See [2].

Lemma 3.6. *We introduce the linear discrete dynamical system* $x(t) = Ax(t-1)$*, where* A *is the* δ*-polynomial* $A = \sum_{i=0}^{\theta} a_i \delta_i$*. If we define* $\mathbf{x}(t)$ *to be the vector*

$$\mathbf{x}(t) = [x(t), x(t-1), \ldots, x(t-\theta)]^T,$$

then $\mathbf{x}(t) = \mathcal{A}\mathbf{x}(t-1)$*, where* $\mathcal{A}$ *is the matrix*

$$\mathcal{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{\theta-1} & a_\theta \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

This vector, $\mathbf{x}(t)$*, is called the state and the above description, the state space description.*

Proof. The proof can be found in any classical textbook of discrete dynamical systems, see for instance [2]. □

Theorem 3.7. *We introduce the linear discrete dynamical system* $x(t) = Ax(t-1)$*, where* A *is the* δ*-polynomial* $A = \sum_{i=0}^{\theta} a_i \delta_i$ *and* $\mathbf{x}(t) = \mathcal{A}\mathbf{x}(t-1)$ *its state-space expression,* $\mathcal{A}$ *and* $\mathbf{x}(t)$ *the matrix and the state vector of Lemma 3.6. Then,*

$$\mathbf{x}(t) = \mathcal{A}^n \mathbf{x}(t-n), \;\; \textit{for all} \;\; n \geq 0,$$

where $\mathcal{A}^n$ *has been calculated with respect to the classical matrix product.*

Proof. We shall work by induction. Let us start with $n = 2$, we have to prove that $\mathbf{x}(t) = \mathcal{A}^2\mathbf{x}(t-2)$. We know that $x(t) = Ax(t-1)$ and thus $x(t) = A(Ax(t-2))= A * Ax(t-2)$. But,

$$A * Ax(t-2) = a_0 Ax(t-2) + \sum_{i=1}^{\theta} a_i(\delta_i * A)x(t-2). \tag{3.3}$$

Since $x(t-1) = Ax(t-2)$, we conclude that

$$(\delta_i * A)x(t-2) = \delta_i * x(t-1) = x(t-1-i).$$

Thus, (3.3) becomes

$$\begin{aligned} x(t) &= a_0 Ax(t-2) + \sum_{i=1}^{\theta} a_i x(t-i-1) \\ &= a_0 \left(\sum_{i=0}^{\theta} a_i\delta_i\right) x(t-2) + \sum_{i=1}^{\theta} a_i x(t-i-1) \\ &= \sum_{i=0}^{\theta-1}(a_0a_i + a_{i+1})x(t-2-i) + a_0a_\theta x(t-2-\theta) \\ &= \left[\sum_{i=0}^{\theta-1}(a_0a_i + a_{i+1})\delta_i + a_0a_\theta\delta_\theta\right] x(t-2). \end{aligned}$$

Going into state-space expression, we get $\mathbf{x}(t) = \mathcal{C}\mathbf{x}(t-2)$, with

$$\mathcal{C} = \begin{bmatrix} a_0^2 + a_1 & a_0a_1 + a_2 & a_0a_2 + a_3 & . & . & . & a_0a_\theta \\ a_0 & a_1 & a_2 & . & . & . & a_\theta \\ 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & 1 & 0 & 0 \end{bmatrix}.$$

Some simple manipulations can prove that $\mathcal{C} = \mathcal{A} \cdot \mathcal{A} = \mathcal{A}^2$ and the theorem has been proved for $n = 2$. Let us now examine the case where, $n = 3$. Obviously,

$$x(t) = A * A * Ax(t-3) = A * A^2x(t-3).$$

But,

$$\begin{aligned} A * A^2x(t-3) &= \left(\sum_{i=0}^{\theta} a_i\delta_i\right) * A^2x(t-3) \\ &= a_0A^2x(t-3) + \sum_{i=1}^{\theta} a_i(\delta_i * A^2)x(t-3). \end{aligned} \tag{3.4}$$

Furthermore, in view of the fact that,

$$x(t-2) = Ax(t-3)$$

and

$$a_2\delta_2 * A * Ax(t-3) = a_2\delta_2 Ax(t-2) = a_2\delta_2 x(t-1) = a_2 x(t-3),$$

we find that

$$a_0 A * Ax(t-3) = a_0 \left(\sum_{i=0}^{\theta} a_i \delta_i \right) * Ax(t-3)$$

and

$$a_1\delta_1 * A^2 x(t-3) = a_1 A * \delta_1 Ax(t-3) = a_1 Ax(t-3).$$

Substituting these results and executing the operations, (3.4) becomes

$$x(t) = (a_0^3 + 2a_0a_1 + a_2)x(t-3) + (a_0^2 a_1 + a_0 a - 2 + a_1^2 + a_3)x(t-4)$$

$$+(a_0^2 a_2 + a_1 a_2 + a_0 a_3 + a_4)x(t-5) + \cdots + (a_0^2 a_\theta + a_1 a_\theta)x(t-3-\theta).$$

So, going into state-space expression, we get $\mathbf{x}(t) = \mathcal{C}\mathbf{x}(t-3)$, with

$$\mathcal{C} = \begin{bmatrix} b_0 & b_1 & b_2 & . & . & . & b_\theta \\ a_0^2 + a_1 & a_0a_1 + a_2 & a_0a_2 + a_3 & . & . & . & a_0 a_\theta \\ a_0 & a_1 & a_2 & . & . & . & a_\theta \\ 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & 1 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} b_0 &= a_0^3 + 2a_0a_1 + a_2, \\ b_1 &= a_0^2 a_1 + a_0 a_2 + a_1^2 + a_3, \\ b_2 &= a_0^2 a_2 + a_1 a_2 + a_0 a_3 + a_4, \\ &\vdots \\ b_\theta &= a_0^2 a_\theta + a_1 a_\theta. \end{aligned}$$

By executing the operations, we deduce easily that $\mathcal{C} = \mathcal{A}^3$ and the result is valid for $n = 3$. Working similarly, we can prove the theorem for any n. The proof is complete. □

4 Series-Similarity

In this section, we examine how a nonlinear system is similar (equivalent) to another one. This can be achieved by transforming the dynamic behavior of the given dynamical system, so that its output to be identical equal with the output of another dynamical system, called the desired one, under the same initial conditions (Section 3). It is of prime interest to work with linear desired systems since this will determine how complicated the original system is, defining henceforth its complexity level. Indeed, a nonlinear system equivalent to a linear one, can be considered as less complex than another one which is not. The whole approach will be relied on a proper transformation of the system, obtained by means of the star product and the D-series. We present now the relevant definitions.

Definition 4.1. A D-series $\mathbf{T}$ is called invertible if we can find another D-series $\mathbf{T}'$, such that $\mathbf{T}' * \mathbf{T} = \begin{bmatrix} \delta_0 \\ \epsilon_0 \end{bmatrix}$.

Definition 4.2. Two pairs of sequences

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \text{ and } \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},$$

are called Series Similar, or briefly S-similar, if there exists a nontrivial invertible D-series $\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$, such that $\mathbf{y}(t) = \mathbf{T}\mathbf{x}(t)$.

The meaning of the above definition is that, by means of T, we can go from $\mathbf{x}(t)$ to $\mathbf{y}(t)$ and vice-versa. Let us now see how we can extend this notion in order D-operators to be involved.

Definition 4.3. Let

$$\mathbf{G} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \text{ and } \mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

be two D-operators. They are called S-similar, if we can find a D-series, $\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$, such that

$$F_1 * [T_1, T_2] = T_1 * [G_1, G_2], \; F_2 * [T_1, T_2] = T_2 * [G_1, G_2],$$

or shortly

$$\mathbf{F} * \mathbf{T} = \mathbf{T} * \mathbf{G}.$$

Theorem 4.4. *Let* $\mathbf{x}(n) = \mathbf{G}\mathbf{x}(n-1)$, $\mathbf{y}(n) = \mathbf{F}\mathbf{y}(n-1)$ *be two systems. The sequences* $\mathbf{x}(n), \mathbf{y}(n)$ *are S-similar, if and only if, the D-operators* $\mathbf{G}, \mathbf{F}$ *are S-similar.*

Proof. Let us suppose that $\mathbf{x}(n)$ and $\mathbf{y}(n)$ are S-similar. This means that $\mathbf{y}(n) = \mathbf{T}\mathbf{x}(n)$ for some invertible D-series $\mathbf{T}$. Hence,

$$\mathbf{F}\mathbf{y}(n-1) = \mathbf{T} * \mathbf{G}\mathbf{x}(n-1),$$

but

$$\mathbf{y}(n-1) = \mathbf{T}\mathbf{x}(n-1),$$

and so,

$$\mathbf{F} * \mathbf{T}\mathbf{x}(n-1) = \mathbf{T} * \mathbf{G}\mathbf{x}(n-1).$$

Using Theorem 3.4, we get

$$\mathbf{F} * \mathbf{T} = \mathbf{T} * \mathbf{G}.$$

The inverse now. The relation

$$\mathbf{x}(n) = \mathbf{G}\mathbf{x}(n-1),$$

implies

$$\mathbf{T} * \mathbf{x}(n) = \mathbf{T} * \mathbf{G}\mathbf{x}(n-1),$$

but

$$\mathbf{T} * \mathbf{G} = \mathbf{F} * \mathbf{T},$$

and so,

$$\mathbf{T} * \mathbf{x}(\mathbf{n}) = \mathbf{F} * \mathbf{T}\mathbf{x}(n-1).$$

Setting,

$$\mathbf{w}(n) = \mathbf{T}\mathbf{x}(n-1),$$

we get

$$\mathbf{w}(n) = \mathbf{F}\mathbf{w}(n-1),$$

but

$$\mathbf{y}(n) = \mathbf{F}\mathbf{y}(n-1),$$

from which it follows that,

$$\mathbf{w}(n) = \mathbf{y}(n).$$

Hence,

$$\mathbf{y}(n) = \mathbf{T}\mathbf{x}(n-1)$$

and the sequences are S-similar. The proof is complete. □

Theorem 4.5. *S-similarity is an equivalence relation among D-operators.*

Proof. We shall prove that the three properties of an equivalence relation are satisfied. First,

$$\mathbf{G} * \begin{bmatrix} \delta_0 \\ \epsilon_0 \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \epsilon_0 \end{bmatrix} * \mathbf{G},$$

and thus the reflexive property is satisfied, with $\mathbf{T} = \begin{bmatrix} \delta_0 \\ \epsilon_0 \end{bmatrix}$. Secondly, let us suppose that $\mathbf{F}$ and $\mathbf{G}$ are $\mathbf{T}$-similar. This means that $\mathbf{F} * \mathbf{T} = \mathbf{T} * \mathbf{G}$. Since $\mathbf{T}^{-1}$ exists, with $d(\mathbf{T}^{-1}) = 0$, we get

$$\mathbf{T}^{-1} * \mathbf{F} * \mathbf{T} = \mathbf{T}^{-1} * \mathbf{T} * \mathbf{G},$$

or

$$\mathbf{T}^{-1} * \mathbf{F} * \mathbf{T} = \mathbf{G} \text{ and } \mathbf{T}^{-1} * \mathbf{F} * \mathbf{T} * \mathbf{T}^{-1} = \ G * \mathbf{T}^{-1}.$$

Finally,

$$\mathbf{T}^{-1} * \mathbf{F} = \ G * \mathbf{T}^{-1}$$

or

$$G * \mathbf{T}^{-1} = \mathbf{T}^{-1} * \mathbf{F},$$

and so, $\mathbf{G}$ is S-similar to $\mathbf{F}$ by means of the $\mathbf{T}^{-1}$ series and the symmetric property is valid. For the transitive property, we have

$$\mathbf{F} \text{ is } \mathbf{T} - \text{similar to } \mathbf{G} \Rightarrow \mathbf{F} * \mathbf{T} = \mathbf{T} * \mathbf{G},$$

and if

$$\mathbf{G} \text{ is } \mathbf{T} - \text{similar to } \mathbf{U} \Rightarrow \mathbf{F} * \mathbf{\Sigma} = \mathbf{\Sigma} * \mathbf{U}.$$

From the first one, we have

$$\mathbf{F} * \mathbf{T} * \mathbf{\Sigma} = \mathbf{T} * \mathbf{G} * \mathbf{\Sigma}.$$

The second one gives

$$\mathbf{F} * \mathbf{T} * \mathbf{\Sigma} = \mathbf{T} * \mathbf{\Sigma} * \mathbf{U},$$

which means that $\mathbf{F}$ is S-similar to $\mathbf{U}$, with respect to the series $\mathbf{T} * \mathbf{\Sigma}$. All the above ensure that this is an equivalence relation. The proof is complete. □

The most interesting situation is when a system is S-similar with a linear one. In this case, we speak for the linear S-similarity. In other words, let us suppose that we have the given nonlinear D-operator $\mathbf{G}$ and the linear one $\mathbf{L}$. We want to find a D-series $\mathbf{T}$, such that $\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G}$. We must note here the similarity between the latter relation and the well-known eigenvector-eigenvalues equation of linear transformations. In what follows, we shall work with the said problem. First we shall construct a procedure algorithm for solving the problem and then we shall present some theoretical results.

4.1 Computation of the Series $\mathbf{T}$

In this section, we shall establish a procedure dealing with the computation of the series $\mathbf{T}$. Our manipulation philosophy is that we shall solve the problem gradually working step by step. We shall start with the calculation of the linear part of $\mathbf{T}$, working then with the quadratic part, the cubic and so on. At each step the results of the previous step are used in the calculation of the current part.

Let us suppose that the D-operator $\mathbf{G}$ is S-similar to the linear system $\mathbf{L}$ with respect to the series $\mathbf{T}$, i.e., $\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G}$. In order to describe a calculation procedure for the series $\mathbf{T}$, we adopt the notations below, which have been introduced in Subsection 2.3:

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},$$

with

$$L_\theta = L_\theta^{(1,0)} + L_\theta^{(0,1)},\ L_\theta^{(1,0)} = \sum_{i=0}^{\nu} l_{\theta,i}^{(1,0)}\delta_i,\ L_\theta^{(0,1)} = \sum_{i=0}^{\nu} l_{\theta,i}^{(0,1)}\epsilon_i, \quad \theta = 1,2,$$

$$T_\theta = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} T_\theta^{(a,b)},\ T_\theta^{(a,b)} = \sum_{(\mathbf{i},\mathbf{j})\in \mathbf{I}\times\mathbf{J}} t_{\theta,(\mathbf{i},\mathbf{j})}^{(a,b)}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}, \quad \theta = 1,2,$$

$$G_\theta = \sum_{a=0}^{a'}\sum_{b=0}^{b'} G_\theta^{(a,b)},\ G_\theta^{(a,b)} = \sum_{(\mathbf{i},\mathbf{j})\in \mathbf{I}\times\mathbf{J}} g_{\theta,(\mathbf{i},\mathbf{j})}^{(a,b)}\delta_{\mathbf{i}}\epsilon_{\mathbf{j}}, \quad \theta = 1,2.$$

THE PROCEDURE

STEP 0: After substituting the above expressions into the main relation, we successively have

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} * \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} * \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

or equivalently

$$L_1 * [T_1, T_2] = T_1 * [G_1, G_2] \Leftrightarrow (L_1^{(1,0)} + L_1^{(0,1)}) * [T_1, T_2] = T_1 * [G_1, G_2]$$

$$L_2 * [T_1, T_2] = T_2 * [G_1, G_2] \Leftrightarrow (L_2^{(1,0)} + L_2^{(0,1)}) * [T_1, T_2] = T_2 * [G_1, G_2]$$

and finally

$$L_1^{(1,0)} * \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} T_1^{(a,b)} + L_1^{(0,1)} * \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} T_2^{(a,b)} = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} T_1^{(a,b)} * [G_1, G_2]$$

and

$$L_2^{(1,0)} * \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} T_1^{(a,b)} + L_2^{(0,1)} * \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} T_2^{(a,b)} = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} T_2^{(a,b)} * [G_1, G_2].$$

STEP 1 – THE LINEAR PART: The linear parts of the series $\mathbf{T}_\theta$, $\theta = 1, 2$ are those for which $a + b = 1$, a, b positive integers. By equating those linear terms in the above equation, we get the relations

$$\begin{aligned} L_1^{(1,0)} * T_1^{(1,0)} + L_1^{(0,1)} * T_2^{(1,0)} &= T_1^{(1,0)} * G_1^{(1,0)} + T_1^{(0,1)} * G_2^{(1,0)}, \\ L_1^{(1,0)} * T_1^{(0,1)} + L_1^{(0,1)} * T_2^{(0,1)} &= T_1^{(1,0)} * G_1^{(0,1)} + T_1^{(0,1)} * G_2^{(0,1)}, \\ L_2^{(1,0)} * T_1^{(1,0)} + L_2^{(0,1)} * T_2^{(1,0)} &= T_2^{(1,0)} * G_1^{(1,0)} + T_2^{(0,1)} * G_2^{(1,0)}, \\ L_2^{(1,0)} * T_1^{(0,1)} + L_2^{(0,1)} * T_2^{(0,1)} &= T_2^{(1,0)} * G_1^{(0,1)} + T_2^{(0,1)} * G_2^{(0,1)}. \end{aligned} \tag{4.1}$$

This is a homogeneous system of four equations with unknowns the quantities

$$T_1^{(1,0)},\ T_2^{(1,0)},\ T_1^{(0,1)},\ T_2^{(0,1)}.$$

STEP 2 – THE QUADRATIC PART: Let us work now with the quadratic parts $T_\theta^{(a,b)}$ of the series $\mathbf{T}_\theta$, $\theta = 1, 2$. That is when $a + b = 2$, a, b positive integers. The equations here will arise by equating the coefficients of the $\delta_i\delta_j$-terms, that is when $a = 2$, $b = 0$, the $\epsilon_i\epsilon_j$-terms, that is when $a = 0$, $b = 2$ and the $\delta_i\epsilon_j$-terms, that is when $a = 1$, $b = 1$. We shall first work with the $\delta_i\delta_j$-terms.

$$\begin{aligned} L_1^{(1,0)} * T_1^{(2,0)} + L_1^{(0,1)} * T_2^{(2,0)} &= T_1^{(1,0)} * G_1^{(2,0)} + T_1^{(0,1)} * G_2^{(2,0)} \\ &+T_1^{(2,0)} * G_1^{(1,0)} + T_1^{(0,2)} * G_2^{(1,0)} + T_1^{(1,1)} * [G_1^{(1,0)}, G_2^{(1,0)}], \\ L_2^{(1,0)} * T_1^{(2,0)} + L_2^{(0,1)} * T_2^{(2,0)} &= T_2^{(1,0)} * G_1^{(2,0)} + T_2^{(0,1)} * G_2^{(2,0)} \\ &+T_2^{(2,0)} * G_1^{(1,0)} + T_2^{(0,2)} * G_2^{(1,0)} + T_2^{(1,1)} * [G_1^{(1,0)}, G_2^{(1,0)}]. \end{aligned} \tag{4.2}$$

For the $\epsilon_i\epsilon_j$-terms, we have

$$\begin{aligned} L_1^{(1,0)} * T_1^{(0,2)} + L_1^{(0,1)} * T_2^{(0,2)} &= T_1^{(1,0)} * G_1^{(0,2)} + T_1^{(0,1)} * G_2^{(0,2)} \\ &+T_1^{(2,0)} * G_1^{(0,1)} + T_1^{(0,2)} * G_2^{(0,1)} + T_1^{(1,1)} * [G_1^{(0,1)}, G_2^{(0,1)}], \\ L_2^{(1,0)} * T_1^{(0,2)} + L_2^{(0,1)} * T_2^{(0,2)} &= T_2^{(1,0)} * G_1^{(0,2)} + T_2^{(0,1)} * G_2^{(0,2)} \\ &+T_2^{(2,0)} * G_1^{(0,1)} + T_2^{(0,2)} * G_2^{(0,1)} + T_2^{(1,1)} * [G_1^{(0,1)}, G_2^{(0,1)}], \end{aligned} \tag{4.3}$$

and for the $\delta_i\epsilon_j$-terms, we have

$$\begin{aligned} L_1^{(0,1)} * T_1^{(1,1)} + L_1^{(0,1)} * T_2^{(1,1)} &= T_1^{(0,1)} * G_1^{(1,1)} + T_1^{(0,1)} * G_2^{(1,1)} \\ &+T_1^{(1,1)} * [G_1^{(0,1)}, G_2^{(1,0)}] + T_1^{(1,1)} * [G_1^{(1,0)}, G_2^{(0,1)}], \\ L_2^{(0,1)} * T_1^{(1,1)} + L_2^{(0,1)} * T_2^{(1,1)} &= T_2^{(0,1)} * G_1^{(1,1)} + T_2^{(0,1)} * G_2^{(1,1)} \\ &+T_2^{(1,1)} * [G_1^{(0,1)}, G_2^{(1,0)}] + T_2^{(1,1)} * [G_1^{(1,0)}, G_2^{(0,1)}]. \end{aligned} \tag{4.4}$$

The unknown polynomials are

$$T_1^{(2,0)}, T_2^{(2,0)}, T_1^{(0,2)}, T_2^{(0,2)}, T_1^{(1,1)}, T_2^{(1,1)}.$$

We transform now the above equations to relations among the coefficients of the unknowns polynomials. Let us start with an arbitrary term of the form $\delta_i\delta_j$. Equations (4.2) will give

$$\left[\sum_{a*(b_1,b_2)=(i,j)} \left(l_{\phi,a}^{(1,0)} t_{1,(b_1,b_2)}^{(2,0)} + l_{\phi,a}^{(0,1)} t_{2,(b_1,b_2)}^{(2,0)}\right)\right] \delta_i\delta_j$$

$$= \left[\sum_{a*(b_1,b_2)=(i,j)} \left(t_{\phi,a}^{(1,0)} g_{1,(b_1,b_2)}^{(2,0)} + t_{\phi,a}^{(0,1)} g_{2,(b_1,b_2)}^{(2,0)}\right) + \sum_{(a_1,a_2)*(b_1,b_2)=(i,j)} t_{\phi,(a_1,a_2)}^{(2,0)}\, g_{1,b_1}^{(1,0)}\, g_{1,b_2}^{(1,0)}\right.$$

$$+ \sum_{\substack{\mathbf{b}=(b_1,b_2)\in(\cup J_m)^2 \\ (a_1,a_2)*\mathbf{b}=(i,j)}} t_{\phi,(a_1,a_2)}^{(2,0)}\, g_{1,b_1}^{(1,0)} g_{1,b_2}^{(1,0)} + \sum_{\substack{\mathbf{b}=(b_1,b_2)\in(\cup J_m)^2 \\ (a_1,a_2)*\mathbf{b}=(i,j)}} t_{\phi,(a_1,a_2)}^{(0,2)}\, g_{2,b_1}^{(1,0)} g_{2,b_2}^{(1,0)}$$

$$\left. + \sum_{\substack{\mathbf{b}=(b_1,b_2)\in J_1\times J_1' \\ (a_1,a_2)*\mathbf{b}=(i,j)}} t_{\phi,(a_1,a_2)}^{(1,1)}\, g_{1,b_1}^{(1,0)} g_{2,b_2}^{(1,0)}\right] \delta_i\delta_j, \quad \phi = 1,2. \tag{4.5}$$

Note that, the star product operation among the indices has been defined in Section 2. For an arbitrary term of the form $\epsilon_i\epsilon_j$, (4.3) will give

$$\left[\sum_{a*(b_1,b_2)=(i,j)} \left(l_{\phi,a}^{(1,0)} t_{1,(b_1,b_2)}^{(0,2)} + l_{\phi,a}^{(0,1)} t_{2,(b_1,b_2)}^{(0,2)}\right)\right] \epsilon_i\epsilon_j$$

$$= \left[\sum_{a*(b_1,b_2)=(i,j)} \left(t_{\phi,a}^{(1,0)} g_{1,(b_1,b_2)}^{(0,2)} + t_{\phi,a}^{(0,1)} g_{2,(b_1,b_2)}^{(0,2)}\right) + \sum_{(a_1,a_2)*(b_1,b_2)=(i,j)} t_{\phi,(a_1,a_2)}^{(2,0)} g_{1,b_1}^{(0,1)} g_{1,b_2}^{(0,1)}\right.$$

$$+ \sum_{\substack{\mathbf{b}=(b_1,b_2)\in(\cup J_m)^2 \\ (a_1,a_2)*\mathbf{b}=(i,j)}} t_{\phi,(a_1,a_2)}^{(2,0)} g_{1,b_1}^{(0,1)} g_{1,b_2}^{(0,1)} + \sum_{\substack{\mathbf{b}=(b_1,b_2)\in(\cup J_m)^2 \\ (a_1,a_2)*\mathbf{b}=(i,j)}} t_{\phi,(a_1,a_2)}^{(0,2)} g_{2,b_1}^{(0,1)} g_{2,b_2}^{(0,1)}$$

$$\left. + \sum_{\substack{\mathbf{b}=(b_1,b_2)\in J_1\times J_1' \\ (a_1,a_2)*\mathbf{b}=(i,j)}} t_{\phi,(a_1,a_2)}^{(1,1)} g_{1,b_1}^{(0,1)} g_{2,b_2}^{(0,1)}\right] \epsilon_i\epsilon_j, \quad \phi = 1,2. \tag{4.6}$$

Finally, for the arbitrary term $\delta_i\epsilon_j$, (4.4) will give

$$\sum_{a*(b_1,b_2)=(i,j)} \left(l_{\phi,a}^{(1,0)} t_{1,(b_1,b_2)}^{(1,1)} + l_{\phi,a}^{(0,1)} t_{2,(b_1,b_2)}^{(1,1)} \right) \quad \delta_i\epsilon_j$$

$$= \left[\sum_{a*(b_1,b_2)=(i,j)} \left(t_{\phi,a}^{(1,0)} g_{1,(b_1,b_2)}^{(1,1)} + t_{\phi,a}^{(0,1)} g_{2,(b_1,b_2)}^{(1,1)} \right) + \right.$$

$$\left. + \sum_{\substack{\mathbf{b}=(b_1,b_2)\in J_1\times J_1' \\ (a_1,a_2)*\mathbf{b}=(i,j)}} \left(t_{\phi,(a_1,a_2)}^{(1,1)} g_{1,b_1}^{(1,0)} g_{2,b_2}^{(0,1)} + t_{\phi,(a_1,a_2)}^{(1,1)} g_{1,b_1}^{(0,1)} g_{2,b_2}^{(1,0)} \right) \right] \delta_i\epsilon_j, \qquad (4.7)$$

$\phi = 1, 2$. The above equations, despite their apparent complexity, form a system of linear equations with unknowns the coefficients of the polynomials

$$T_1^{(2,0)},\ T_2^{(2,0)},\ T_1^{(0,2)},\ T_2^{(0,2)},\ T_1^{(1,1)},\ T_2^{(1,1)}.$$

STEP 3 – THE k–PART: Let us work with the k-degree terms of the relation $\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G}$, that is terms of the form

$$\delta_{i_1}\delta_{i_2}\cdots\delta_{i_n}\epsilon_{j_1}\epsilon_{j_2}\cdots\epsilon_{j_m}, \quad n+m=k.$$

In the series $\mathbf{T}$ they will be produced by the parts

$$T_i^{(k,0)}, T_i^{(k-1,1)}, T_i^{(k-2,2)}, \ldots, T_i^{(k-b,b)}, \ldots, T_i^{(0,k)},\ i=1,2.$$

These are $2(k+1)$ series, which are unknown and have to be determined. For each of them, we form an equation. Therefore, we have a system of $2(k+1)$ equations, with $2(k+1)$ unknowns. A such equation devoted to the term (n,m) has the form

$$L_i^{(1,0)} * T_1^{(n,m)} + L_i^{(0,1)} * T_2^{(n,m)}$$

$$= \sum_{\substack{ax_1+bx_2=n \\ ay_1+by_2=m \\ n+m=k \\ a,b,x_1,y_1,x_2,y_2\in\mathbf{N}}} T_i^{(a,b)} * \left[G_1^{(x_1,y_1)}, G_2^{(x_2,y_2)} \right], \quad i=1,2.$$

Analyzing the sum on the right-hand side of the above equation, we get

$$L_i^{(1,0)} * T_1^{(n,m)} + L_i^{(0,1)} * T_2^{(n,m)} = \sum_{\substack{a+b<k \\ a(x_1+y_1)+b(x_2+y_2)=k \\ a,b,x_1,y_1,x_2,y_2\in\mathbf{N}}} T_i^{(a,b)} * \left[G_1^{(x_1,y_1)}, G_2^{(x_2,y_2)} \right]$$

$$+ \sum_{\substack{a+b=k \\ a(x_1+y_1)+b(x_2+y_2)=k \\ a,b,x_1,y_1,x_2,y_2 \in \mathbf{N}}} T_i^{(a,b)} * \left[G_1^{(x_1,y_1)}, G_2^{(x_2,y_2)} \right], \quad i = 1,2$$

or

$$L_i^{(1,0)} * T_1^{(n,m)} + L_i^{(0,1)} * T_2^{(n,m)} - T_i^{(n,m)} * \left[G_1^{(1,0)}, G_2^{(0,1)} \right]$$

$$-T_i^{(m,n)} * \left[G_1^{(0,1)}, G_2^{(1,0)} \right] - \sum_{\substack{a+b=k, a \neq n, b \neq m \\ a(x_1+y_1)+b(x_2+y_2)=k \\ a,b,x_1,y_1,x_2,y_2 \in \mathbf{N}}} T_i^{(a,b)} * \left[G_1^{(x_1,y_1)}, G_2^{(x_2,y_2)} \right]$$

$$= \sum_{\substack{a+b<k \\ a(x_1+y_1)+b(x_2+y_2)=k \\ a,b,x_1,y_1,x_2,y_2 \in \mathbf{N}}} T_i^{(a,b)} * \left[G_1^{(x_1,y_1)}, G_2^{(x_2,y_2)} \right], \tag{4.8}$$

$i = 1,2$. This is a pair of equations, devoted to the polynomial $T^{(n,m)}, n + m = k$. Applying it to each one of the following cases,

$$(k,0), (k-1,1), \ldots, (0,k),$$

and working as in Step 2, (4.5), (4.6), (4.7), we should get a system of equations which will help us to determine the coefficients of the series $T_i^{(a,b)}, a + b = k$. It is named the basic k-degree system, which determines the requested series $\mathbf{T}$.

Theorem 4.6. *Let us suppose that the D-operator $\mathbf{G}$ is S-similar to the linear system $\mathbf{L}$ with respect to the series $\mathbf{T}$, i.e., $\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G}$. Then, the successive solution of the pair of equations (4.8), $k = 1, 2, \ldots$, with respect to the quantities $T_i^{(n,m)}$, $n + m = k$, determines the series $\mathbf{T}$.*

Proof. Obvious. □

4.2 The Structure of the Series T

In this section, we present two theorems dealing with the structure of the series $T_\theta = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} T_\theta^{(a,b)}$, $\theta = 1,2$. We examine when each of the terms $T_\theta^{(a,b)}$ is a polynomial or a series, too. In other words, we check whether $\mathbf{T}$ consists either from an infinite number of polynomials (series of polynomials), or from an infinite number of series (series of series). In the first case, we speak for a polynomial solution, in the second case for a series solution. Both cases will be used in the next section when the notion of complexity level will be established.

Before we present the main theorem concerning the polynomial solution, a nomenclature is needed. The matrix of the polynomial coefficients of the system (4.1) is denoted by Q_1. That is,

$$Q_1 = \begin{pmatrix} L_1^{(1,0)} - G_1^{(1,0)} & L_1^{(0,1)} & -G_2^{(1,0)} & 0 \\ L_2^{(1,0)} & L_2^{(0,1)} - G_1^{(1,0)} & 0 & -G_2^{(1,0)} \\ -G_1^{(0,1)} & 0 & L_1^{(1,0)} - G_2^{(0,1)} & L_1^{(0,1)} \\ 0 & -G_1^{(0,1)} & L_2^{(1,0)} & L_2^{(0,1} - G_2^{(0,1)} \end{pmatrix}.$$

The matrix of the linear system, formed by the equations (4.5),(4.6), (4.7), is denoted by Q_2. The corresponding augmented matrix is denoted by $Q_2^*(T_\theta^{(a,b)})$, $a+b=1$, or shortly $Q_2^*(\mathbf{T})$, to indicate the dependence from the polynomials $T_\theta^{(1,0)}, T_\theta^{(0,1)}, \theta = 1,2$ The matrix of the linear system, formed by the coefficients of the equations (4.8), is denoted by Q_k. The corresponding augmented matrix is denoted by $Q_k^*(T_\theta^{(a,b)})$, $a+b<k$, $\theta = 1,2$, or shortly $Q_k^*(\mathbf{T})$, to indicate the dependence from the previous polynomials $T_\theta^{(a,b)}$.

The solution set of the linear system (4.1), if any, is denoted by Γ. The set $\mathcal{S}_2$ is defined as

$$\mathcal{S}_2 = \left\{ T_\theta^{(a,b)} \in \Gamma, a+b=1 : \operatorname{rank}(Q_2) = \operatorname{rank}\left(Q_2^*(T_\theta^{(a,b)}) \right) \right\}.$$

Henceforth, the set $\mathcal{S}_k$ is defined as

$$\mathcal{S}_k = \left\{ T_\theta^{(a,b)} \in \Gamma, a+b<k : \operatorname{rank}(Q_k) = \operatorname{rank}\left(Q_k^*(T_\theta^{(a,b)}) \right), j = 1,2,\ldots,k \right\}.$$

Theorem 4.7. *Let* $\mathbf{G}$ *be a* D*-operator and* $\mathbf{L}$ *a linear one. Let* $\mathbf{T}$ *be the series which solves the* S*-similarity problem, i.e.,* $\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G}$*. If* $|Q_1| = 0$ *and* $\cap_{k=2}^{\infty} \mathcal{S}_k \neq \emptyset$*, then the series* $\mathbf{T}$ *is a series of polynomials.*

Proof. Let us start our analysis with the case $k = 1$, that is the linear case. Using the analysis of the previous section, we see that (4.1) is a homogeneous system of four equations with unknowns the quantities $T_1^{(1,0)}, T_2^{(1,0)}, T_1^{(0,1)}, T_2^{(0,1)}$. Taking under consideration the fact that in the linear case the star product coincides with the usual product among linear polynomials and the linear δ or ϵ-polynomials coincide with the usual univariate polynomials (Proposition 3.5), we get that the condition $|Q_1| = 0$, guarantees the polynomial solvability of the above system. In fact it has an infinite set of solutions denoted by Γ. We introduce a specific $\mathbf{T} \in \cap_{k=1}^{\infty} \mathcal{S}_k$. This means that $\operatorname{rank}(Q_k) = \operatorname{rank}(Q_k^*(\mathbf{T}))$ for each k. Let us go now to the quadratic case, that is when $k = 2$. The equations (4.5), (4.6), (4.7), form a system of linear equations. The relation $\operatorname{rank}(Q_2) = \operatorname{rank}(Q_2^*(\mathbf{T}))$ ensures the solution of the system and thus the existence of polynomial quadratic solutions. Using this solution, we substitute it to the equations of the cubic terms. The assumption $\operatorname{rank}(Q_3) = \operatorname{rank}(Q_3^*(\mathbf{T}))$ helps us, as before, to find a cubic polynomial solution. Working inductively, we prove the theorem. □

Remark 4.8. The assumption $\bigcap_{k=2}^{\infty} \mathcal{S}_k \neq \emptyset$, of the previous theorem, is a theoretical one and it cannot be easily checked. Nevertheless, when we face a concrete problem, we have predetermined the number of terms of the series $\mathbf{T}$ we are going to work with and thus, we know in advance the maximum degree term, appeared in our calculations, let us say r. Then we construct a system of equations of the form (4.8), for $k = 1, 2, \ldots, r$ the solvability of which can be checked with regular methods.

Remark 4.9. The equation $\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G}$ resembles with the classical equation of Linear Algebra for finding eigenvectors and eigenvalues. Therefore, all the linear D-polynomials $\mathbf{L}$ for which the condition $|Q_1| = 0$ is satisfied, form a set called the eigenpolynomials of $\mathbf{G}$. These are all the linear D-polynomials for which the S-similarity problem may be solved for a given $\mathbf{G}$. It can be calculated by computing the determinant $|Q_1|$ and working the coefficients. Analogously, the series $\mathbf{T}$, is called the eigenseries of $\mathbf{G}$.

The next theorems are dealing with the series solution. The first, says that a series solution of the linear part implies a global series solution. The second, provides us with the proper assumptions for a series solution.

Corollary 4.10. *If the linear part of the equations* (4.8) *accepts a series as solution, that is the quantities*

$$T_\theta^{(a,b)},\ a+b=1,\ \theta=1,2,$$

are linear series, then no one of the quantities

$$T_\theta^{(a,b)},\ a+b=k,\ k>1,\ \theta=1,2,$$

can be a polynomial.

Proof. Let us work, for the sake of simplicity, with the quadratic case

$$T^{(n,m)},\ n+m=2.$$

We suppose that the requested quantities

$$T^{(n,m)},\ n+m=2,$$

are finite polynomials but the quantity $T^{(1,0)}$, which has already been calculated, is not. Working with relation (4.5) we see that in the right-hand side, due to the factor

$$t_\varphi^{(1,0)} g_{1,(b_1,b_2)}^{(2,0)},$$

there is an infinite number of terms $\delta_i \delta_j$, whilst the left hand side there is only a finite number, due to the finiteness of the factors

$$l_{\varphi,a}^{(1,0)} t_{1,(b_1,b_2)}^{(2,0)} + \cdots .$$

This is a contradiction. Working in the same way, we can prove the argument for any series $T^{(n,m)}$. □

By equating the coefficients of the first terms of the equations (4.5), (4.6), (4.7), that is the coefficients of the terms δ_0^2, ϵ_0^2, $\delta_0\epsilon_0$, we form a linear system with unknowns the quantities

$$t_{1,(0,0)}^{(2,0)},\ t_{2,(0,0)}^{(2,0)},\ t_{1,(0,0)}^{(0,2)},\ t_{2,(0,0)}^{(0,2)},\ t_{1,(0,0)}^{(1,1)},\ t_{2,(0,0)}^{(1,1)}.$$

We denote it by C_2. Inductively, we can define the system of the first terms for the equations (4.8), denoted by C_k.

Theorem 4.11. *Let* $\mathbf{G}$ *be a given* D*-operator and* $\mathbf{L}$ *a linear one. We construct the quantity*

$$\mathcal{A} = \begin{bmatrix} l_{1,0}^{(1,0)} - g_{1,0}^{(1,0)} & l_{1,0}^{(0,1)} & -g_{2,0}^{(1,0)} & 0 \\ l_{1,0}^{(1,0)} & l_{2,0}^{(0,1)} - g_{1,0}^{(1,0)} & 0 & -g_{2,0}^{(1,0)} \\ -g_{1,0}^{(0,1)} & 0 & l_{1,0}^{(1,0)} - g_{2,0}^{(0,1)} & l_{1,0}^{(0,1)} \\ 0 & -g_{1,0}^{(0,1)} & l_{2,0}^{(1,0)} & l_{2,0}^{(0,1)} - g_{2,0}^{(0,1)} \\ l_{1,1}^{(1,0)} - g_{1,1}^{(1,0)} & l_{1,1}^{(0,1)} & -g_{2,1}^{(1,0)} & 0 \\ l_{2,1}^{(1,0)} & l_{2,1}^{(0,1)} - g_{1,1}^{(1,0)} & 0 & -g_{2,1}^{(1,0)} \\ -g_{1,1}^{(0,1)} & 0 & l_{1,1}^{(1,0)} - g_{2,1}^{(0,1)} & l_{1,1}^{(0,1)} \\ 0 & -g_{1,1}^{(0,1)} & l_{2,1}^{(1,0)} & l_{2,1}^{(0,1)} - g_{2,1}^{(0,1)} \\ & \vdots & & \vdots \\ l_{1,\nu}^{(1,0)} - g_{1,\nu}^{(1,0)} & l_{1,\nu}^{(0,1)} & -g_{2,\nu}^{(1,0)} & 0 \\ l_{2,\nu}^{(1,0)} & l_{2,\nu}^{(0,1)} - g_{1,\nu}^{(1,0)} & 0 & -g_{2,\nu}^{(1,0)} \\ -g_{1,\nu}^{(0,1)} & 0 & l_{1,\nu}^{(1,0)} - g_{2,\nu}^{(0,1)} & l_{1,\nu}^{(0,1)} \\ 0 & -g_{1,\nu}^{(0,1)} & l_{2,\nu}^{(1,0)} & l_{2,\nu}^{(0,1)} - g_{2,\nu}^{(0,1)} \end{bmatrix}.$$

The entries of this matrix are the coefficients of the linear polynomials appeared in $\mathbf{L}$ *and the linear part of* $\mathbf{G}$. *If* $\operatorname{rank}(\mathcal{A}) < 4$ *and all the "first-terms" systems* C_k *are solvable, then the problem* $\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G}$ *accepts a series-solution.*

Before we present the proof of the theorem, we need to establish the next two lemmas.

Lemma 4.12. *We introduce a collection of* 4×4 *matrices* $A_1, A_2, \ldots, A_k$. *We construct the block matrix with those matrices as a column. If*

$$\operatorname{rank} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} < 4,$$

then

1. $\operatorname{rank}\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_\theta \end{bmatrix} < 4$, *for each* $\theta = 1, 2, \ldots, k-1$.

2. The systems $A_1\vec{x} = 0$, $A_2\vec{x} = 0$, ..., $A_k\vec{x} = 0$, *have common nontrivial solutions.*

Proof. If $\operatorname{rank}\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_\theta \end{bmatrix} = 4$, then there must be at least one 4×4 sub-matrix, named A_σ with $\det(A_\sigma) \neq 0$ and thus the rank of the original matrix would be equal to 4, a contradiction. Now, from the previous result, we get that $\operatorname{rank}(A_j) < 4$ for each j, which means that all the systems $A_\phi\vec{x} = 0$ have infinite nontrivial solutions. Furthermore, we can rewrite all those systems to a single system of the form

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} \vec{x} = \vec{0}$$

The assumption of the lemma guarantees the solution of this system, too. □

Lemma 4.13. *We introduce the linear* δ*-polynomials* $A_k = \sum_{i=0}^{\mu} a_{k,i}\delta_i$, $B_k = \sum_{i=0}^{\mu} b_{k,i}\delta_i$, $\Gamma_k = \sum_{i=0}^{\mu} \gamma_{k,i}\delta_i$, $\Theta_k = \sum_{i=0}^{\mu} \theta_{k,i}\delta_i$, $k = 1, 2, 3, 4$. *We construct the* 4×4 *matrices*

$$\mathcal{A}_0 = [a_{k,0}, b_{k,0}, \gamma_{k,0}, \theta_{k,0}],$$
$$\mathcal{A}_1 = [a_{k,1}, b_{k,1}, \gamma_{k,1}, \theta_{k,1}],$$
$$\cdots$$
$$\mathcal{A}_\mu = [a_{k,\mu}, b_{k,\mu}, \gamma_{k,\mu}, \theta_{k,\mu}],$$

and the matrix

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_\mu \end{bmatrix}.$$

If $\operatorname{rank}(\mathcal{A}) < 4$, *then we can find linear* δ*-series*

$$X = \sum_{i=0}^{\infty} x_i\delta_i, \Psi = \sum_{i=0}^{\infty} \psi_i\delta_i, Z = \sum_{i=0}^{\infty} z_i\delta_i, \Omega = \sum_{i=0}^{\infty} w_i\delta_i$$

such that

$$A_1 * X + B_1 * \Psi + \Gamma_1 * Z + \Theta_1 * \Omega = 0,$$
$$A_2 * X + B_2 * \Psi + \Gamma_2 * Z + \Theta_2 * \Omega = 0,$$
$$A_3 * X + B_3 * \Psi + \Gamma_3 * Z + \Theta_3 * \Omega = 0,$$
$$A_4 * X + B_4 * \Psi + \Gamma_4 * Z + \Theta_4 * \Omega = 0.$$

Proof. By executing the star products, we get the next four equations

$$\begin{gathered}
(a_{k,0}x_0 + b_{k,0}\psi_0 + \gamma_{k,0}z_0 + \theta_{k,0}w_0)\delta_0 \\
+(a_{k,1}x_0 + a_{k,0}x_1 + b_{k,1}\psi_0 + b_{k,0}\psi_1 + \gamma_{k,1}z_0 + \gamma_{k,0}z_1 + \theta_{k,1}w_0 + \theta_{k,0}w_1)\delta_1 \\
+ \left(\sum_{i=0}^{2} a_{k,2-i}x_i + \sum_{i=0}^{2} b_{k,2-i}\psi_i + \sum_{i=0}^{2} \gamma_{k,2-i}z_i + \sum_{i=0}^{2} \theta_{k,2-i}w_i \right) \delta_2 \\
+ \cdots \\
+ \left(\sum_{i=0}^{\mu} a_{k,\mu-i}x_i + \sum_{i=0}^{\mu} b_{k,\mu-i}\psi_i + \sum_{i=0}^{\mu} \gamma_{k,\mu-i}z_i + \sum_{i=0}^{\mu} \theta_{k,\mu-i}w_i \right) \delta_\mu \\
+ \cdots \\
+ \left(\sum_{i=\rho-\mu}^{\rho} a_{k,\rho-i}x_i + \sum_{i=\rho-\mu}^{\rho} b_{k,\rho-i}\psi_i + \sum_{i=\rho-\mu}^{\rho} \gamma_{k,\rho-i}z_i + \sum_{i=\rho-\mu}^{\rho} \theta_{k,\rho-i}w_i \right) \delta_\rho \\
+ \cdots = 0, \quad k = 1, 2, 3, 4.
\end{gathered}$$

The first part is the obtained δ-series and δ_ρ an arbitrary term of it with $\rho > \mu$. To take its coefficient, we used the fact that $a_{k,\xi} = b_{k,\xi} = \gamma_{k,\xi} = \theta_{k,\xi} = 0$, for any $\xi > \mu$. By equating the coefficients of the terms δ_i to zero, we get

$$a_{k,0}x_0 + b_{k,0}\psi_0 + \gamma_{k,0}z_0 + \theta_{k,0}w_0 = 0, \tag{4.9}$$

$$\begin{gathered}
a_{k,1}x_0 + a_{k,0}x_1 + b_{k,1}\psi_0 + b_{k,0}\psi_1 + \gamma_{k,1}z_0 + \gamma_{k,0}z_1 + \theta_{k,1}w_0 + \theta_{k,0}w_1 = 0, \\
\cdots \\
\sum_{i=0}^{\mu} a_{k,\mu-i}x_i + \sum_{i=0}^{\mu} b_{k,\mu-i}\psi_i + \sum_{i=0}^{\mu} \gamma_{k,\mu-i}z_i + \sum_{i=0}^{\mu} \theta_{k,\mu-i}w_i = 0, \\
\cdots \\
\sum_{i=\rho-\mu}^{\rho} a_{k,\rho-i}x_i + \sum_{i=\rho-\mu}^{\rho} b_{k,\rho-i}\psi_i + \sum_{i=\rho-\mu}^{\rho} \gamma_{k,\rho-i}z_i + \sum_{i=\rho-\mu}^{\rho} \theta_{k,\rho-i}w_i = 0 \\
\cdots .
\end{gathered}$$

Equivalently, by using the notation $\vec{x}_\lambda = [x_\lambda, \psi_\lambda, z_\lambda, w_\lambda]^T$, $\lambda = 0, 1, 2, \ldots, \rho$, where T stands for transpose, we have

$$\mathcal{A}_0 \vec{x}_0 = \vec{0}, \tag{4.10}$$

$$\mathcal{A}_1 \vec{x}_0 + \mathcal{A}_0 \vec{x}_1 = \vec{0}, \tag{4.11}$$

$$\cdots,$$

$$\mathcal{A}_\mu \vec{x}_0 + \mathcal{A}_{\mu-1} \vec{x}_1 + \mathcal{A}_{\mu-2} \vec{x}_2 + \cdots + \mathcal{A}_1 \vec{x}_{\mu-1} + \mathcal{A}_0 \vec{x}_\mu = \vec{0}, \tag{4.12}$$

$$\cdots,$$

$$\mathcal{A}_\rho \vec{x}_0 + \mathcal{A}_{\rho-1} \vec{x}_1 + \mathcal{A}_{\rho-2} \vec{x}_2 + \cdots + \mathcal{A}_\mu \vec{x}_{\rho-\mu} = \vec{0} \tag{4.13}$$

$$\cdots,$$

where we have used the fact that $\mathcal{A}_\rho = 0$, for $\rho > \mu$. System (4.10) is a homogeneous linear system. Since $\operatorname{rank}(\mathcal{A}) < 4$, we conclude, by means of Lemma 4.12, that $\operatorname{rank}(\mathcal{A}_0) < 4$, and thus the system (4.10) has nontrivial solutions. Let us go now to the system (4.11). By Lemma 4.12, we get that $\operatorname{rank}\begin{bmatrix}\mathcal{A}_0\\ \mathcal{A}_1\end{bmatrix} < 4$, which means that there is at least one $\vec{x}_0$ with $\mathcal{A}_0\vec{x}_0 = \vec{0}$ and $\mathcal{A}_1\vec{x}_0 = \vec{0}$ and hence (4.11) is transformed to $\mathcal{A}_0\vec{x}_1 = \vec{0}$. The condition $\operatorname{rank}(\mathcal{A}_0) < 4$ guarantees that (4.11) has nontrivial solutions with respect to $\vec{x}_1$. We work now with (4.13). The condition $\operatorname{rank}(\mathcal{A}) < 4$ implies that the homogeneous systems $\mathcal{A}_k\vec{x}_{\mu-k} = \vec{0}$, $k = 1, 2, \ldots, \mu$ have common solutions and thus (4.13) is transformed to $\mathcal{A}_0\vec{x}_\mu = \vec{0}$, which is solvable, since $\operatorname{rank}(\mathcal{A}_0) < 4$. Therefore $\vec{x}_\mu$ is determined. Working inductively, we can find all the terms of the unknown series. □

Proof of Theorem 4.11. Working as in proof of Theorem 4.7, we take the same expansive expressions for the relations $\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G}$, as before. The D-series $\mathbf{T}$ is upon request. By using Lemma 4.13, where the polynomials A, B, Γ, Δ have been replaced by the linear parts of the above equations, the first condition of the current theorem guarantees that the linear part accepts a series solution. The solvability of the system C_2 implies that we can solve the system of (4.5), (4.6), (4.7), with respect to the coefficients of the terms δ_0^2, ϵ_0^2, $\delta_0\epsilon_0$. Then, after substituting the above solution, we can find the coefficients of the terms δ_1^2, ϵ_1^2, $\delta_0\epsilon_1$, $\delta_1\epsilon_0$. Working inductively, we can define all the terms of the quadratic series. The solvability of the system C_3 provides us with the cubic series. Continuing with the solvability of the next systems C_k, we can obtain the series of any degree and thus, the final series $\mathbf{T}$ can be determined. □

Remark 4.14. The assumption about the solvability of the systems C_k cannot be checked for an infinite number of series. Nevertheless, as before, when we face a concrete problem, we have predetermined the number of terms of the series $\mathbf{T}$ we are going to work with and thus, we know in advance the maximum degree term, appeared in our calculations, let us say r. In this case, we have a finite number of systems C_k, $k = 2, \ldots, r$, the solvability of which can be checked after finite steps.

T-series	Complexity Degree	
	L Stable	**L** Unstable
A polynomial	0	0+
An invertible, convergence, simple series	1	1+
A convergence simple series	1.5	1.5+
A simple series	2	2+
An invertible, convergence, series of series	3	3+
A convergence series of series	3.5	3.5+
A series of series	4	4+

Table 5.1: Complexity Degrees

5 Levels of Model Complexity

Complex systems appears in many fields of contemporary science and different communities have different aspects about complexity and how they ranked it [5]. In this section, we shall try to approach this issue for $2D$-polynomial – discrete – systems, using the mathematical tools developed previously. Specifically, we have described a procedure for checking the equivalence of a nonlinear discrete system with a linear one. This was achieved via a D-series, named $\mathbf{T}$. The construction of $\mathbf{T}$ determines the kind of the model complexity or how "hard" the nonlinearity is. If, for instance, $\mathbf{T}$ converges, then we speak for a "light" complexity, otherwise for a "strong" one. If $\mathbf{T}$ is a simple series or consists from an infinite sum of series (series of series), this will influence the kind of complexity since checking convergence in the latter case, is a very difficult task. The nature of $\mathbf{L}$ plays also important role. If, for instance it is stable then the level of complexity is less than the level of complexity which corresponds to an unstable $\mathbf{L}$. We summarize the different cases of complexity degrees in Table 5.1.

It should be noted here that the above classification is arbitrary in nature and represents the authors view. It has been arisen from the way the problem is described in this paper and it is based on the algebraic approach. Obviously, other approaches can give other classifications by using other criteria The stability of the linear systems can be checked by the well-known classical theory. The convergence of the series is an analytic question, depending on the values of the coefficients of the terms of the series and it is difficult to be studied. To this purpose, we use certain theorems. To determine the type of the series is the task of the methodology developed in this work. Sometimes, parameters are presented in the coefficients. Various values of the parameters define various kinds of complexity degree. Usually, we seek if there are values that give the simplest one. Finally, we should note here, that it is sufficient to find at least one stable linear system, equivalent with the original nonlinear one, to have complexity degree 0, 1, 1.5, etc., while we have to prove that there are no equivalent stable linear systems to have complexity degree 0+, 1+, etc. How and when this happens will be the subject of future research.

6 Examples

We give now two examples, the first deals with a linear system and the second with a nonlinear one. The aim of those examples are to indicate how the previous theory works in practise.

Example 6.1. Just to point out the compatibility of the current method with the classical ones, we start with the linear case. We introduce the linear systems

$$\left.\begin{aligned} x(t) &= x(t-1) + 2x(t-2) + \frac{1}{2}y(t-1) \\ y(t) &= \frac{7}{2}x(t-1) - 2y(t-1) + 2y(t-2) \end{aligned}\right\}$$

and

$$\left.\begin{aligned} u(t) &= -\frac{3}{2}u(t-1) + 2u(t-2) - 3v(t-1) \\ v(t) &= -u(t-1) + \frac{1}{2}v(t-1) + 2v(t-2) \end{aligned}\right\}.$$

We want to study their S-similarity. Using the algebraic tools, we have developed, they are described as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \delta_0 + 2\delta_1 + \frac{1}{2}\epsilon_0 \\ \frac{7}{2}\delta_0 - 2\epsilon_0 + 2\epsilon_1 \end{bmatrix} \begin{bmatrix} x(t-1) \\ y(t-1) \end{bmatrix},$$

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}\delta_0 + 2\delta_1 - 3\epsilon_0 \\ -\delta_0 + \frac{1}{2}\epsilon_0 + 2\epsilon_1 \end{bmatrix} \begin{bmatrix} u(t-1) \\ v(t-1) \end{bmatrix}$$

and shortly $\mathbf{x}(t) = \mathbf{G}\mathbf{x}(t-1)$, $\hat{\mathbf{x}}(t) = \mathbf{L}\hat{\mathbf{x}}(t-1)$. We want to find linear polynomials T_1, T_2, such that the following equations hold:

$$\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G} \Rightarrow \begin{cases} L_1 * [T_1, T_2] = T_1 * [G_1, G_2] \\ L_2 * [T_1, T_2] = T_2 * [G_1, G_2]. \end{cases}$$

For the sake of the simplification of the computation, we arbitrarily set

$$T_1 = w_{1,0}\delta_0 + w_{1,1}\delta_1 + h_{1,0}\epsilon_0 + h_{1,1}\epsilon_1$$

$$T_2 = w_{2,0}\delta_0 + w_{2,1}\delta_1 + h_{2,0}\epsilon_0 + h_{2,1}\epsilon_1.$$

We could of course, take any other expression for the polynomials T_1, T_2 with more delays, but the computational effort would be harder. By equating the coefficients and

Repeats	G	L	Transformed - G
2	7.5	-97.3	-97.3
5	105.625	-1729.63	-1729.63
10	1102.92	12477.2	12477.2

Table 6.1: Linear Systems

solving the corresponding system of equations, we get $w_{1,0} = h_{1,0} - 6h_{2,0}$, $w_{1,1} = h_{1,1} - 6h_{2,1}$ $w_{2,0} = -2h_{1,0} + 5h_{2,0}$, $w_{2,1} = -2h_{1,1} + 5h_{2,1}$ and thus a transformation which solves the problem is

$$T_1 = (h_{1,0} - 6h_{2,0})\delta_0 + (h_{1,1} - 6h_{2,1})\delta_1 + h_{1,0}\epsilon_0 + h_{1,1}\epsilon_1$$

and

$$T_2 = (-2h_{1,0} + 5h_{2,0})\delta_0 + (-2h_{1,1} + 5h_{2,1})\delta_1 + h_{2,0}\epsilon_0 + h_{2,1}\epsilon_1,$$

with $h_{ij} \in \mathbf{R}$. To check the validity of the method let us proceed as follows: First, let us assign some initial conditions, for instance $x(0) = 3, y(0) = 1, x(1) = 2, y(1) = -1$. We find then, their transformed values with respect to the T-polynomials, and then we put these values as initial values to the other system, specifically, we set

$$u(1) = h_{1,0} + 4h_{1,1} - 12h_{2,0} - 18h_{2,1},$$

$$v(1) = -4h_{1,0} - 6h_{1,1} + 9h_{2,0} + 16h_{2,1},$$

$$u(2) = \frac{37}{2}h_{1,0} + h_{1,1} - 45h_{2,0} - 12h_{2,1},$$

$$v(2) = -15h_{1,0} - 4h_{1,1} + \frac{97}{2}h_{2,0} + 9h_{2,1}.$$

By executing the operations symbolically, we see that the results are identical. Just to take our approach one step forward, let us give the next numerical values to the free parameters $h_{i,j}$,$h_{1,0} = 1, h_{1,1} = -1.5, h_{2,0} = 2.3, h_{2,1} = 0.9$. Then, the results we take are presented in Table 6.1. Since the solution we found is a pure polynomial, the complexity degree in this case is 0 or $0+$, depending from the stability or not of the linear system **L**.

Example 6.2. Let us consider now the nonlinear system

$$x(t+1) = x(t) + y(t) - x^2(t)$$

$$y(t+1) = x(t)$$

we want to examine if it can be S-similar, in other words equivalent, with the next linear system (the " target ") and thus to study its complexity degree. The linear system is

$$z(n+1) = z(n) - z(n-1) + w(n) + \frac{1}{2}(1 - \sqrt{5})w(n-1)$$

$$w(n+1) = z(n) + \frac{1}{2}(1+\sqrt{5})z(n-1) + w(n-1).$$

Using the D-operators, we get the next descriptions for both of them: $\mathbf{x}(t+1) = \mathbf{G}\mathbf{x}(t)$, $\hat{\mathbf{x}}(t+1) = \mathbf{L}\hat{\mathbf{x}}(t)$, where

$$\mathbf{G} = \begin{bmatrix} \delta_0 + \epsilon_0 - \delta_0^2 \\ \delta_0 \end{bmatrix} \text{ and } \mathbf{L} = \begin{bmatrix} \delta_0 - \delta_1 + \epsilon_0 + \frac{1}{2}(1-\sqrt{5})\epsilon_1 \\ \delta_0 + \frac{1}{2}(1+\sqrt{5})\delta_1 + \epsilon_1 \end{bmatrix}.$$

First of all, we see that $|Q_1| = 0$ and thus the problem, accordingly to Theorem 4.7, if it has a solution, it would be a polynomial series. That is a series with a finite number of first degree terms, a finite number of quadratic terms, cubic terms and so on. To calculate the series T_1, T_2, we use the equations

$$\mathbf{L} * \mathbf{T} = \mathbf{T} * \mathbf{G} \Rightarrow \begin{cases} L_1 * [T_1, T_2] = T_1 * [G_1, G_2] \\ L_2 * [T_1, T_2] = T_2 * [G_1, G_2]. \end{cases}$$

By choosing a proper $\mathbf{T}$, so that the second assumption of the Theorem 4.7, to be satisfied, we can calculate the series T_1, T_2, by following the procedure of the previous section. We finally take

$$\begin{aligned} T_1 = (\Gamma + A)\delta_0 &+ \left(-\frac{1}{2}(1+\sqrt{5})\Gamma + \Delta - \frac{1}{2}(1+\sqrt{5})A + B\right)\delta_1 + A\epsilon_0 \\ &+ \left(\Gamma - \frac{1}{2}(1+\sqrt{5})A + B\right)\epsilon_1 + \frac{1}{2}A\delta_0^2 + \frac{1}{2}\Gamma\epsilon_0^2 + (A+\Gamma)\delta_0\epsilon_0 \\ &- \frac{1}{6}A\delta_0^3 - \frac{1}{6}\Gamma\epsilon_0^3 + \frac{1}{2}(A+\Gamma)\delta_0\epsilon_0^2 + \frac{1}{2}(3A+\Gamma)\delta_0^2\epsilon_0 + \cdots \end{aligned}$$

and

$$\begin{aligned} T_2 &= A\delta_0 + B\delta_1 + \Gamma\epsilon_0 + \Delta\epsilon_1 \\ &+ \frac{1}{2}\Gamma_0^2 + \frac{1}{2}(A-\Gamma)\epsilon_0^2 + A\delta_0\epsilon_0 - \frac{1}{6}\Gamma\delta_0^3 + \frac{1}{6}(\Gamma - A)\epsilon_0^3 \\ &+ \frac{1}{2}A\delta_0\epsilon_0^2 + \left(\frac{1}{2}A + \Gamma\right)\delta_0^2\epsilon_0 + \cdots, \end{aligned}$$

where A, B, Γ, Δ arbitrary parameters taking real values. The complexity degree of the model depends from the values of those parameters. If we are able to find certain values of these parameters which can guarantee the convergence as well as the invertibility of the series (This means $A \neq 0$ and $A + \Gamma \neq 0$), then, the complexity degree of the model will be either 1 or $1+$, depending from the stability or not of the linear system $\mathbf{L}$. If invertibility cannot be guaranteed but convergence is valid, then we have complexity degree equal to 1.5 or $1.5+$. In the current case, the linear system is unstable and hence, we have complexity degree either $1+$ or $1.5+$. If, instead, we could prove the S-similarity property with a stable linear system, then the complexity degree would be 1 or 1.5.

7 Concluding Remark

In this paper, we presented a methodology for deciding if a nonlinear polynomial s discrete system, with two components, is equivalent with a linear one. This equivalence means that the two such systems have the same dynamic behavior. The solution of the problem is achieved by the use of transformations described by proper series. The whole approach is relied on an algebraic framework, developed by the authors and no analytical tools are used. Finally, the entire issue is applied to the systems complexity problem in order to provide us with different complexity levels, depending from the structure of the transformation series and the linear systems.

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