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Stability, Periodicity and Neimark–Sacker Bifurcation of Certain Homogeneous Fractional Difference Equations

Mirela Garić-Demirović, **Mehmed Nurkanović**
and **Zehra Nurkanović**
University of Tuzla
Department of Mathematics
Tuzla, 75 000, Bosnia and Herzegovina
mirela.garic@untz.ba
mehmed.nurkanovic@untz.ba
zehra.nurkanovic@untz.ba

Abstract

We investigate the local and global character of the unique equilibrium point, the existence and the local stability of the period-two solutions of certain homogeneous fractional difference equation with quadratic terms. The local stability and global attractivity results of the minimal period-two solution in one special case are given. Also, we investigate the bifurcation of a fixed point of the map associated to the equation in the special case where the eigenvalues are complex conjugate numbers on the unit circle.

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1 Introduction and Preliminaries

In this paper, we investigate the local and global character of the equilibrium point and the existence of period-two solutions of the difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_nx_{n-1} + Cx_{n-1}^2}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2}, \quad n = 0, 1, \ldots \tag{1.1}$$

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where the parameters A, B, C, a, b, c are positive numbers and where the initial conditions x_{-1} and x_0 are arbitrary nonnegative real numbers such that $x_{-1}+x_0>0$. Also, we investigate the special case of (1.1) when $A=C=0$.

Equation (1.1) is the special case of a general second order quadratic fractional equation of the form

$$x_{n+1}=\frac{Ax_n^2+Bx_nx_{n-1}+Cx_{n-1}^2+Dx_n+Ex_{n-1}+F}{ax_n^2+bx_nx_{n-1}+cx_{n-1}^2+dx_n+ex_{n-1}+f},\quad n=0,1,\ldots \tag{1.2}$$

with nonnegative parameters and $A+B+C>0$, $a+b+c+d+e+f>0$ and initial conditions such that $ax_n^2+bx_nx_{n-1}+cx_{n-1}^2+dx_n+ex_{n-1}+f>0, n=0,1,\ldots$. Several global asymptotic results for some special cases of (1.2) were obtained in [14–16,28,31].

One interesting special case of (1.1) is the following difference equation studied in [4, 18, 19], when $c=C=0$:

$$x_{n+1}=\frac{Dx_n+Ex_{n-1}}{dx_n+ex_{n-1}},\quad n=0,1,\ldots \tag{1.3}$$

which represents discretization of the differential equation model in biochemical networks, see [8]. Also, the special case of (1.1) when $a=A=B=0$ is the linear fractional difference equation whose global dynamics is described in [18]. Notice that (1.3) is also the special case of the linear fractional difference equation

$$x_{n+1}=\frac{Dx_n+Ex_{n-1}+F}{dx_n+ex_{n-1}+f},\quad n=0,1,\ldots \tag{1.4}$$

(which was investigated in great detail in [18]) with well-known but very complicated dynamics, such as Lyness' equation (see [20]).

Equation (1.1) can be written in the form

$$x_{n+1}=\frac{A\left(x_n/x_{n-1}\right)^2+B\left(x_n/x_{n-1}\right)+C}{a\left(x_n/x_{n-1}\right)^2+b\left(x_n/x_{n-1}\right)+c},\quad n=0,1,\ldots$$

and one can take the advantage of this auxiliary equation to describe the dynamics of (1.1) (see [5, 6, 15, 16, 31]).

The first systematic study of global dynamics of (1.2) in a special case where $A=C=D=a=c=d=0$ was performed in [1, 2].

In [9] and [10], we gave more precisely the dynamics in two special cases of (1.1) where the right-hand side of (1.1) is decreasing in x_n and increasing in x_{n-1} and where we could have applied the theory of monotone maps to give global dynamics. Also, see [11, 12, 21–23, 25–27, 29] for an application of the monotone maps techniques to some competitive systems of linear fractional difference equations.

The special case of (1.1) when $A=B=0$, $C=1$, i.e.,

$$x_{n+1}=\frac{x_{n-1}^2}{ax_n^2+bx_nx_{n-1}+cx_{n-1}^2},\quad n=0,1,\ldots \tag{1.5}$$

was investigated in [9]. It was shown that equation (1.5) exhibits three types of global behavior characterized by the existence of a unique positive equilibrium point and one or two minimal period-two solutions, one of which is locally stable and the other is a saddle point. The unique feature of the equation is the coexistence of an equilibrium point and the minimal period-two solutions both being locally asymptotically stable. This new phenomenon is caused by the presence of quadratic terms and did not exist in the case of (1.4).

The special case of (1.1) when $B = C = 0$, i.e.,

$$x_{n+1} = \frac{Ax_n^2 + Cx_{n-1}^2}{ax_n^2 + bx_nx_{n-1}}, \quad n = 0, 1, \ldots \tag{1.6}$$

was studied in [10]. It was shown that (1.6) exhibits three types of global behavior characterized by the existence of a unique equilibrium point and a minimal period-two solution, which stable manifold serves the boundary of the basins of attraction of locally stable equilibrium and points at infinity $(0, \infty)$ and $(\infty, 0)$. In fact, the equation exhibits period-two bifurcation studied in great details in [22].

Following the approach from [18], we can divide (1.1) into 49 cases of types (k, m), where type (k, m) means that special case has k terms in the numerator and m terms in the denominator. Notice the following fact: if we use substitution $x_n = \frac{1}{u_n}$, then each of the equations of the type (3,1) transforms into an equation of the type (1,3), each of the equations of the type (2,1) transforms into an equation of the type (1,2), and three of seven equations of the type (2,2) transform into the other equations of the type (2,2). In this paper we present the local and global character of the unique equilibrium point, the existence and the local stability of the period-two solutions of (1.1), with positive parameters, i.e., of the equation of the type (3,3). The local stability analysis indicates some possible scenarios for (1.1): global attractivity of the unique equilibrium point, Neimark–Sacker bifurcation and period-doubling bifurcation, see [13, 17, 24]. This means that the techniques used in [3, 14, 18, 20, 22, 23, 30] can be applied here.

The global attractivity results obtained specifically for complicated cases of (1.2) are the following theorems (see [7]).

Theorem 1.1. *Assume that* (1.2) *has the unique equilibrium* $\bar{x}$. *If the following condition holds*

$$\frac{(|A - a\bar{x}| + |B - b\bar{x}| + |C - c\bar{x}|)(U + \bar{x}) + |D - d\bar{x}| + |E - e\bar{x}|}{(a + b + c)L^2 + (d + e)L + f} < 1 \tag{1.7}$$

where L *and* U *are lower and upper bounds of all solutions of* (1.2), *then* $\bar{x}$ *is globally asymptotically stable.*

Theorem 1.2. *Assume that* (1.2) *has the unique equilibrium* $\bar{x}$. *If the following condition holds*

$$(|A-a\bar{x}|+|B-b\bar{x}|+|C-c\bar{x}|)(M+\bar{x})+|D-d\bar{x}|+|E-e\bar{x}| < (a+b+c)m^2+(d+e)m+f \tag{1.8}$$

where $m = \min\{\bar{x}, x_{-1}, x_0\}$ *and* $M = \max\{\bar{x}, x_{-1}, x_0\}$ *are lower and upper bounds of specific solution of* (1.2)*, then* $\bar{x}$ *is globally asymptotically stable on the interval* $[m, M]$*.*

Theorem 1.1 and Theorem 1.2 can be used efficiently to obtain global stability results for the special cases of (1.1), in particular, for some equations of types (2,2), (3,3) (see Section 2) and (1,3) (see Subsection 3.1).

The paper is organized as follows. Section 2 gives the local stability analysis of the unique positive equilibrium point of (1.1) and some global attractivity results in some special cases. In Section 3, we investigate the existence of the minimal period-two solutions of (1.1) and of the some special cases of (1.1). Subsection 3.1 gives the local stability and the global attractivity results of the minimal period-two solution of (1.1) in special case when $A = C = 0$, $B = 1$. In Section 4, we consider the bifurcation of a fixed point of the map associated to (1.1), when $B = C = 0$, $A = 1$, in the case where the eigenvalues are complex conjugate numbers on the unit circle.

2 Stability of the Positive Equilibrium Point of (1.1)

In this section, we investigate the local and global stability analysis of the positive equilibrium point of (1.1) (with some special cases of (1.1)).

It is clear that (1.1) has a unique equilibrium point $\overline{x} = \frac{A+B+C}{a+b+c}$. If we denote

$$f(u, v) = \frac{Au^2 + Buv + Cv^2}{au^2 + buv + cv^2},$$

then the linearized equation associated with (1.1) about the equilibrium point $\overline{x}$ is of the form

$$y_{n+1} = sy_n + ty_{n-1},$$

where

$$s = -t = \frac{\partial f}{\partial u}(\overline{x}, \overline{x}) = \frac{Ab - Ba + 2(Ac - Ca) + Bc - Cb}{(A + B + C)(a + b + c)}.$$

Theorem 2.1. *Equation* (1.1) *has a unique positive equilibrium point* $\overline{x} = \frac{A+B+C}{a+b+c}$*.*

i) If

$$A(a + 3b + 5c) + B(-a + b + 3c) > C(3a + b - c)$$

and

$$A(c - a) < (2a + b)B + (3a + 2b + c)C,$$

then the equilibrium point $\overline{x}$ *is locally asymptotically stable.*

ii) If $A(a + 3b + 5c) + B(-a + b + 3c) < C(3a + b - c)$*, then the equilibrium point* $\overline{x}$ *is a saddle point.*

iii) If $A(c-a) > (2a+b)B + (3a+2b+c)C$, then the equilibrium point $\overline{x}$ is a repeller.

iv) If $A(a+3b+5c) + B(-a+b+3c) = C(3a+b-c)$, then the equilibrium point $\overline{x}$ is nonhyperbolic with eigenvalues $\lambda_{1,2} \in \left\{-1, \frac{1}{2}\right\}$. If

$$A(c-a) = (2a+b)B + (3a+2b+c)C,$$

then the equilibrium point $\overline{x}$ is nonhyperbolic with eigenvalues $\lambda_{1,2} = \frac{1 \pm i\sqrt{3}}{2}$.

Proof. The characteristic equation at the equilibrium point is of the form

$$\lambda^2 - s\lambda - t = 0. \tag{2.1}$$

i) Equilibrium point $\overline{x}$ is locally asymptotically stable if

$$|s| < 1 - t < 2 \Leftrightarrow |s| < 1 + s < 2 \Leftrightarrow -\frac{1}{2} < s < 1$$
$$\Leftrightarrow \left\{ \begin{array}{c} A(a+3b+5c) + B(-a+b+3c) > C(3a+b-c) \\ \wedge \\ A(c-a) < (2a+b)B + (3a+2b+c)C \end{array} \right\}.$$

ii) Equilibrium point $\overline{x}$ is a saddle point if

$$\left\{|s| > |1-t| \wedge s^2 + 4t > 0\right\} \Leftrightarrow \left\{s^2 > (1+s)^2 \wedge s(s-4) > 0\right\}$$
$$\Leftrightarrow \left\{s < -\frac{1}{2} \wedge s(s-4) > 0\right\},$$

i.e.,

$$s < -\frac{1}{2} \Leftrightarrow A(a+3b+5c) + B(-a+b+3c) < C(3a+b-c).$$

iii) Equilibrium point $\overline{x}$ is a repeller if

$$\{|s| < |1-t| \wedge |t| > 1\} \Leftrightarrow \left\{s^2 < (1+s)^2 \wedge |s| > 1\right\} \Leftrightarrow \left\{s > -\frac{1}{2} \wedge |s| > 1\right\}$$
$$\Leftrightarrow s > 1 \Leftrightarrow A(c-a) > (2a+b)B + (3a+2b+c)C.$$

iv) Equilibrium point $\overline{x}$ is nonhyperbolic if

$$\{|s| = |1-t| \vee (t = -1 \wedge |s| \leq 2)\} \Leftrightarrow \left(s = -\frac{1}{2} \vee s = 1\right)$$

$$\Leftrightarrow \left\{ \begin{array}{c} A(a+3b+5c) + B(-a+b+3c) = C(3a+b-c) \\ \vee \\ A(c-a) = (2a+b)B + (3a+2b+c)C \end{array} \right\}.$$

If $s=1$, i.e., $A\left(c-a\right)=\left(2a+b\right)B+\left(3a+2b+c\right)C$, then the characteristic equation (2.1) becomes

$$\lambda^2-\lambda+1=0,$$

with eigenvalues $\lambda_{1,2}=\frac{1\pm i\sqrt{3}}{2}$.

If $s=-\frac{1}{2}$, i.e., $A\left(a+3b+5c\right)+B\left(-a+b+3c\right)=C\left(3a+b-c\right)$, then the characteristic equation (2.1) is of the form

$$2\lambda^2+\lambda-1=0,$$

with eigenvalues $\lambda_1=-1$ and $\lambda_2=\frac{1}{2}$. □

This leads to the following global attractivity result for (1.1), and for some its special cases.

Theorem 2.2. *(i) Consider* (1.1)*, where all coefficients are positive, subject to the condition*

$$\Lambda\left(U\left(a+b+c\right)+A+B+C\right)<\left(a+b+c\right)^3L^2,$$

where

$$\Lambda=\left|A\left(b+c\right)-a\left(B+C\right)\right|+\left|B\left(a+c\right)-b\left(A+C\right)\right|+\left|C\left(a+b\right)-c\left(A+B\right)\right|,$$

and $L=\frac{\min\{A,B,C\}}{\max\{a,b,c\}}$, $U=\frac{\max\{A,B,C\}}{\min\{a,b,c\}}$. *Then* $\bar{x}=\frac{A+B+C}{a+b+c}$ *is globally asymptotically stable.*

(ii) Consider (1.1)*, where* $C=c=0$*, and all other coefficients are positive, subject to the condition*

$$2|Ab-Ba|\left(U\left(a+b\right)+A+B\right)<\left(a+b\right)^3L^2,$$

where $L=\frac{\min\{A,B\}}{\max\{a,b\}}, U=\frac{\max\{A,B\}}{\min\{a,b\}}$. *Then* $\bar{x}=\frac{A+B}{a+b}$ *is globally asymptotically stable.*

(iii) Consider (1.1)*, where* $B=b=0$*, and all other coefficients are positive, subject to the condition*

$$2|Ac-Ca|\left(U\left(a+c\right)+A+C\right)<(a+c)^3L^2,$$

where $L=\frac{\min\{A,C\}}{\max\{a,c\}}, U=\frac{\max\{A,C\}}{\min\{a,c\}}$. *Then* $\bar{x}=\frac{A+C}{a+c}$ *is globally asymptotically stable.*

(iv) Consider (1.1)*, where* $A=a=0$*, and all other coefficients are positive, subject to the condition*

$$2|Bc-Cb|\left(U\left(b+c\right)+B+C\right)<(b+c)^3L^2,$$

where $L=\frac{\min\{B,C\}}{\max\{b,c\}}, U=\frac{\max\{B,C\}}{\min\{b,c\}}$. *Then* $\bar{x}=\frac{B+C}{b+c}$ *is globally asymptotically stable.*

Proof. In view of Theorem 1.1, we need to find the lower and upper bounds for all solutions of (1.1) for $n \geq 1$.

(i) In this case the lower and upper bounds for all solutions of (1.1) for $n \geq 1$ are derived as:

$$x_{n+1} \geq \frac{\min\{A,B,C\}}{\max\{a,b,c\}} \frac{x_n^2 + x_n x_{n-1} + x_{n-1}^2}{x_n^2 + x_n x_{n-1} + x_{n-1}^2} \geq \frac{\min\{A,B,C\}}{\max\{a,b,c\}} = L > 0,$$

and

$$x_{n+1} \leq \frac{\max\{A,B,C\}}{\min\{a,b,c\}} \frac{x_n^2 + x_n x_{n-1} + x_{n-1}^2}{x_n^2 + x_n x_{n-1} + x_{n-1}^2} \leq \frac{\max\{A,B,C\}}{\min\{a,b,c\}} = U.$$

The condition (1.7) is of the form

$$\Lambda \left(U \left(a + b + c \right) + A + B + C \right) < \left(a + b + c \right)^3 L^2,$$

where

$$\Lambda = \left| A \left(b + c \right) - a \left(B + C \right) \right| + \left| B \left(a + c \right) - b \left(A + C \right) \right| + \left| C \left(a + b \right) - c \left(A + B \right) \right|.$$

The proof of the other cases is analogous. □

3 The Existence of the Period-two Solutions

In this section, we investigate the existence of the minimal period-two solutions of (1.1) and of the some special cases of (1.1).

First, consider the existence of the minimal period-two solutions of (1.1), where all the parameters A, B, C, a, b, c are positive. Assume that (ϕ, ψ) is a minimal period-two solution of (1.1) with $\phi, \psi \in [0, +\infty)$ and $\phi \neq \psi$. Then

$$\phi = \frac{A\psi^2 + B\phi\psi + C\phi^2}{a\psi^2 + b\phi\psi + c\phi^2}, \quad \psi = \frac{A\phi^2 + B\phi\psi + C\psi^2}{a\phi^2 + b\phi\psi + c\psi^2},$$

from which

$$\phi \left(a\psi^2 + b\phi\psi + c\phi^2 \right) = A\psi^2 + B\phi\psi + C\phi^2 \tag{3.1}$$

and

$$\psi \left(a\phi^2 + b\phi\psi + c\psi^2 \right) = A\phi^2 + B\phi\psi + C\psi^2. \tag{3.2}$$

Subtracting (3.1) and (3.2), we obtain

$$\left(b - a - c \right) \phi\psi + c \left(\phi + \psi \right)^2 = \left(C - A \right) \left(\phi + \psi \right). \tag{3.3}$$

Lemma 3.1. *Equation* (1.1) *has a minimal period-two solution* (ϕ, ψ)*, with* $\phi\psi = 0$*, if and only if* $A = 0$*, and then* $(\phi, \psi) = \left(0, \frac{C}{c}\right)$.

Proof. If $\phi = 0$, then equations (3.1) and (3.2) implies

$$A = 0 \text{ and } \psi = \frac{C}{c}.$$

□

Now, suppose that $\phi, \psi \in (0, +\infty)$, $\phi \neq \psi$ and $b - a - c \neq 0$. Then (3.3) implies

$$\phi\psi = \frac{1}{(b-a-c)}(\phi+\psi)\left[(C-A) - c(\phi+\psi)\right]. \tag{3.4}$$

Dividing (3.1) by ϕ and (3.2) by ψ and subtracting them, we have

$$(a-c)(\phi+\psi) = A\frac{(\phi+\psi)^2}{\phi\psi} - A + B - C. \tag{3.5}$$

If we set

$$\phi\psi = x \text{ and } \phi + \psi = y,$$

where $x > 0$ and $y > 0$, then ϕ and ψ are positive and different solutions of the quadratic equation

$$t^2 - yt + x = 0. \tag{3.6}$$

In addition to the conditions $x, y > 0$, it is necessary that $y^2 - 4x > 0$. From (3.4) and (3.5), we obtain the system

$$\begin{array}{c} (b-a-c)\,x = y\,(C - A - cy) \\ x\left[(a-c)\,y + A - B + C\right] = Ay^2, \end{array} \tag{3.7}$$

from which

$$x = \frac{Ay^2}{(a-c)\,y + A - B + C}.$$

Since $x > 0$, we have

$$(a-c)\,y + A - B + C > 0 \Leftrightarrow (a-c)\,y > B - A - C.$$

The condition $y^2 - 4x > 0$ is equivalent to

$$y^2\frac{(a-c)\,y - 3A - B + C}{(a-c)\,y + A - B + C} > 0,$$

which implies

$$(a-c)\,y - 3A - B + C > 0 \Leftrightarrow (a-c)\,y > 3A + B - C.$$

Since $3A + B - C > B - A - C$, we have $x > 0$ and $y^2 - 4x > 0$ reduce to

$$(a-c)\,y > 3A + B - C. \tag{3.8}$$

From (3.7), since $y > 0$, we have

$$\frac{y\left(C-A-cy\right)}{b-a-c}\left[\left(a-c\right)y+A-B+C\right]=Ay^{2},$$

i.e.,

$$\left\{\begin{array}{c} c\left(a-c\right)y^{2}+\left[A\left(b-a-c\right)+c\left(A-B+C\right)-\left(C-A\right)\left(a-c\right)\right]y \\ -\left(C-A\right)\left(A-B+C\right)=0 \end{array}\right\}. \tag{3.9}$$

Under the assumption $a \neq c$, the roots of the (3.9) are of the form

$$\begin{aligned} y_{\pm} &= \frac{1}{2c\left(a-c\right)}\left[-F\pm\sqrt{F^{2}+4c\left(a-c\right)\left(C-A\right)\left(A-B+C\right)}\right] \\ &= \frac{1}{2c\left(a-c\right)}\left[-F\pm\sqrt{D}\right], \end{aligned}$$

where

$$\begin{aligned} F &= A\left(b-a-c\right)+c\left(A-B+C\right)-\left(C-A\right)\left(a-c\right) \\ &= bA-aC+c\left(-A-B+2C\right), \end{aligned} \tag{3.10}$$

and

$$D=F^{2}+4c\left(a-c\right)\left(C-A\right)\left(A-B+C\right).$$

Notice that from (3.6), we have

$$t_{\pm}=\frac{y}{2}\left(1\pm\sqrt{\frac{\left(a-c\right)y-3A-B+C}{\left(a-c\right)y+A-B+C}}\right), \tag{3.11}$$

i.e., (1.1) has one or two minimal period-two solutions of the form

$$\phi_1=\tfrac{y_+}{2}\left(1+\sqrt{\tfrac{(a-c)y_+-3A-B+C}{(a-c)y_++A-B+C}}\right),\quad \psi_1=\tfrac{y_+}{2}\left(1-\sqrt{\tfrac{(a-c)y_+-3A-B+C}{(a-c)y_++A-B+C}}\right), \tag{3.12}$$

or

$$\phi_2=\tfrac{y_-}{2}\left(1+\sqrt{\tfrac{(a-c)y_--3A-B+C}{(a-c)y_-+A-B+C}}\right),\quad \psi_2=\tfrac{y_-}{2}\left(1-\sqrt{\tfrac{(a-c)y_--3A-B+C}{(a-c)y_-+A-B+C}}\right). \tag{3.13}$$

Since $y > 0$, we have the following cases:

$$\{a-c>0,\left(C-A\right)\left(A-B+C\right)>0\}\Rightarrow y_{+}>0\ (y_{-}<0), \tag{3.14}$$

$$\{a-c>0,\left(C-A\right)\left(A-B+C\right)<0,D>0,F<0,\}\Rightarrow y_{+}>y_{-}>0, \tag{3.15}$$

$$\{a-c<0,F>0,\left(C-A\right)\left(A-B+C\right)>0,D>0\}\Rightarrow y_{-}>y_{+}>0, \tag{3.16}$$

$$\{a-c<0,(C-A)(A-B+C)<0\} \Rightarrow y_- > 0\ (y_+ < 0), \tag{3.17}$$

$$\{D=0 \wedge \operatorname{sgn} F = -\operatorname{sgn}(a-c)\} \Rightarrow y_+ = y_- = \tfrac{-F}{2c(a-c)}, \tag{3.18}$$

$$\{(C-A)(A-B+C)=0 \wedge \operatorname{sgn} F = -\operatorname{sgn}(a-c)\} \Rightarrow y_- = \tfrac{-F}{c(a-c)}\ , y_+ = 0. \tag{3.19}$$

We have to check the condition (3.8) in all of this situations. To this end, consideration will continue depending on the sign of the expression $3A+B-C$ assuming $D>0$.

On the other hand, we can find the minimal period-two solutions for some special cases of (1.1) (see cases 13–17 in Table 3.1).

In Table 3.1, we have all situations when (1.1), with positive parameters, has the minimal period-two solutions. In other cases, (1.1) has no minimal period-two solutions.

3.1 Local Stability when $A = C = 0, B = 1$

Consider the equation

$$x_{n+1} = \frac{Bx_nx_{n-1}}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{3.20}$$

which is special case of (1.1) with $A = C = 0$. As we see in Table 3.1, case 17, (3.20) has only one minimal period-two solution of the form

$$\phi = \frac{B}{2(a-c)}\left(1+\sqrt{\frac{a-b-3c}{a-b+c}}\right), \psi = \frac{B}{2(a-c)}\left(1-\sqrt{\frac{a-b-3c}{a-b+c}}\right), \tag{3.21}$$

when $a-b-3c>0$. It is interesting because we can investigate the local stability of the minimal period-two solution.

Assume that $B = 1$. By substitution

$$\begin{cases} x_{n-1} = u_n \\ x_n = v_n, \end{cases}$$

(3.20) becomes the system

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{u_nv_n}{av_n^2 + bu_nv_n + cu_n^2} \end{aligned} . \tag{3.22}$$

The map T corresponding to (3.22) is of the form

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ g(u,v) \end{pmatrix},$$

1.	$3A+B-C\leq 0,\ a>c$	Only one P2 solution of the form (3.12).
2.	$3A+B-C<0,\ a<c,$ $D>0,\ M<0,\ N\neq 0$	
3.	$3A+B-C<0,$ $a<c,\ D>0,\ N=0$	
4.	$3A+B-C>0,\ a>c,$ $(C-A)(A-B+C)>0,$ $N>0,\ M<0$	
5.	$3A+B-C>0,\ a>c,$ $(C-A)(A-B+C)>0,\ N\leq 0$	
6.	$3A+B-C>0,\ a>c,$ $(C-A)(A-B+C)<0,$ $D>0,\ F<0,\ M<0$	
7.	$C=A,\ a-b-3c>0$	Only one P2 solution generated by $y_{-}=-\frac{F}{c(a-c)}$
8.	$B=A+C,\ C(a-c)>A(b+2c)$	
9.	$D=0,\ a>c,$ $aC>c(5A+B)+bA,$ $aC>c(-A-B+2C)+bA$	Only one P2 solution generated by $y_{+}=y_{-}=-\frac{F}{2c(a-c)}$
10.	$D=0,\ a<c,$ $c(-A-B+2C)+bA>aC$ $aC>c(5A+B)+bA$	
11.	$3A+B-C<0,\ a<c,$ $D>0,\ N<0,\ M>0$	Two P2 solutions of the form (3.12) and (3.13)
12.	$3A+B-C>0, a>c,$ $(C-A)(A-B+C)<0,$ $F<0,\ D>0,\ N<0,\ M>0$	
13.	$b=a+c,\ a(C-A)>c(2A+b)$	Only one P2 solution: $\phi=\frac{C-A}{2c}\left(1+\widetilde{K}\right),$ $\psi=\frac{C-A}{2c}\left(1-\widetilde{K}\right),$ $\widetilde{K}=\sqrt{1-\frac{4cA}{(a-c)(C-A)+c(A-B+C)}}$
14.	$a=c, 3A+B-C<0$	Only one P2 solution: $\phi=\widetilde{P}\left(1+\widetilde{P}_1\right),$ $\psi=\widetilde{P}\left(1-\widetilde{P}_1\right),$ $\widetilde{P}=\frac{1}{2}\frac{(C-A)(A-B+C)}{Ab+a(C-A-B)},$ $\widetilde{P}_1=\sqrt{\frac{-3A-B+C}{A-B+C}}.$

<table>
<tr><td>15.</td><td>$a\left(B-a\right)+b\left(C-B\right)>3\left(Ab-aC\right)>0,$
$c=0,\ \left(C-A\right)\left(A-B+C\right)>0.$</td><td rowspan="2">Only one P2 solution:
$\phi=\frac{y}{2}\left(1+\widetilde{S}\right),$
$\psi=\frac{y}{2}\left(1-\widetilde{S}\right),$
$y=\frac{(C-A)(A-B+C)}{Ab-aC},$
$\widetilde{S}=\sqrt{\frac{ay-3A-B+C}{ay+A-B+C}}.$</td></tr>
<tr><td>16.</td><td>$a\left(B-a\right)+b\left(C-B\right)<3\left(Ab-aC\right)<0,$
$c=0,\ \left(C-A\right)\left(A-B+C\right)<0.$</td></tr>
<tr><td>17.</td><td>$A=C=0, a-b-3c>0$</td><td>Only one P2 solution:
$\phi=\frac{B}{2(a-c)}\left(1+\widetilde{S}_1\right),$
$\psi=\frac{B}{2(a-c)}\left(1-\widetilde{S}_1\right),$
$\widetilde{S}_1=\sqrt{\frac{a-b-3c}{a-b+c}}.$</td></tr>
</table>

Table 3.1: Existence of the minimal period-two solutions.

where $g\left(u,v\right)=\dfrac{uv}{av^2+buv+cu^2}$. The second iteration of the map T is

$$T^2\begin{pmatrix} u \\ v \end{pmatrix}=T\begin{pmatrix} v \\ g\left(u,v\right) \end{pmatrix}=\begin{pmatrix} g\left(u,v\right) \\ g\left(v,g\left(u,v\right)\right) \end{pmatrix}=\begin{pmatrix} F\left(u,v\right) \\ G\left(u,v\right) \end{pmatrix},$$

where

$$F\left(u,v\right)=g\left(u,v\right)\ \ \text{and}\ \ G\left(u,v\right)=\frac{vF\left(u,v\right)}{aF^2\left(u,v\right)+bvF\left(u,v\right)+cv^2}.$$

The Jacobian matrix of the map T^2 is

$$J_{T^2}\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{pmatrix},$$

where

$$\begin{aligned}
\frac{\partial F}{\partial u}&=\frac{\partial g}{\partial u}=\frac{v\left(av^2+buv+cu^2\right)-uv\left(bv+2cu\right)}{\left(av^2+buv+cu^2\right)^2}=\frac{av^3-cu^2v}{\left(av^2+buv+cu^2\right)^2},\\
\frac{\partial F}{\partial v}&=\frac{\partial g}{\partial v}=\frac{u\left(av^2+buv+cu^2\right)-uv\left(bu+2av\right)}{\left(av^2+buv+cu^2\right)^2}=\frac{cu^3-auv^2}{\left(av^2+buv+cu^2\right)^2},\\
\frac{\partial G}{\partial u}&=\frac{v\left(cv^2-aF^2\left(u,v\right)\right)\frac{\partial F}{\partial u}}{\left(aF^2\left(u,v\right)+bvF\left(u,v\right)+cv^2\right)^2},\\
\frac{\partial G}{\partial v}&=\frac{aF^3\left(u,v\right)+cv^3\frac{\partial F}{\partial v}-avF^2\left(u,v\right)\frac{\partial F}{\partial v}-cv^2F\left(u,v\right)}{\left(aF^2\left(u,v\right)+bvF\left(u,v\right)+cv^2\right)^2}.
\end{aligned}$$

Theorem 3.2. *Assume that* $a-b-3c>0, B=1$ *and that* $\{\phi,\psi\}$ *is a unique periodic solution of minimal period-two of* (3.20)*, given by* (3.21)*.*

(i) If $\Lambda = 4a^2b+7a^2c+ac^2+b^2c+c^3-4abc-3ab^2-a^3>0$*, then* $\{\phi,\psi\}$ *is locally asymptotically stable solution.*

(ii) If $\Lambda = 4a^2b+7a^2c+ac^2+b^2c+c^3-4abc-3ab^2-a^3<0$*, then* $\{\phi,\psi\}$ *is a saddle.*

(iii) If $\Lambda = 4a^2b+7a^2c+ac^2+b^2c+c^3-4abc-3ab^2-a^3=0$*, then* $\{\phi,\psi\}$ *is nonhyperbolic.*

Proof. We have that

$$1=\frac{\psi}{a\psi^2+b\phi\psi+c\phi^2}, \tag{3.23}$$

$$1=\frac{\phi}{a\phi^2+b\phi\psi+c\psi^2}, \tag{3.24}$$

$$F(\phi,\psi)=\phi,$$

and

$$\frac{\partial F}{\partial u}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\frac{\psi\left(a\psi^2-c\phi^2\right)}{\left(a\psi^2+b\phi\psi+c\phi^2\right)^2}\overset{((3.23))}{=}\frac{a\psi^2-c\phi^2}{a\psi^2+b\phi\psi+c\phi^2}=\psi\left(a-c\frac{\phi^2}{\psi^2}\right),$$

$$\frac{\partial F}{\partial v}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\frac{\phi\left(c\phi^2-a\psi^2\right)}{\left(a\psi^2+b\phi\psi+c\phi^2\right)^2}\overset{((3.23))}{=}\frac{\phi\left(c\phi^2-a\psi^2\right)}{\psi^2}=-\phi\left(a-c\frac{\phi^2}{\psi^2}\right).$$

Similarly, by (3.23) and (3.24), we obtain

$$\begin{aligned}\frac{\partial G}{\partial u}\begin{pmatrix}\phi\\ \psi\end{pmatrix}&=-\psi^2\left(a-c\frac{\psi^2}{\phi^2}\right)\left(a-c\frac{\phi^2}{\psi^2}\right),\\ \frac{\partial G}{\partial v}\begin{pmatrix}\phi\\ \psi\end{pmatrix}&=\frac{\psi\left(2a\phi+b\psi\right)\phi-\left(2a\phi+b\psi\right)\psi^2\frac{\partial F}{\partial v}\binom{\phi}{\psi}}{\left(a\phi^2+b\psi\phi+c\psi^2\right)^2}\\ &=\phi\left(a-c\frac{\psi^2}{\phi^2}\right)+\left(a-c\frac{\phi^2}{\psi^2}\right)\left(a-c\frac{\psi^2}{\phi^2}\right)\phi\psi.\end{aligned}$$

The Jacobian matrix of the map T^2 at the point (ϕ,ψ) is of the form

$$J_{T^2}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\frac{\partial F}{\partial u}\binom{\phi}{\psi} & \frac{\partial F}{\partial v}\binom{\phi}{\psi}\\ \frac{\partial G}{\partial u}\binom{\phi}{\psi} & \frac{\partial G}{\partial v}\binom{\phi}{\psi}\end{pmatrix}.$$

The corresponding characteristic equation is $\lambda^2 - p\lambda + q = 0$, where

$$\begin{aligned} p &= \frac{\partial F}{\partial u}\begin{pmatrix}\phi\\ \psi\end{pmatrix} + \frac{\partial G}{\partial v}\begin{pmatrix}\phi\\ \psi\end{pmatrix} \\ &= \psi\left(a - c\frac{\phi^2}{\psi^2}\right) + \phi\left(a - c\frac{\psi^2}{\phi^2}\right) + \left(a - c\frac{\phi^2}{\psi^2}\right)\left(a - c\frac{\psi^2}{\phi^2}\right)\phi\psi \\ &= \alpha + \beta + \alpha\beta, \end{aligned}$$

with

$$\alpha = \psi\left(a - c\frac{\phi^2}{\psi^2}\right), \quad \beta = \phi\left(a - c\frac{\psi^2}{\phi^2}\right), \tag{3.25}$$

and

$$q = \det J_{T^2}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \alpha\beta\left(1+\alpha\right) - \alpha^2\beta = \alpha\beta,$$

because

$$\begin{aligned} \frac{\partial F}{\partial u}\begin{pmatrix}\phi\\ \psi\end{pmatrix} &= \alpha, & \frac{\partial F}{\partial v}\begin{pmatrix}\phi\\ \psi\end{pmatrix} &= -\frac{\phi}{\psi}\alpha, \\ \frac{\partial G}{\partial u}\begin{pmatrix}\phi\\ \psi\end{pmatrix} &= -\frac{\psi}{\phi}\alpha\beta, & \frac{\partial G}{\partial v}\begin{pmatrix}\phi\\ \psi\end{pmatrix} &= \left(1+\alpha\right)\beta. \end{aligned}$$

Notice that

$$\begin{aligned} \alpha &= \psi\left(a - c\frac{\phi^2}{\psi^2}\right) = a\psi - c\frac{\phi^2}{\psi} \overset{(3.23)}{=} a\psi - \frac{1}{\psi}\left(\psi - a\psi^2 - b\phi\psi\right) = 2a\psi + b\phi - 1, \\ \beta &= \phi\left(a - c\frac{\psi^2}{\phi^2}\right) = a\phi - c\frac{\psi^2}{\phi} \overset{(3.24)}{=} a\phi - \frac{1}{\phi}\left(\phi - a\phi^2 - b\phi\psi\right) = 2a\phi + b\psi - 1. \end{aligned}$$

(i) We need show that

$$|p| < 1 + q \ \text{ and } \ q < 1.$$

First, we will show that $q + 1 > 0$. Indeed,

$$\begin{aligned} q + 1 &= \alpha\beta + 1 = \left(2a\psi + b\phi - 1\right)\left(2a\phi + b\psi - 1\right) + 1 \\ &= \left(4a^2 + b^2\right)\phi\psi + 2ab\left(\phi^2 + \psi^2\right) - \left(2a+b\right)\left(\phi+\psi\right) + 2 \\ &= \left(2a - b\right)^2\phi\psi + 2ab\left(\phi+\psi\right)^2 - \left(2a+b\right)\left(\phi+\psi\right) + 2 \\ &\overset{(3.21)}{=} \frac{c\left(2a-b\right)^2}{\left(a-b+c\right)\left(a-c\right)^2} + \frac{2ab}{\left(a-c\right)^2} - \frac{2a+b}{a-c} + 2 \\ &= \frac{ab\left(a-b\right) + c\left(2a^2 - bc\right) + 2c^3}{\left(a-b+c\right)\left(a-c\right)^2} > 0, \end{aligned}$$

because $a - b - 3c > 0 \Rightarrow (a > b \wedge a > c)$.

Using (3.25), we have that

$$\begin{aligned} q<1 &\Leftrightarrow \alpha\beta<1 \Leftrightarrow \phi\psi\left(a^2+c^2-ac\left(\frac{\phi^2}{\psi^2}+\frac{\psi^2}{\phi^2}\right)\right)<1 \\ &\Leftrightarrow a^2+c^2-ac\left(\frac{\phi^2}{\psi^2}+\frac{\psi^2}{\phi^2}\right)<\frac{1}{\phi\psi} \Leftrightarrow a^2+c^2-\frac{1}{\phi\psi}<ac\left(\frac{\phi^2}{\psi^2}+\frac{\psi^2}{\phi^2}\right). \end{aligned}$$

Since

$$2ac<ac\left(\tfrac{\phi^2}{\psi^2}+\tfrac{\psi^2}{\phi^2}\right),$$

we need to show that

$$a^2+c^2-\frac{1}{\phi\psi}<2ac.$$

Namely,

$$\begin{aligned} a^2+c^2-\frac{1}{\phi\psi}<2ac &\Leftrightarrow \frac{1}{\phi\psi}>a^2+c^2-2ac \Leftrightarrow \frac{(a-c)^2(a+c-b)}{c}>(a-c)^2 \\ &\Leftrightarrow a+c-b>c \Leftrightarrow a-b>0, \end{aligned}$$

which is satisfied because $a-b-3c>0$.

Similarly,

$$\begin{aligned} p<1+q &\Leftrightarrow \alpha+\beta<1 \Leftrightarrow 2a(\phi+\psi)+b(\phi+\psi)-2<1 \Leftrightarrow (\phi+\psi)(2a+b)<3 \\ &\Leftrightarrow 2a+3b<3(a-c) \Leftrightarrow a-b-3c>0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} p+q+1>0 &\Leftrightarrow \alpha+\beta+2\alpha\beta+1>0 \\ &\Leftrightarrow 1+\left(8a^2+2b^2\right)\phi\psi+4ab\left(\phi^2+\psi^2\right)-(2a+b)(\phi+\psi)>0 \\ &\Leftrightarrow 1+2(2a-b)^2\phi\psi+4ab(\phi+\psi)^2-(2a+b)(\phi+\psi)>0 \\ &\Leftrightarrow 1+\frac{2c(2a-b)^2}{(a-b+c)(a-c)^2}+\frac{4ab}{(a-c)^2}-\frac{2a+b}{a-c} \\ &\Leftrightarrow 4a^2b+7a^2c+ac^2+b^2c+c^3-4abc-3ab^2-a^3>0. \end{aligned}$$

(ii) The minimal period-two solution is a saddle if

$$|p|>|1+q| \Leftrightarrow (p-q-1)(p+q+1)>0.$$

Since $p-q-1<0$, we have that $p+q+1<0$, i.e.,

$$4a^2b+7a^2c+ac^2+b^2c+c^3-4abc-3ab^2-a^3<0.$$

(iii) Analogously, the minimal period-two solution is nonhyperbolic if

$$\begin{aligned} |p|=|1+q| &\Leftrightarrow (p-q-1)(p+q+1)=0 \Leftrightarrow p+q+1=0 \\ &\Leftrightarrow 4a^2b+7a^2c+ac^2+b^2c+c^3-4abc-3ab^2-a^3=0. \end{aligned}$$

□

Remark 3.3. We see that (3.20) has very complicated dynamics, particularly in the case c), where the equilibrium point is unstable, but minimal period-two solution $\{\phi, \psi\}$ can be stable, nonhyperbolic or unstable (a saddle), see Figures 3.1–3.5. The plots are produced by Dynamica 3 [20].

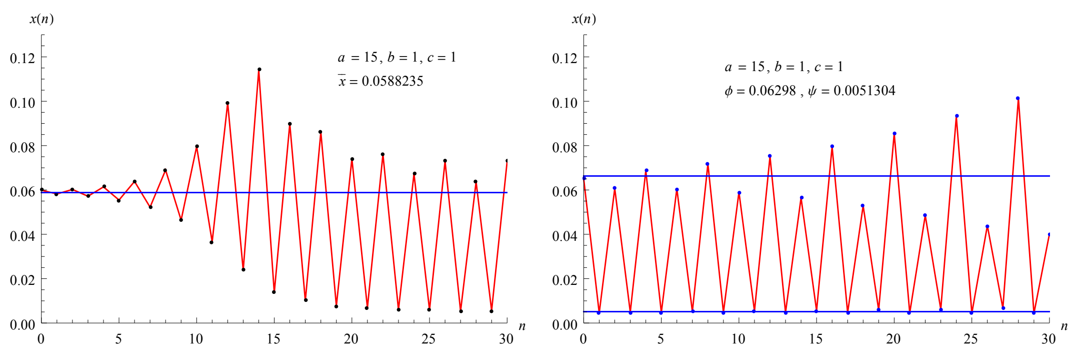
Remark 3.4. Notice that for the unique equilibrium point of (3.20), under the assumption $B = 1$, the following statements hold (see Theorem 2.1):

a) If $a - b - 3c < 0$, then the unique equilibrium point $\overline{x} = \frac{1}{a+b+c}$ is locally asymptotically stable.

b) If $a - b - 3c = 0$, then the unique equilibrium point $\overline{x} = \frac{1}{a+b+c}$ is nonhyperbolic with eigenvalues $\lambda_{1,2} \in \left\{-1, \frac{1}{2}\right\}$.

c) If $a - b - 3c > 0$, then the unique equilibrium point $\overline{x} = \frac{1}{a+b+c}$ is a saddle (i.e., $\overline{x}$ is unstable).

This leads to the following global asymptotic stability result for (3.20) (see Theorem 1.2).



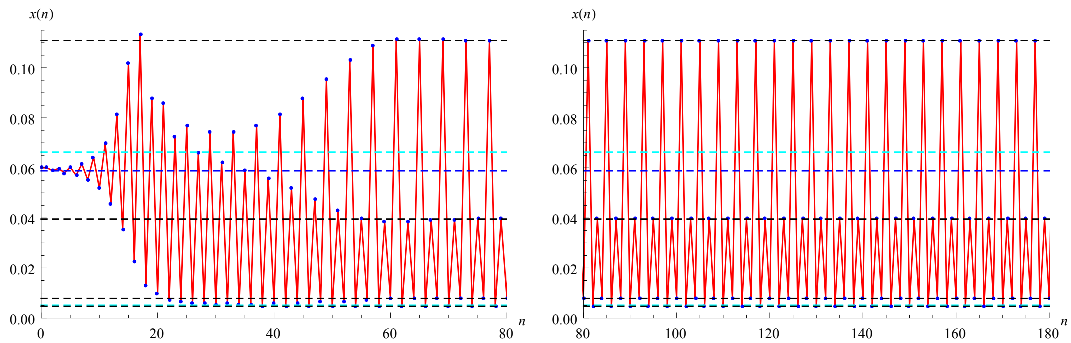
(a) (b)

Figure 3.1: a) For $x_{-1} = 0.0599$ and $x_0 = 0.0582$, $\overline{x} = \frac{1}{17}$ is unstable. b) The minimal period-two solution $\{\phi, \psi\}$ is unstable when $x_{-1} = 0.065$ and $x_0 = 0.0045$ (here is $\Lambda < 0$).

Theorem 3.5. *Consider* (3.20) *with* $B = 1$ *and* $a - b - 3c < 0$*, subject to the condition*

$$2(a + c)(1 + M(a + b + c)) < (a + b + c)^3 m^2,$$

where $m = \min\{\overline{x}, x_{-1}, x_0\}$ *and* $M = \max\{\overline{x}, x_{-1}, x_0\}$ *are lower and upper bounds of specific solution of* (3.20)*. Then, the unique equilibrium point* $\overline{x} = \frac{1}{a+b+c}$ *is globally asymptotically stable on the interval* $[m, M]$.



(a) (b)

Figure 3.2: The minimal period-four solution $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ where $\varphi_1 = 0.0046861$, $\varphi_2 = 0.034535$, $\varphi_3 = 0.11087$, $\varphi_4 = 0.0078329$, is stable, when $a = 15$, $b = 1$, $c = 1$, $x_{-1} = x_0 = 0.06$.

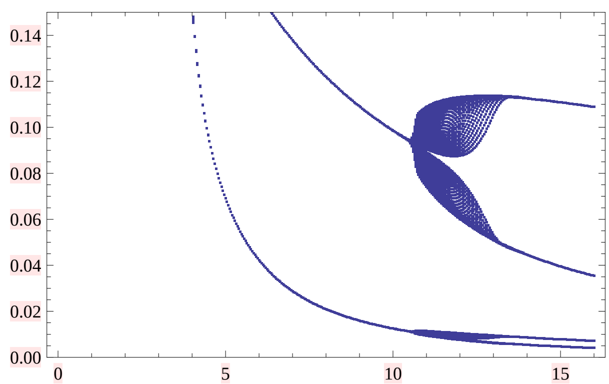


Figure 3.3: The minimal period-four solution on the bifurcation diagram for $b = 1$, $c = 1$ and $a \in (0, 16)$.

4 Neimark–Sacker Bifurcation

Consider (1.1) when $B = C = 0$ and assume that $A = 1$, that is consider the equation

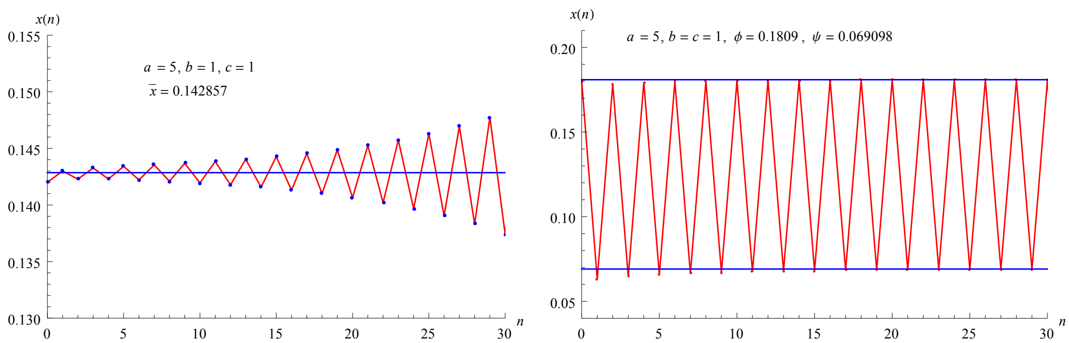
$$x_{n+1} = \frac{x_n^2}{ax_n^2 + bx_nx_{n-1} + cx_{n-1}^2}. \tag{4.1}$$

This equation has a unique equilibrium point $\overline{x} = \frac{1}{a+b+c}$.

Lemma 4.1. *If $a > c$, then equilibrium point $\overline{x}$ is locally asymptotically stable. If $a = c$, then equilibrium point $\overline{x}$ is nonhyperbolic with eigenvalues $\lambda_\pm = \frac{1 \pm i\sqrt{3}}{2}$. If $a < c$, then equilibrium point $\overline{x}$ is a repeller.*

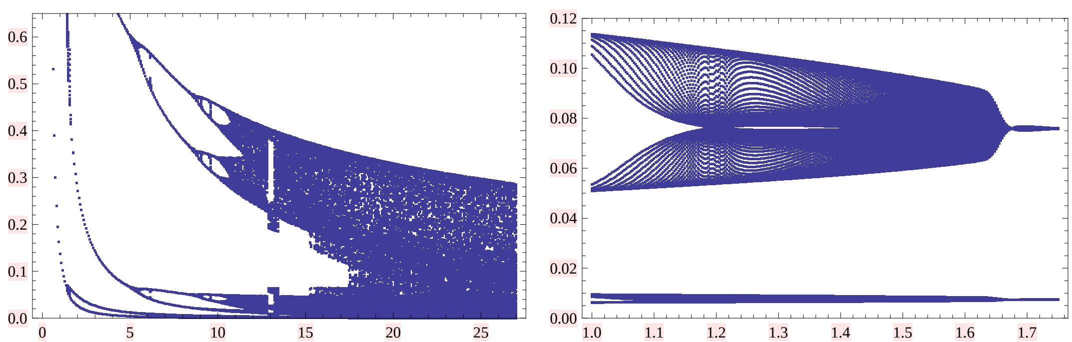
Proof. See Lemma 2.1. □

Unfortunately, in a global sense, we can give only the following result.



(a) (b)

Figure 3.4: a) $\overline{x} = \frac{1}{7}$ is unstable when $x_{-1} = 0.142$ and $x_0 = 0.143$; b) the minimal period-two solution $\{\phi, \psi\}$ is stable, when $x_{-1} = 0.18$ and $x_0 = 0.063$ (here is $\Lambda > 0$).



(a) (b)

Figure 3.5: Two bifurcation diagrams of a typical solution of (3.20): a) for $a \in (0.0, 30.0)$, $b = 0.2$, $c = 0.1$; b) for $b \in (1.0, 1.8)$, $a = 13$, $c = 1$.

Theorem 4.2. *If*

$$A\,(b+c)\,(M\,(a+b+c)+A) < (a+b+c)^2\, m^2,$$

where $m = \min\{\overline{x}, x_{-1}, x_0\}$ *and* $M = \max\{\overline{x}, x_{-1}, x_0\}$ *are lower and upper bounds of specific solution of* (4.1)*, then the unique equilibrium* $\overline{x}$ *is globally asymptotically stable on the interval* $[m, M]$.

Proof. See Theorem 1.2. □

Now, we consider bifurcation of a fixed point of map associated to (4.1) in the case where the eigenvalues are complex conjugate numbers on the unit circle. For this, we need the following result.

Theorem 4.3 (Poincaré–Andronov–Hopf Bifurcation for Maps)**.** *Let*

$$F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to F(\lambda, x)$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, 0) = 0$ *for λ near some fixed λ_0;*

(ii) $DF(\lambda, 0)$ *has two nonreal eigenvalues $\mu(\lambda)$ and $\overline{\mu(\lambda)}$ for λ near λ_0, $|\mu(\lambda_0)| = 1$;*

(iii) $\dfrac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) \neq 0$ *at $\lambda = \lambda_0$;*

(iv) $\mu^k(\lambda_0) \neq 1$ *for $k = 1, 2, 3, 4$.*

Then there is a smooth λ-dependent change of coordinate bringing f into the form

$$F(\lambda, x) = \mathcal{F}(\lambda, x) + O(||x||^5)$$

and there are smooth function $a(\lambda), b(\lambda)$ and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} |\mu(\lambda)|r - a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{4.2}$$

If $a(\lambda_0) > 0$ and $d\,(\lambda_0) > 0$ $(d\,(\lambda_0) < 0)$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda < \lambda_0$ $(\lambda > \lambda_0)$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda < \lambda_0$ $(\lambda > \lambda_0)$. Furthermore, $\Gamma(\lambda_0) = 0$.
If $a(\lambda_0) < 0$ and $d\,(\lambda_0) > 0$ $(d\,(\lambda_0) < 0)$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then α-limit set of x_0 is the origin if $\lambda > \lambda_0$ $(\lambda < \lambda_0)$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$ $(\lambda < \lambda_0)$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $F\,(\lambda, x)$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\overline{\mu(\lambda)} = \alpha(\lambda) - i\beta(\lambda)$ satisfying $(\alpha(\lambda))^2 + (\beta(\lambda))^2 = 1$ and $\beta(\lambda) \neq 0$. By putting the linear part of such a map into Jordan canonical form, we may assume F to have the following form near the origin

$$F\,(\lambda, x) = \begin{pmatrix} \alpha(\lambda) & -\beta(\lambda) \\ \beta(\lambda) & \alpha(\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1\,(\lambda, x_1, x_2) \\ g_2\,(\lambda, x_1, x_2) \end{pmatrix}. \tag{4.3}$$

Then the coefficient $a(\lambda_0)$ of the cubic term in (4.2) in polar coordinate is equal to

$$a\,(\lambda_0) = \mathrm{Re}\left(\tfrac{(1-2\mu(\lambda_0))\overline{\mu(\lambda_0)}^2}{1-\mu(\lambda_0)}\xi_{11}\xi_{20}\right) + \tfrac{1}{2}\left|\xi_{11}\right|^2 + \left|\xi_{02}\right|^2 - \mathrm{Re}\left(\overline{\mu\,(\lambda_0)}\xi_{21}\right), \tag{4.4}$$

where

$$\xi_{20} = \tfrac{1}{8}\left(\tfrac{\partial^2 g_1(0,0)}{\partial x_1^2} - \tfrac{\partial^2 g_1(0,0)}{\partial x_2^2} + 2\tfrac{\partial^2 g_2(0,0)}{\partial x_1 \partial x_2} + i\left(\tfrac{\partial^2 g_2(0,0)}{\partial x_1^2} - \tfrac{\partial^2 g_2(0,0)}{\partial x_2^2} - 2\tfrac{\partial^2 g_1(0,0)}{\partial x_1 \partial x_2}\right)\right), \tag{4.5}$$

$$\xi_{11} = \frac{1}{4}\left(\frac{\partial^2 g_1(0,0)}{\partial x_1^2} + \frac{\partial^2 g_1(0,0)}{\partial x_2^2} + i\left(\frac{\partial^2 g_2(0,0)}{\partial x_1^2} + \frac{\partial^2 g_2(0,0)}{\partial x_2^2}\right)\right), \tag{4.6}$$

$$\xi_{02} = \frac{1}{8}\left(\frac{\partial^2 g_1(0,0)}{\partial x_1^2} - \frac{\partial^2 g_1(0,0)}{\partial x_2^2} - 2\frac{\partial^2 g_2(0,0)}{\partial x_1 \partial x_2} + i\left(\frac{\partial^2 g_2(0,0)}{\partial x_1^2} - \frac{\partial^2 g_2(0,0)}{\partial x_2^2} + 2\frac{\partial^2 g_1(0,0)}{\partial x_1 \partial x_2}\right)\right), \tag{4.7}$$

and

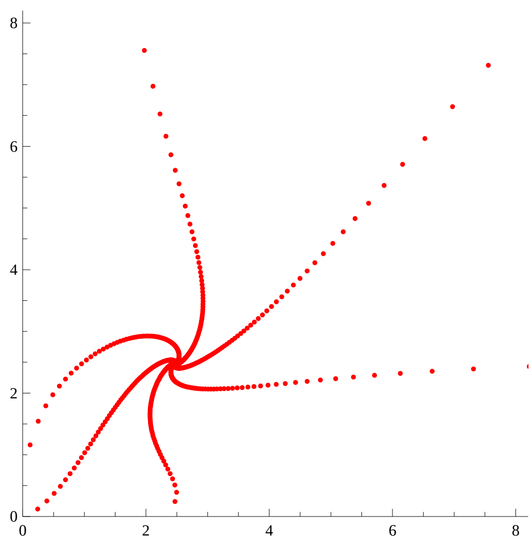
$$\xi_{21} = \frac{1}{16}\left(\frac{\partial^3 g_1}{\partial x_1^3} + \frac{\partial^3 g_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 g_2}{\partial x_1^2 \partial x_2} + \frac{\partial^3 g_2}{\partial x_2^3} + i\left(\frac{\partial^3 g_2}{\partial x_1^3} + \frac{\partial^3 g_2}{\partial x_1 \partial x_2^2} - \frac{\partial^3 g_1}{\partial x_1^2 \partial x_2} - \frac{\partial^3 g_1}{\partial x_2^3}\right)\right). \tag{4.8}$$

Theorem 4.4. *Let*

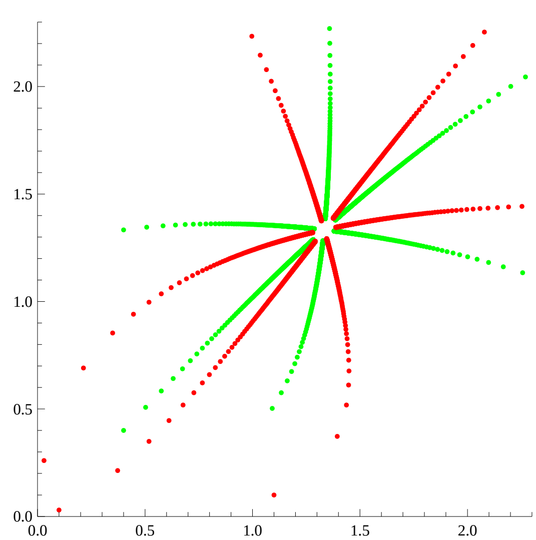
$$a_0 = c \quad \textit{and} \quad \overline{x} = \frac{1}{a+b+c}.$$

Then there is a neighborhood U of the equilibrium point $\overline{x}$ and a $\rho > 0$ such that for $|a - a_0| < \rho$ and $x_0, x_{-1} \in U$, the ω-limit set of a solution of (4.1)*, with initial condition x_0, x_{-1}, is the equilibrium point $\overline{x}$ if $a > a_0$ and belongs to a closed invariant C^1 curve $\Gamma(a)$ encircling $\overline{x}$ if $a < a_0$. Furthermore, $\Gamma(a_0) = 0$.*

Visual illustrations of Theorem 4.4 are given in Figures 4.1–4.3.



(a)



(b)

Figure 4.1: a) Phase portrait when $a = 0.055$, $b = 0.3$, $c = 0.05$ and $x_-1 = 16.1$, $x_0 = 16.1$. b) Phase portraits when $a = 0.25$, $b = 0.25$, $c = 0.25$, $x_-1 = 0.4$, $x_0 = 0.4$ (red) and $x_-1 = 1.1$, $x_0 = 0.1$ (green).

Proof. In order to apply Theorem 4.3, we make a change of variable $y_n = x_n - \overline{x}$. Then, the new equation is given by

$$y_{n+1} = \frac{(y_n + \overline{x})^2}{a(y_n + \overline{x})^2 + b(y_n + \overline{x})(y_{n-1} + \overline{x}) + c(y_{n-1} + \overline{x})^2} - \overline{x}.$$

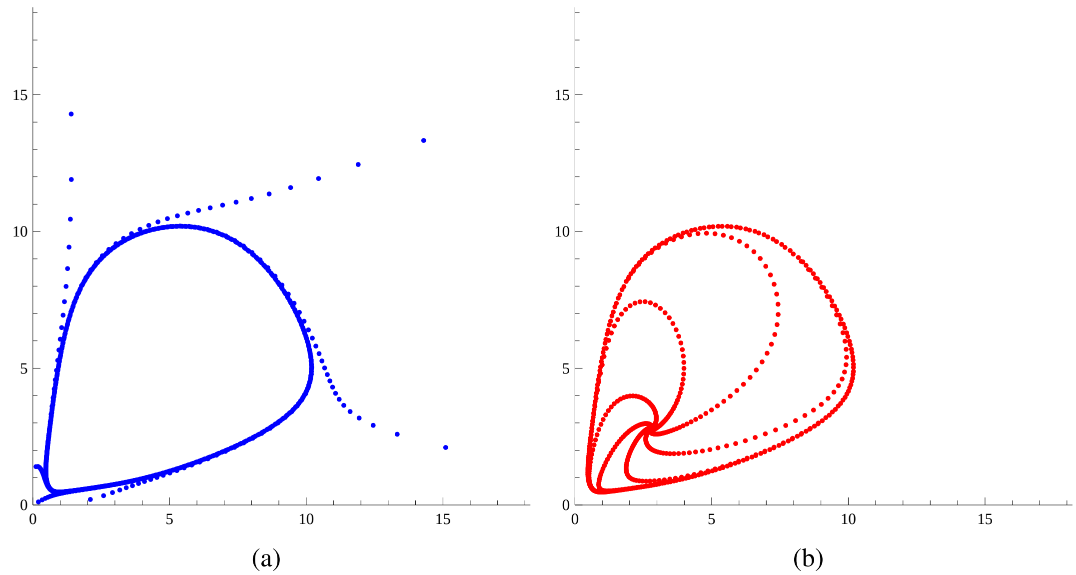


Figure 4.2: a) Phase portrait when $a = 0.05$, $b = 0.25$, $c = 0.06$, $x_-1 = 15.1$, $x_0 = 2.1$. b) Phase portrait when $a = 0.05$, $b = 0.25$, $c = 0.06$, $x_-1 = 2.8$, $x_0 = 2.8$.

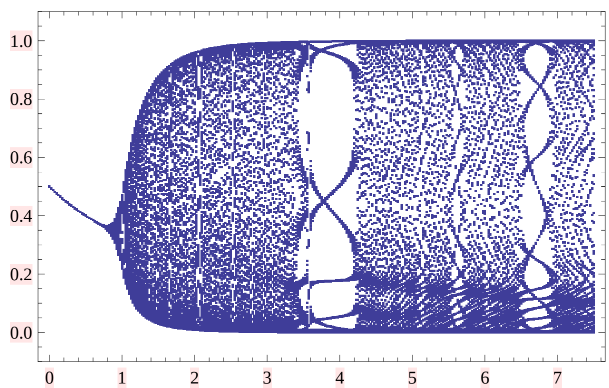


Figure 4.3: Bifurcation diagram in (c, x) plane for $A = 1$, $a = 1$, $b = 1$.

Set $u_n = y_{n-1}$ and $v_n = y_n$ for $n = 0, 1, \ldots$. The previous equation is equivalent with the system

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{(v_n+\overline{x})^2}{a(v_n+\overline{x})^2+b(u_n+\overline{x})(v_n+\overline{x})+c(u_n+\overline{x})^2} - \overline{x}. \end{aligned} \tag{4.9}$$

Let F be the function defined by

$$F(u, v) = \begin{pmatrix} v \\ \frac{(v+\overline{x})^2}{a(v+\overline{x})^2+b(v+\overline{x})(u+\overline{x})+c(u+\overline{x})^2} - \overline{x} \end{pmatrix}.$$

Then $F(u, v)$ has the unique fixed point $(0, 0)$ and the Jacobian matrix of $F(u, v)$ is given by

$$J_F(u,v)=\begin{pmatrix} 0 & 1 \\ R & S \end{pmatrix},$$

where

$$R=\frac{-(bv+2cu+b\overline{x}+2c\overline{x})(v+\overline{x})^2}{(buv+2av\overline{x}+bu\overline{x}+bv\overline{x}+2cu\overline{x}+av^2+cu^2+a\overline{x}^2+b\overline{x}^2+c\overline{x}^2)^2},$$
$$S=\frac{(bv+2cu+b\overline{x}+2c\overline{x})(v+\overline{x})(u+\overline{x})}{(buv+2av\overline{x}+bu\overline{x}+bv\overline{x}+2cu\overline{x}+av^2+cu^2+a\overline{x}^2+b\overline{x}^2+c\overline{x}^2)^2},$$

and

$$J_F(0,0)=\begin{pmatrix} 0 & 1 \\ \frac{-(b+2c)}{a+b+c} & \frac{b+2c}{a+b+c} \end{pmatrix}.$$

The eigenvalues of $J_F(0,0)$ are $\mu(a)$ and $\overline{\mu}(a)$:

$$\mu_{\pm}(a)=\frac{b+2c\pm i\sqrt{(b+2c)(4a+3b+2c)}}{2(a+b+c)}.$$

Since $\mu(a)=\alpha(a)+i\beta(a)$ and $\overline{\mu}(a)=\alpha(a)-i\beta(a)$, we have that

$$\alpha(a)=\frac{b+2c}{2(a+b+c)} \text{ and } \beta(a)=\frac{\sqrt{(b+2c)(4a+3b+2c)}}{2(a+b+c)}.$$

Assume that F has the following form near the origin

$$F(a,u,v)=\begin{pmatrix} 0 & 1 \\ \frac{-(b+2c)}{a+b+c} & \frac{b+2c}{a+b+c} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}+\begin{pmatrix} f_1(a,u,v) \\ f_2(a,u,v) \end{pmatrix}.$$

Then

$$\begin{pmatrix} v \\ \frac{(v+\overline{x})^2}{a(v+\overline{x})^2+b(v+\overline{x})(u+\overline{x})+c(u+\overline{x})^2}-\overline{x} \end{pmatrix}=F(a,u,v), \tag{4.10}$$

from which

$$f_1(a,u,v)=0,$$
$$f_2(a,u,v)=\frac{(v+\overline{x})^2}{a(v+\overline{x})^2+b(v+\overline{x})(u+\overline{x})+c(u+\overline{x})^2}-\overline{x}+\frac{b+2c}{a+b+c}u-\frac{b+2c}{a+b+c}v.$$

Let $a_0=c$. For $a=a_0$, we obtain

$$\overline{x}=\frac{1}{b+2c} \text{ and } J_0=\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues of $J_F(0,0)$ are $\mu_{\pm}(a_0)=\frac{1\pm i\sqrt{3}}{2}$, and the eigenvectors corresponding to $\mu(a)$ i $\overline{\mu(a)}$ are $v(a_0)$ i $\overline{v(a_0)}$, where

$$v(a_0)=\left(\frac{1-i\sqrt{3}}{2},1\right).$$

Note that

$$
\begin{aligned}
&|\mu(a_0)| = 1,\ \mu(a_0) = \frac{1}{2} + i\frac{\sqrt{3}}{2},\ \mu^2(a_0) = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\\
&\mu^3(a_0) = -1,\ \mu^4(a_0) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.
\end{aligned}
$$

For $a = a_0$ and $\overline{x} = \frac{1}{b+2c}$, (4.10) has the form

$$
F(u,v) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix},
$$

where

$$
h_1(u,v) = f_1(a_0,u,v) = 0
$$

and

$$
\begin{aligned}
h_2(u,v) &= f_2(a_0,u,v)\\
&= \frac{(bu + cu + cv + 2bcuv + bcu^2 + bcv^2 + b^2uv + 2c^2u^2 + 2c^2v^2)(u - v)}{bu + bv + 2cu + 2cv + 2bcuv + bcu^2 + bcv^2 + b^2uv + 2c^2u^2 + 2c^2v^2 + 1}.
\end{aligned}
$$

Hence (for $a = a_0$), (4.9) is equivalent to

$$
\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} h_1(u_n,v_n) \\ h_2(u_n,v_n) \end{pmatrix}.
$$

Let

$$
\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix},
$$

where

$$
P = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 0 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{2\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \end{pmatrix}.
$$

Then (4.9) is equivalent to

$$
\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + P^{-1} H\left(P \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}\right),
$$

where

$$
H \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix}.
$$

Let

$$
G \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix} = P^{-1} H\left(P \begin{pmatrix} u \\ v \end{pmatrix}\right).
$$

Now, if we set $u_1 = u(b+2c)+1$, $v_1 = (b+2c)\left(\frac{1}{2}u + \frac{1}{2}v\sqrt{3}\right)$, then

$$g_1(u,v) = \frac{1}{2}v\sqrt{3} - \frac{1}{b+2c} - \frac{1}{2}u + \frac{u_1^2}{cu_1^2 + cv_1^2 + bu_1v_1},$$
$$g_2(u,v) = -\tfrac{\sqrt{3}}{3}g_1(u,v).$$

By straightforward calculation, we obtain

$$(g_1)_{uu} = -\frac{b+3c}{2},\ \ (g_1)_{uv} = \tfrac{\sqrt{3}}{2}c,\quad (g_1)_{vv} = \tfrac{3}{2}(b+c),$$

$$(g_1)_{uuu} = \tfrac{3}{4}(b+4c)(b+2c),\quad (g_1)_{uuv} = \tfrac{\sqrt{3}}{4}b(b+2c),$$
$$(g_1)_{uvv} = -\tfrac{3}{4}(b+4c)(b+2c),\quad (g_1)_{vvv} = -\tfrac{9\sqrt{3}}{4}b(b+2c),$$
$$(g_2)_{uu} = \frac{\sqrt{3}(b+3c)}{6},\ \ (g_2)_{uv} = -\tfrac{1}{2}c,\quad (g_2)_{vv} = -\tfrac{\sqrt{3}}{2}(b+c),$$

$$(g_2)_{uuu} = -\tfrac{\sqrt{3}}{4}(b+4c)(b+2c),\quad (g_2)_{uuv} = -\tfrac{1}{4}b(b+2c),$$
$$(g_2)_{uvv} = \tfrac{\sqrt{3}}{4}(b+4c)(b+2c),\quad (g_2)_{vvv} = \tfrac{9}{4}b(b+2c),$$

and furthermore

$$\xi_{20} = -\tfrac{1}{4}\left(b+2c - i\tfrac{\sqrt{3}}{3}b\right),\ \ \xi_{11} = b\left(1 - i\tfrac{\sqrt{3}}{3}\right),$$
$$\xi_{02} = \tfrac{1}{4}\left(-(b+c) + i\tfrac{\sqrt{3}}{3}(b+3c)\right),$$
$$\xi_{21} = \frac{(b+2c)b}{8}\left(1 + i\sqrt{3}\right),$$

$$\begin{aligned} a(\lambda_0) &= \operatorname{Re}\left[\left(i\tfrac{\sqrt{3}}{2} + \tfrac{3}{2}\right)\left(1 - i\tfrac{\sqrt{3}}{3}\right)\tfrac{b}{4}\left(b+2c - i\tfrac{\sqrt{3}}{3}b\right)\right] + \tfrac{2}{3}b^2 \\ &+ \tfrac{1}{12}\left(b^2 + 3bc + 3c^2\right) - \operatorname{Re}\left(\frac{(b+2c)b}{4}\right) \\ &= \tfrac{b}{4}(b+2c) + \tfrac{2}{3}b^2 + \tfrac{1}{12}\left(b^2+3bc+3c^2\right) = \tfrac{1}{4}\left(3bc + 4b^2 + c^2\right) > 0. \end{aligned}$$

We can see that

$$|\mu(a)| = \alpha^2(a) + \beta^2(a) = \frac{(b+2c)^2}{4(a+b+c)^2} + \frac{(b+2c)(4a+3b+2c)}{4(a+b+c)^2} = \frac{b+2c}{a+b+c},$$

and so that

$$\left(\frac{d}{da}|\mu(a)|\right)_{a=a_0} = \left(\frac{d}{da}\left(\frac{b+2c}{a+b+c}\right)\right)_{a=a_0} = \left(\frac{-(b+2c)}{(a+b+c)^2}\right)_{a=a_0} = \frac{-1}{b+2c} < 0.$$

□

Based on our simulations, we pose the following conjecture.

Conjecture 4.5. In all considered equations the equilibrium is globally asymptotically stable whenever is locally stable.

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