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Equations with Separated Variables on Time Scales

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Abstract

We show that the well-known theory for classical ordinary differential equations with separated variables is not valid in case of equations on time scales. Namely, the uniqueness of solutions does not depend on the convergence of appropriate integrals.

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1 Introduction

Dynamic equations on time scales have become very interesting for many mathematicians in recent years (see [2, 3, 5], and references for others).

The main aim of this paper is to verify how the theory of dynamic equations with separated variables (see [6]) changes if we generalize classical equations to equations on time scales. We show that classical theorems are not true even under some additional assumptions.

We introduce some basic definitions (see [2]).

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Definition 1.1. A time scale is an arbitrary nonempty closed set of the real numbers.

Let $\mathbb{T}$ be a time scale. We denote $I_{\mathbb{T}} := I \cap \mathbb{T}$ for an arbitrary $I \subset \mathbb{R}$.
Now we define some operators.

Definition 1.2. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf \{s : s \in \mathbb{T}, s > t\},$$

$\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if $\sup \mathbb{T} < \infty$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup \{s : s \in \mathbb{T}, s < t\},$$

$\rho(\min \mathbb{T}) = \min \mathbb{T}$ if $\inf \mathbb{T} > -\infty$.

Definition 1.3. Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa = \overline{\mathbb{T} \setminus \{\sup \mathbb{T}\}}$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that for any given $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \text{for all} \quad s \in U_{\mathbb{T}}.$$

If $f^\Delta(t) = g(t)$ for every $t \in [a, b]_{\mathbb{T}}$, then we denote

$$f(b) = f(a) + \int_a^b g(t)\Delta t.$$

Let us introduce some special class of time scales that are the most important for this paper.

Definition 1.4. If there exists a sequence $\{t_n\}_{n=1}^\infty \subset \mathbb{T}$ such that $t_n \searrow 0$ and $\sigma(t_n) > t_n$, then we say that $\mathbb{T}$ is a DNC time scale at 0 (dense not classical).

In this paper, we assume that $\mathbb{T}$ means a DNC time scale and let, for simplicity, $\min \mathbb{T} = 0$. Moreover, we fix the symbol $\{t_n\}$ for a sequence that has the property mentioned in the above definition. We study the uniqueness of solutions of the following initial value problem for equations with separated variables

$$\begin{cases} x^\Delta(t) = f(t)g(x(t)), & t \in \mathbb{T}, \\ x(0) = 0, \end{cases} \tag{1.1}$$

where $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

We say that a function $z : \mathbb{T} \to \mathbb{R}$ is a solution of the problem (1.1) if $z(0) = 0$ and $z^\Delta(t) = f(t)g(z(t))$ for all $t \in \mathbb{T}$.

Rewriting from [6] the uniqueness of the zero solution of (1.1) (if $g(0) = 0$) in the classical case depends on divergence of the integral

$$\int_0^\varepsilon \frac{dx}{g(x)}.$$

Concerning solutions of the problem (1.1) in a local sense, all time scales can be generally divided into three following classes:

(i) $[0, a] \subset \mathbb{T}$ for some $a \in \mathbb{R}$;

(ii) $\sigma(0) > 0$;

(iii) $\mathbb{T}$ is a DNC time scale at 0.

In the first case (i), we are dealing with the classical theory of ordinary differential equations with separated variables. In the second case (ii), regardless of the time scale, the problem (1.1) has always exactly one solution in a local sense.

The DNC time scales are very wide class of different time scales. In this paper, we find different examples of the problem (1.1), where independently from any DNC time scale and from convergence of appropriate integral it may have exactly one solution or it may have more than one solution. In some cases it may even have no solution at all.

The following closed sets are some examples of DNC-time scales:

- $\{0\} \cup \{\tau_n : \quad n \in \mathbb{N}\}, \quad \tau_n \searrow 0$;
- $\{0\} \cup \bigcup\limits_{n\in\mathbb{N}} \left[\frac{1}{n}, \frac{1}{n} + \frac{1}{n^2}\right]$;
- $\{0\} \cup \left\{\frac{1}{n} : \quad n \in \mathbb{N}\right\} \cup \left\{\frac{1}{n} + \frac{1}{nk} : \quad n, k \in \mathbb{N}, \quad k > n\right\}$;
- the Cantor set.

In the theory of equations $x^{\Delta}(t) = F(t, x(t))$ on time scales it is natural to consider not necessarily continuous but so called rd-continuous functions F. It means such functions F that for any continuous $x : \mathbb{T} \to \mathbb{R}$ the function $\varphi(t) = F(t, x(t))$ has the following property:

φ is continuous at $t \in \mathbb{T}$ if $\sigma(t) = t$ and $\lim\limits_{s\to t} \varphi(s)$ is finite if $\rho(t) = t < \sigma(t)$ (see [2]).

Unfortunately, the Peano type theorem is not valid for equations with rd-continuous right-hand-side (see [1,4] for counterexample on time scale $\{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$).

We present below that for every DNC time scale $\mathbb{T}$ there exists a problem (1.1) without any solution, however the right-hand-side of the equation is an rd-continuous function with separated variables.

Namely, for any DNC time scale $\mathbb{T}$, we find an rd-continuous function of the form $f(t)g(x)$ such that the problem (1.1) has no solution.

Example 1.5. Let us define $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by the formulas

$$f(t) = \begin{cases} t, & \text{for } t = t_n, \\ 0, & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} 1, & \text{for } x = 0, \\ 0, & \text{for } x \neq 0. \end{cases}$$

It is not difficult to verify that the function given by the product of functions f and g as follows $F(t,x) = f(t)g(x)$, is rd-continuous. Suppose that $x(0) = 0$, $x^{\Delta}(t) = f(t)g(x(t))$ for all $t \in [0,\delta)_{\mathbb{T}}$ and some $\delta > 0$. Then $x(t) \geq 0$ on $[0,\delta)_{\mathbb{T}}$ since $x^{\Delta}(t) \geq 0$. Moreover, $x(t) > 0$ for $t > 0$, because if $x(t_n) = 0$, then $x^{\Delta}(t_n) > 0$. Hence for $t > 0$, we have $x^{\Delta}(t) = 0$, and therefore x is a constant positive function on $[0,\delta)_{\mathbb{T}}$. It means that $x(0) > 0$. We obtain the contradiction to the assumption that $x(0) = 0$.

We are able to prove only the following result corresponding to the classical theory of equations with separated variables.

Theorem 1.6. *Assume that the integral*

$$\int_0^{\epsilon} \frac{dx}{G(x)}$$

is divergent for $\epsilon \neq 0$*, the function* $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* f *is* Δ*–integrable and nonnegative on* $\mathbb{T}$*,* $xg(x) > 0$ *for* $x \neq 0$ *and*

$$G(x) = \begin{cases} \max\{g(y) : y \in [0,x]\}, & \textit{for } x > 0, \\ \min\{g(y) : y \in [x,0]\}, & \textit{for } x < 0. \end{cases}$$

Then the problem (1.1) *has only a trivial solution.*

Proof. Suppose that $z : [0,\delta]_{\mathbb{T}} \to \mathbb{R}$ is a nonzero solution of (1.1), then the function z does not change the sign. Assume that $z(T) > 0$ for some $T \in (0,\delta]_{\mathbb{T}}$, then there exists $t_0 \in [0,T)_{\mathbb{T}}$ such that $z(t) > 0$ for $t \in (t_0,T]_{\mathbb{T}}$ and $z(t_0) = 0$.

Let us assume, that $\sigma(t_0) > t_0$. Since $g(0) = 0$, we have $z^{\Delta}(t_0) = f(t_0)g(z(t_0)) = f(t_0)g(0) = 0$. Consequently $z(\sigma(t_0)) = 0$. This contradicts the definition of t_0. If $[t_0,h] \subset \mathbb{T}$ for some $h \in (t_0,T]$, then we obtain $z \equiv 0$ on $[t_0,h]$ from a well-known theorem for equations with separated variables (see [6], page 45) and from the fact, that

$$\int_0^{\varepsilon} \frac{dx}{g(x)} \geq \int_0^{\varepsilon} \frac{dx}{G(x)} = \infty$$

for $\varepsilon > 0$. The above result means that there exists a sequence $\{s_n\} \subset \mathbb{T}$ such that $s_n \searrow t_0$ and $\sigma(s_n) > s_n$ ($\mathbb{T}$ is DNC at t_0). Let us define $\bar{z} : \text{conv}\mathbb{T} \to \mathbb{R}$ and $\bar{\rho} : \text{conv}\mathbb{T} \to \mathbb{T}$ by the formulae

$$\bar{z}(t) = \begin{cases} z(t), & \text{for} \quad t \in \mathbb{T}, \\ \dfrac{z(s)(\sigma(s)-t) + z(\sigma(s))(t-s)}{\sigma(s)-s} & \text{for} \quad t \in (s,\sigma(s)),\ s \in \mathbb{T}, \end{cases}$$

$$\bar{\rho}(t) = \max\{s \in \mathbb{T} : s \leq t\}.$$

For each $\tau \in \mathbb{T}$ such that $\sigma(\tau) > \tau$, we have

$$\int_{\tau}^{\sigma(\tau)} \frac{z^{\Delta}(s)}{g(z(s))}\Delta s = \int_{\tau}^{\sigma(\tau)} \frac{\bar{z}^{\Delta}(\tau)}{g(z(\tau))}\Delta s = \int_{\tau}^{\sigma(\tau)} \frac{\bar{z}'(s)}{g(z(\bar{\rho}(s)))}\Delta s = \int_{\tau}^{\sigma(\tau)} \frac{\bar{z}'(s)}{g(z(\bar{\rho}(s)))}ds,$$

because $\bar{z}'$ is a constant function on the set $(\tau, \sigma(\tau))$.

Suppose that $y : \mathbb{T} \to \mathbb{R}$ is an integrable function. Let us define a new function $\bar{y} : \text{conv}\mathbb{T} \to \mathbb{R}$ by $\bar{y}|_{\mathbb{T}} = y$ and

$$\bar{y}(s) = y(t) \quad \text{for} \quad s \in (t, \sigma(t)),\ t \in \mathbb{T}.$$

Then

$$\int_{t_0}^{t} y(s)\Delta s = \int_{t_0}^{t} \bar{y}(s)ds, \quad t \in \mathbb{T}.$$

Indeed, we put

$$\bar{F}(t) = \int_{t_0}^{t} \bar{y}(s)ds, \quad t \in \text{conv}\mathbb{T},$$

$$F(t) = \int_{t_0}^{t} y(s)\Delta s \quad t \in \mathbb{T},$$

$$\text{and} \quad \tilde{F} = \bar{F}\Big|_{\mathbb{T}}.$$

For $t \in \mathbb{T}$, we have

$$\tilde{F}^{\Delta}(t) = \bar{F}'_{+}(t) = \bar{y}(t) = y(t) = F^{\Delta}(t),$$

where $\bar{F}'_{+}$ means the right-hand-side derivative of the function $\bar{F}$.

Of course $\tilde{F}(t_0) = 0 = F(t_0)$, so we can conclude that

$$\bar{F}(t) = F(t), \quad \text{for} \quad t \in \mathbb{T}.$$

Now put

$$y(s) = \frac{z^{\Delta}(s)}{g(z(s))} \quad \text{for} \quad s \in \mathbb{T},$$

and

$$\bar{y}(s) = \frac{\bar{z}'_{+}(s)}{g(z(\bar{\rho}(s)))} \text{ for } s \in \text{conv}\mathbb{T}.$$

Now we have

$$\int_{t_0}^{t} \frac{z^{\Delta}(s)}{g(z(s))}\Delta s = \int_{t_0}^{t} \frac{\bar{z}'_{+}(s)}{g(z(\bar{\rho}(s)))}ds = \int_{t_0}^{t} \frac{\bar{z}'(s)}{g(z(\bar{\rho}(s)))}ds.$$

The last equality follows from the fact, that $\bar{z}'(s)$ can be not defined only for $s \in \mathbb{T}$ such that $\rho(s) \neq s$ or $s \neq \sigma(s)$. The set of such s is countable. Therefore, the value of the integrand on that set does not affect the value of the Riemann integral. Then

$$\int_{t_0}^{t} f(s)\Delta s = \int_{t_0}^{t} \frac{z^{\Delta}(s)}{g(z(s))}\Delta s = \int_{t_0}^{t} \frac{\bar{z}'(s)}{g(z(\bar{\rho}(s)))}ds \geq$$
$$\geq \int_{t_0}^{t} \frac{\bar{z}'(s)}{G(z(\bar{\rho}(s)))}ds \geq \int_{t_0}^{t} \frac{\bar{z}'(s)}{G(\bar{z}(s))}ds = \int_{0}^{\bar{z}(t)} \frac{dx}{G(x)} = \infty$$

for $t \in (t_0, T]$, which contradicts the integrability of function f. Analogously, we obtain a contradiction if $z(T) < 0$ for some $T > 0$. □

2 Results

We show that in general there is no correspondence of the uniqueness of solutions of (1.1) with integrability of the function $\dfrac{1}{g(x)}$ and the sign of $xg(x)$ for $x \neq 0$.

Proposition 2.1. *There exists a* Δ*-integrable function* f*, such that* $\int_0^{\delta} f(t)\Delta t > 0$ *for any* $\delta > 0$*,* $f(t) \geq 0$*, a continuous function* g*, such that* $g(x) > 0$ *for* $x \neq 0$*,*

$$\int_0^{\epsilon} \frac{dx}{g(x)} < \infty$$

for $\varepsilon \neq 0$*, and the problem* (1.1) *has only trivial solution.*

Proposition 2.2. *There exists a continuous function* g *such that* $g(x) > 0$ *for* $x > 0$ *and*

$$\int_0^{\epsilon} \frac{dx}{g(x)} = \infty$$

for $\varepsilon > 0$*, but the problem* (1.1) *has a positive solution on* $\mathbb{T}\setminus\{0\}$ *for* $f(t) \equiv 1$*.*

Remark 2.3. One may prove an analogous result for negative solutions in a similar manner.

Proposition 2.4. *There exists a continuous function* g *such that* $xg(x) < 0$ *for* $x \neq 0$ *and the integral*

$$\int_0^{\epsilon} \frac{dx}{g(x)}$$

is divergent for $\varepsilon \neq 0$*, but the problem* (1.1) *for* $f(t) \equiv 1$ *has a nontrivial solution.*

3 Proofs

Proof of Proposition 2.1. Let

$$f(t) = \begin{cases} 0, & \text{for } t \neq t_n, \\ \dfrac{s_{n-1} - s_n}{\sigma(t_n) - t_n}, & \text{for } t = t_n, \end{cases}$$

where $s_1 = \dfrac{1}{2}$, $s_k = s_{k-1}^2$. Then

$$\int_0^{t_n} f(t)\Delta t = \sum_{j=n+1}^{\infty} f(t_j)(\sigma(t_j) - t_j) =$$
$$= \sum_{j=n+1}^{\infty} (s_{j-1} - s_j) = s_n > 0.$$

Suppose that $x(0) = 0$, $x^{\Delta}(t) = f(t)\sqrt{|x(t)|}$ and $x(t) \not\equiv 0$ on $[0, \delta)$ for any $\delta > 0$. Then $x(t) > 0$ for $t > 0$. Moreover, the function x is constant on any set $[\sigma(t_n), t_{n-1})_{\mathbb{T}}$, since $f(t) = 0$ for $t \in [\sigma(t_n), t_{n-1})_{\mathbb{T}}$.

Let us define the time scale

$$\mathbb{T}_0 = \{0\} \cup \{s_n\}_{n=1}^{\infty}$$

and function $z : \mathbb{T}_0 \to \mathbb{R}$ by the formula $z(0) = 0$, $z(s_n) = x(t_n)$. We can see, that

$$z^{\Delta}(s_n) = \frac{x(t_{n-1}) - x(t_n)}{s_{n-1} - s_n} = \frac{x(\sigma(t_n)) - x(t_n)}{s_{n-1} - s_n} =$$
$$= \frac{x^{\Delta}(t_n)}{f(t_n)} = \sqrt{z(s_n)},$$

hence

$$z(s_{n-1}) = z(s_n) + (s_{n-1} - s_n)\sqrt{z(s_n)}.$$

Let us substitute $y_n = \sqrt{4z(t_n)}$. Then we obtain

$$y_n = \sqrt{(s_{n-1} - s_n)^2 + y_{n-1}^2} - (s_{n-1} - s_n) < y_{n-1},$$

from the fact, that $\sqrt{a^2 + b^2} < a + b$ for $a, b > 0$. Moreover

$$y_n - y_{n+1} = y_n + s_n - s_{n+1} - \sqrt{(s_n - s_{n+1})^2 + y_n^2} < s_n - s_{n+1},$$

hence

$$y_n = \sum_{j=n}^{\infty} (y_j - y_{j+1}) < \sum_{j=n}^{\infty} (s_n - s_{n+1}) = s_n.$$

Consequently

$$z(s_n) < s_n^2.$$

Suppose that

$$z(s_n) < s_n^\alpha \tag{3.1}$$

for all natural n and some $\alpha \geq 2$. Since $s_{n+1} = s_n^2$, we have

$$z(s_n) = z(s_{n+1}) + (s_n - s_{n+1})\sqrt{z(s_{n+1})} < s_{n+1}^\alpha + (s_n - s_{n+1})s_{n+1}^{\alpha/2} =$$
$$= s_n^{2\alpha} + (s_n - s_n^2)s_n^\alpha = s_n^{\alpha+1}(s_n^{\alpha-1} + 1 - s_n) \leq s_n^{\alpha+1}.$$

This implies that (3.1) is satisfied for every $\alpha \geq 2$ and $z(s_n) = 0$, so $x(t_n) = 0$. We obtain that $x(t) \equiv 0$ on $[0, \delta]_{\mathbb{T}}$ for some $\delta > 0$, and then we see, that $x(t) \equiv 0$ on $\mathbb{T}$. □

It is obvious that if $\mathbb{T} = \mathbb{T}_0$, then the function f is continuous, namely $f(t) \equiv 1$.

Proof of Proposition 2.2. Let $z : \mathbb{T} \to \mathbb{R}$ be an arbitrary Δ-differentiable function such that $z(0) = z^\Delta(0) = 0$, and z^Δ is a strictly increasing and continuous function, for example

$$z(t) = \int_0^t s\Delta s.$$

Let us define

$$Z = z(\mathbb{T}) \cup \bigcup_{n\in\mathbb{N}} \left\{ \frac{z(t_n) + z(\sigma(t_n))}{2} \right\} \cup \left\{ x \in \mathbb{R} : x > \sup_{\mathbb{T}} z^\Delta(t) \right\}.$$

For $x \notin Z$, $x > 0$, let us define

$$a(x) = \max \{y \in Z : y < x\},$$
$$b(x) = \min \{y \in Z : y > x\}.$$

We define function $g : \mathbb{R} \to \mathbb{R}$ as follows

$$g(x) = \begin{cases} 0, & \text{for } x = 0, \\ z^\Delta(t), & \text{for } x = z(t),\ t \in \mathbb{T}, \\ z_n, & \text{for } x = \dfrac{z(t_n) + z(\sigma(t_n))}{2}, \\ g(\sup\limits_{\mathbb{T}} z^\Delta(t)), & \text{for } x \geq \sup\limits_{\mathbb{T}} z^\Delta(t), \\ \dfrac{g(a(x))(b(x) - x) + g(b(x))(x - a(x))}{b(x) - a(x)} & \text{for } x \notin Z,\ x > 0, \\ -g(-x), & \text{for } x < 0. \end{cases}$$

Of course, $z^\Delta(t) = g(z(t))$, $t \in \mathbb{T}$, $z(0) = 0$. Let us suppose that $z_n > 0$. Then for $x \neq 0$, we have $xg(x) > 0$, and we can choose $\{z_n\}$ such that the integral

$$\int_0^\epsilon \frac{dx}{g(x)}$$

is divergent for every $\varepsilon \neq 0$. □

Proof of Proposition 2.4. Let $a_1 = 1$ and

$$a_{k+1} = \frac{1}{k+1} \min \{a_k, \sigma(t_k) - t_k, \sigma(t_{k+1}) - t_{k+1}\},$$

$$x_k = (-1)^k a_k.$$

If $\sigma(t_{k+1}) = t_k$, then $y_k = x_{k+1}$, but if $\sigma(t_{k+1}) < t_k$, then y_k is close enough to x_{k+1} in a such way, that

$$A = \frac{t_k - \sigma(t_{k+1})}{y_k - x_{k+1}} \cdot \frac{x_k - x_{k+1}}{\sigma(t_k) - \sigma(t_{k+1})} \geq 2,$$

and

$$a_{k+3} < (-1)^{k+1} y_k < a_{k+1}.$$

We prove that there exists a continuous function $z : [0, t_1]_{\mathbb{T}} \to \mathbb{R}$, such that $z(0) = z^{\Delta}(0) = 0$, and

$$z(t_k) = y_k, \quad z(\sigma(t_k)) = x_k$$

for $k \in \mathbb{N}$. Furthermore, if $\sigma(t_{k+1}) < t_k$, then z^{Δ} is a continuous, monotonic and a sign constant function on $[\sigma(t_{k+1}), t_k]_{\mathbb{T}}$. Let

$$\begin{aligned} t &= s t_k + (1-s)\sigma(t_{k+1}), \\ y(s) &= \frac{z(t) - z(\sigma(t_{k+1}))}{z(t_k) - z(\sigma(t_{k+1}))}. \end{aligned} \tag{3.2}$$

Constructing the definition of the function z on $[\sigma(t_{k+1}), t_k]_{\mathbb{T}} \cup \{\sigma(t_k)\}$, where $\sigma(t_k) = s_0 t_k + (1 - s_0)\sigma(t_{k+1})$, comes down to finding a function $y : [0, 1]_{\mathbb{T}'} \cup \{s_0\} \to \mathbb{R}$, such that $y(0) = 0$, $y(1) = 1$, $y^{\Delta}(1) = A$, where $\mathbb{T}'$ is the image of $\mathbb{T}$ with the above change of variables (3.2). The following conditions for the function y are needed: the function y^{Δ} is continuous, monotonic, and a constant sign function on $[0, 1]_{\mathbb{T}'}$.

If the point 1 is left-scattered in $\mathbb{T}'$ (it means that t_k is left-scattered in $\mathbb{T}$), then $y(s) = s$ for $s \in [0, 1]$, $y(s_0) = A(s_0 - 1) + 1$. Suppose that point 1 is left-dense in $\mathbb{T}'$. Then

$$\lim_{n \to \infty} \int_0^1 \tau^n \Delta\tau = 0.$$

Let $n \in \mathbb{N}$ be, such that

$$\int_0^1 \tau^n \Delta\tau < \frac{1}{A}.$$

The function y satisfying the necessary assumptions is the following

$$y(s) = \frac{1 - A\int_0^1 \tau^n \Delta\tau}{1 - \int_0^1 \tau^n \Delta\tau} s + \frac{A - 1}{1 - \int_0^1 \tau^n \Delta\tau} \int_0^s \tau^n \Delta\tau.$$

Let us define

$$Z = \{0\} \cup \bigcup_{k \in \mathbb{N}} \operatorname{conv}\{x_{k+1}, y_k\} \cup \bigcup_{k \in \mathbb{N}} \left\{ \frac{x_{k+3} + y_k}{2} \right\} \cup (-\infty, x_1] \cup [x_2, \infty),$$

and

$$a(x) = \max \{y \in Z : y < x\},$$

$$b(x) = \min \{y \in Z : y > x\}$$

for $x \notin Z$. Now we define $g : [x_1, x_2] \to \mathbb{R}$ as

$$g(x) = \begin{cases} z^{\Delta}(z^{-1}(x)), & \text{for } x \in \operatorname{conv}\{x_{k+1}, y_k\}, \\ 0, & \text{for } x = 0, \\ z_k, & \text{for } x = \dfrac{x_{k+3} + y_k}{2}, \\ \dfrac{g(a(x))(b(x) - x) + g(b(x))(x - a(x))}{b(x) - a(x)} & \text{for } x \notin Z, \end{cases}$$

where $z_k y_k > 0$, $\lim_{k\to\infty} z_k = 0$. Moreover, we put $g(x) = g(x_1)$ for $x \leq x_1$, and $g(x) = g(x_2)$ for $x \geq x_2$. Of course, the function g is continuous on $\mathbb{R}\setminus\{0\}$ and

$$z^{\Delta}(t) = g(z(t)), \quad t \in \mathbb{T}\setminus\{0\}.$$

We can choose $\{z_n\}$, such that the following integral

$$\int_0^{\varepsilon} \frac{dx}{g(x)}$$

is divergent for every $\varepsilon \neq 0$. Moreover, for all $x \neq 0$, we have $xg(x) < 0$. It remains to show that $z^{\Delta}(0) = 0$, and that z^{Δ} is continuous at 0. If $k \to \infty$, then from the fact, that

$$|z^{\Delta}(t_k)| = \frac{|x_k - y_k|}{\sigma(t_k) - t_k} = \frac{a_k + |y_k|}{\sigma(t_k) - t_k} < \frac{a_k + a_{k+1}}{\sigma(t_k) - t_k} \leq \frac{2}{k} \to 0,$$

we obtain

$$\lim_{t\to 0} |z^{\Delta}(t)| \leq \lim_{k\to\infty} |z^{\Delta}(t_k)| = 0.$$

Moreover

$$|z^{\Delta}(0)| = \lim_{t\to 0} |\frac{z(t)}{t}| \leq \lim_{k\to\infty} |\frac{x_k}{\sigma(t_k)}| \leq \lim_{t\to\infty} \frac{a_k}{\sigma(t_k) - t_k} = 0.$$

Consequently, functions z and $x \equiv 0$ are solutions of the problem

$$\begin{cases} x^{\Delta}(t) = g(x(t)), & t \in [0, t_1]_{\mathbb{T}}, \\ x(0) = 0. \end{cases}$$

$\square$

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