International Journal of Difference Equations
ISSN 0973-6069, Volume 11, Number 2, pp. 163–178 (2016)
http://campus.mst.edu/ijde

Comparison Theorems and Asymptotic Behavior of Solutions of Caputo Fractional Equations

Baoguo Jia
Sun Yat-Sen University
School of Mathematics and Computational Science
Guangzhou, 510275, China
mcsjbg@mail.sysu.edu.cn

Lynn Erbe, Allan Peterson
University of Nebraska-Lincoln
Department of Mathematics
lerbe2@math.unl.edu
apeterson1@math.unl.edu

Abstract

We consider the ν-th order Caputo nabla fractional equation

$$\nabla_{a^*}^{\nu} x(t) = c(t)x(t), \qquad t \in \mathbb{N}_{a+1} \tag{0.1}$$

and establish theorems in which we compare the solutions x of (0.1) with the solutions of $\nabla_{a^*}^{\nu} x(t) = bx(t)$, where b is a constant. We obtain the following asymptotic results.

Theorem A. Assume $0 < \nu < 1$ and there exists a constant b_2 such that $0 < b_2 \leq c(t) < 1$. Then the solutions of the equation (0.1) with $x(a) > 0$ satisfy

$$\lim_{t \to \infty} x(t) = +\infty.$$

Theorem B. Assume $0 < \nu < 1$ and there exists a constant b_1 such that $c(t) \leq b_1 < 0$. Then the solutions of the equation (0.1) with $x(a) > 0$ satisfy

$$\lim_{t \to \infty} x(t) = 0.$$

Received April 18, 2016; Accepted May 5, 2016
Communicated by Martin Bohner

This shows that the solutions of the Caputo nabla fractional equation $\nabla^{\nu}_{a^*}x(t) = cx(t)$, $0 < \nu < 1$, have similar asymptotic behavior to the solutions of the first order nabla difference equation $\nabla x(t) = cx(t)$, $|c| < 1$.

AMS Subject Classifications: 39A12, 39A70.
Keywords: Caputo nabla fractional difference, Power rule, Rising function.

1 Introduction

Discrete fractional calculus has generated much interest in recent years. Some of the work has employed the forward or delta difference. We refer the readers to the papers [1] and [5], for example, and more recently [6,7]. Probably more work has been developed for the backward or nabla difference and we refer the readers to the manuscript [9] and the paper [8]. There has been some work to develop relations between the forward and backward fractional operators, Δ^{ν} and ∇^{ν} [2] and fractional calculus on time scales [5].

In [12], the authors consider the comparison theorems and asymptotic behavior of solutions of nabla and delta fractional equations. In this paper, we continue our study of comparison theorems and asymptotic behavior of solutions of certain discrete Caputo nabla fractional equations.

We note that there is a substantial difference between the proofs of Theorem B and the main results in [12] (see Theorem B and Theorem D in [12]).

We are concerned with the following so-called ν-th order Caputo nabla fractional difference equation.

$$\nabla^{\nu}_{a^*}x(t) = c(t)x(t), \qquad t \in \mathbb{N}_{a+1}. \tag{1.1}$$

Our Theorems A and B, show that the solutions of the Caputo nabla fractional difference equation $\nabla^{\nu}_{a^*}x(t) = c(t)x(t), 0 < \nu < 1, t \in \mathbb{N}_a$ have similar asymptotic behavior to the solutions of the first order nabla difference equation $\nabla x(t) = bx(t)$, $|b| < 1$.

2 Asymptotic Behavior, $1 > c(t) \geq b_1 > 0$

Let $\Gamma(x)$ denote the gamma function. Then we define the rising function (see [10]) by

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)},$$

for those values of t and r such that the right hand side of this equation is well defined. We also use the standard extensions of the domain of this rising function by defining it to be zero when the numerator is well defined, but the denominator is not defined. We will be interested in functions defined on sets of the form

$$\mathbb{N}_a := \{a, a+1, a+2, \cdots\},$$

where $a \in \mathbb{R}$. The nabla fractional Taylor monomials of degree ν based at a (see [6]) are defined by

$$H_\nu(t,a) := \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}.$$

First we define the nabla fractional sum (see [6]) as follows.

Definition 2.1. (Nabla Fractional Sum) Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ be given and $\mu > 0$. Then we define

$$\nabla_a^{-\mu} f(t) := \int_a^t H_{\mu-1}(t,\rho(s)) f(s) \nabla s,$$

for $t \in \mathbb{N}_{a+1}$, where by convention $\nabla_a^{-\mu} f(a) = 0$.

Next we define the Caputo nabla fractional difference in terms of the nabla fractional sum as follows:

Definition 2.2. Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $N-1 < \mu \leq N$, where $N \in \mathbb{N}_1$. Then the μ-th Caputo nabla fractional difference of f is defined by

$$\nabla_{a^*}^{\mu} f(t) := \nabla_a^{-(N-\mu)} \nabla^N f(t)$$

for $t \in \mathbb{N}_{a+1}$.

Lemma 2.3. *Assume that* $c(t) < 1$, $0 < \nu < 1$. *Then any solution of*

$$\nabla_{a*}^{\nu} x(t) = c(t)x(t), \qquad t \in \mathbb{N}_{a+1} \tag{2.1}$$

satisfying $x(a) > 0$ *is positive on* $\mathbb{N}_a$.

Proof. Using integration by parts (see [6]) and

$$\nabla_s H_{-\nu}(t,s) = -H_{-\nu-1}(t,\rho(s)),$$

we have

$$\begin{aligned}
\nabla_{a*}^{\nu} x(t) &= \nabla_a^{-(1-\nu)} \nabla x(t) \\
&= \int_a^t H_{-\nu}(t,\rho(s)) \nabla x(s) \nabla s \\
&= H_{-\nu}(t,s)x(s)\big|_{s=a}^{t} + \int_a^t H_{-\nu-1}(t,\rho(s)) x(s) \nabla s \\
&= -H_{-\nu}(t,a)x(a) + \sum_{s=a+1}^{t} H_{-\nu-1}(t,\rho(s))x(s).
\end{aligned}$$

Taking $t = a + k$, we have

$$\begin{aligned}
&\nabla_{a*}^{\nu} x(t) = \nabla_{a*}^{\nu} x(a+k) \\
= \;& x(a+k) - \nu x(a+k-1) - \frac{\nu(-\nu+1)}{2!} x(a+k-2) - \cdots \\
- \;& \frac{\nu(-\nu+1)\cdots(-\nu+k-2)}{(k-1)!} x(a+1) - \frac{(-\nu+1)\cdots(-\nu+k-1)}{(k-1)!} x(a).
\end{aligned}$$

Using (2.1), we get

$$\begin{aligned}
&x(a+k) \\
= \;& \frac{1}{1-c(a+k)} \Big[\nu x(a+k-1) + \frac{\nu(-\nu+1)}{2!} x(a+k-2) + \cdots \\
+ \;& \frac{\nu(-\nu+1)\cdots(-\nu+k-2)}{(k-1)!} x(a+1) + \frac{(-\nu+1)\cdots(-\nu+k-1)}{(k-1)!} x(a) \Big].
\end{aligned}$$

Using the strong induction principle, $0 < \nu < 1$ and $x(a) > 0$, it is easy to see that $x(a+k) > 0$, for $k \in \mathbb{N}_0$. □

The following comparison theorem plays an important role in proving our main results.

Theorem 2.4. *Assume* $c_2(t) \leq c_1(t) < 1$, $0 < \nu < 1$, *and* $x(t), y(t)$ *are the solutions of the equations*

$$\nabla_{a^*}^{\nu} x(t) = c_1(t) x(t), \tag{2.2}$$

and

$$\nabla_{a^*}^{\nu} y(t) = c_2(t) y(t), \tag{2.3}$$

respectively, for $t \in \mathbb{N}_{a+1}$ *satisfying* $x(a) \geq y(a) > 0$. *Then*

$$x(t) \geq y(t),$$

for $t \in \mathbb{N}_a$.

Proof. Similar to the proof of Lemma 2.3, taking $t = a + k$, we have

$$\begin{aligned}
&x(a+k) \qquad (2.4) \\
= \;& \frac{1}{1-c_1(a+k)} \Big[\nu x(a+k-1) + \frac{\nu(-\nu+1)}{2!} x(a+k-2) + \cdots \\
+ \;& \frac{\nu(-\nu+1)\cdots(-\nu+k-2)}{(k-1)!} x(a+1) + \frac{(-\nu+1)\cdots(-\nu+k-1)}{(k-1)!} x(a) \Big].
\end{aligned}$$

$$\begin{aligned}
&y(a+k) \qquad (2.5) \\
= \;& \frac{1}{1-c_2(a+k)} \Big[\nu y(a+k-1) + \frac{\nu(-\nu+1)}{2!} y(a+k-2) + \cdots \\
+ \;& \frac{\nu(-\nu+1)\cdots(-\nu+k-2)}{(k-1)!} y(a+1) + \frac{(-\nu+1)\cdots(-\nu+k-1)}{(k-1)!} y(a) \Big].
\end{aligned}$$

We will prove $x(a+k) \geq y(a+k) > 0$ for $k \in \mathbb{N}_0$ by using the principle of strong induction. By assumption $x(a) \geq y(a) > 0$ so the base case holds. Now assume that $x(a+i) \geq y(a+i) > 0$, for $i = 0, 1, , \cdots , k-1$.

Since $c_2(t) \leq c_1(t) < 1$, and

$$\frac{\nu(-\nu+1)\cdots(-\nu+i-1)}{i!} > 0,$$

the base case $k = 1$ for $i = 2, 3, \cdots k-1$, and

$$\frac{(-\nu+1)(-\nu+2)\cdots(-\nu+k-1)}{(k-1)!} > 0,$$

from (2.4), (2.5) we have

$$x(a+k) \geq y(a+k) > 0.$$

This completes the proof.

□

The following definition of the nabla Mittag–Leffler function (see [6] and [3]) is given as follows.

Definition 2.5. For $|p| < 1$, $0 < \nu < 1$, we define the nabla Mittag–Leffler function by

$$E_{p,\nu,0}(t,a) := \sum_{k=0}^{\infty} p^k H_{\nu k}(t,a), \quad t \in \mathbb{N}_a.$$

Remark 2.6. Since $H_0(t,a) = 1$, we have that $E_{0,\nu,0}(t,a) = 1$ and $E_{p,\nu,0}(a,a) = 1$.

The following lemma is taken from [6, Chapter 3].

Lemma 2.7. *Assume* $f : \mathbb{N}_a \to \mathbb{R}$ *and* $0 < \nu < 1$. *Then*

$$\nabla_a^{\nu} f(t) = \int_a^t H_{-\nu-1}(t,\rho(s)) f(s) \nabla s,$$

for $t \in \mathbb{N}_{a+1}$.

Lemma 2.8. *Assume that* $0 < \nu < 1$, $|b| < 1$. *Then*

$$\nabla_{a^*}^{\nu} E_{b,\nu,0}(t,a) = b E_{b,\nu,0}(t,a)$$

for $t \in \mathbb{N}_{a+1}$.

Proof. From Definition 2.1 and 2.2, using integration by parts, we have

$$\begin{aligned}
&\nabla_{a^*}^{\nu} E_{b,\nu,0}(t,a) \\
&= \int_a^t H_{-\nu}(t,\rho(s)) \nabla E_{b,\nu,0}(s,a) \nabla s \\
&= [H_{-\nu}(t,s) E_{b,\nu,0}(s,a)]_{s=a}^t + \int_a^t H_{-\nu-1}(t,\rho(s)) E_{b,\nu,0}(s,a) \nabla s \\
&= -H_{-\nu}(t,a) + \int_a^t H_{-\nu-1}(t,\rho(s)) \sum_{k=0}^{\infty} b^k H_{\nu k}(s,a) \nabla s, \qquad (2.6)
\end{aligned}$$

where we use $H_{-\nu}(t,t) = 0$ and $E_{b,\nu,0}(a,a) = 1$. In the following, we first prove that the infinite series

$$H_{-\nu-1}(t,\rho(s)) \sum_{k=0}^{\infty} b^k H_{\nu k}(s,a), \qquad (2.7)$$

for each fixed t, is uniformly convergent for $s \in [a,t]$.

We will first show that

$$|H_{-\nu-1}(t,\rho(s))| = \left| \frac{\Gamma(-\nu+t-s)}{\Gamma(t-s+1)\Gamma(-\nu)} \right| \le 1$$

for $a \le s \le t$. For $s = t$ we have that

$$|H_{-\nu-1}(t,\rho(t))| = 1.$$

Now assume that $a \le s < t$, then

$$\begin{aligned}
\left| \frac{\Gamma(-\nu+t-s)}{\Gamma(t-s+1)\Gamma(-\nu)} \right| &= \left| \frac{(t-s-\nu-1)(t-s-\nu-2)\cdots(-\nu)}{(t-s)!} \right| \\
&= \left| \frac{t-s-(\nu+1)}{t-s} \right| \left| \frac{t-s-1-(\nu+1)}{t-s-1} \right| \cdots \left| \frac{-\nu}{1} \right| \\
&\le 1.
\end{aligned}$$

Also consider

$$\begin{aligned}
H_{\nu k}(s,a) &= \frac{\Gamma(\nu k+s-a)}{\Gamma(s-a)\Gamma(\nu k+1)} \\
&= \frac{(\nu k+s-a-1)\cdots(\nu k+1)}{(s-a-1)!}.
\end{aligned}$$

Note that for large k it follows that

$$\begin{aligned}
H_{\nu k}(s,a) &\le (\nu k+s-a-1)^{s-a-1} \\
&\le (\nu k+t-a-1)^{t-a-1}
\end{aligned}$$

for $a \leq s \leq t$. Applying the root test to the infinite series in (2.7) we get that for each fixed t

$$\lim_{k\to\infty} \sqrt[k]{b^k(\nu k + t - a - 1)^{t-a-1}} = |b| < 1.$$

Hence for each fixed t the infinite series in (2.7) is uniformly convergent for $s \in [a, t]$. So from (2.6), integrating term by term, we get, (using Lemma 2.7 and $\nabla_a^\nu H_{\nu k}(s,a)) = H_{\nu k-\nu}(s,a)$), that

$$\begin{aligned}
\nabla_{a*}^\nu E_{b,\nu,0}(t,a) &= -H_{-\nu}(t,a) + \sum_{k=0}^{\infty} b^k \int_a^t H_{-\nu-1}(t,\rho(s))H_{\nu k}(s,a)\nabla s \\
&= -H_{-\nu}(t,a) + \sum_{k=0}^{\infty} b^k \nabla_a^\nu H_{\nu k}(t,a) \\
&= -H_{-\nu}(t,a) + \sum_{k=0}^{\infty} b^k H_{\nu k-\nu}(t,a) \\
&= \sum_{k=1}^{\infty} b^k H_{\nu k-\nu}(t,a) \\
&= bE_{b,\nu,0}(t,a),
\end{aligned}$$

where we also use $H_0(t,a) = 1$. This completes the proof. □

With the aid of Lemma 2.8, we may now give a rigorous proof of the following result.

Lemma 2.9. *Assume that* $0 < \nu < 1$, $|b| < 1$. *Then* $E_{b,\nu,0}(t,a)$ *is the unique solution of the Caputo nabla fractional IVP*

$$\begin{aligned}
&\nabla_{a^*}^\nu x(t) = bx(t), \qquad t \in \mathbb{N}_{a+1} \\
&x(a) = 1.
\end{aligned} \tag{2.8}$$

Proof. If $b = 0$, then

$$E_{0,\nu,0}(t,a) = 1.$$

From [6, Chapter 3], for any constant C, we have $\nabla_{a^*}^\nu C = 0$. So

$$\nabla_{a^*}^\nu E_{0,\nu,0}(t,a) = 0.$$

Now assume $b \neq 0$. From Lemma 2.8, we have

$$\nabla_{a^*}^\nu E_{b,\nu,0}(t,a) = bE_{b,\nu,0}(t,a).$$

The proof of the uniqueness is straightforward (see [6]). This completes the proof. □

Note next that the following lemma, given in Podlubny [13], is useful in proving asymptotic properties of certain fractional Taylor monomials and certain nabla Mittag–Leffler functions.

Lemma 2.10. *Assume* $\Re(z) > 0$. *Then*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.$$

The following lemma gives an asymptotic property for certain nabla fractional Taylor monomials.

Lemma 2.11. *Assume that* $0 < \nu < 1$. *Then we have*

$$\lim_{t\to\infty} H_{\nu k}(t,a) = \infty, \quad \textit{for} \quad k \geq 1,$$
$$\lim_{t\to\infty} H_{\nu k}(t,a) = 1, \quad \textit{for} \quad k = 0.$$

Proof. Taking $t = a + n$, $n \geq 0$, we have

$$\begin{aligned}\lim_{t\to\infty} H_{\nu k}(t,a) &= \lim_{n\to\infty} H_{\nu k}(a+n,a) = \lim_{n\to\infty} \frac{n^{\overline{\nu k}}}{\Gamma(\nu k+1)} \\ &= \lim_{n\to\infty} \frac{\Gamma(\nu k+n)}{\Gamma(n)\Gamma(\nu k+1)} \\ &= \lim_{n\to\infty} \frac{(\nu k+n-1)(\nu k+n-2)\cdots(\nu k+1)}{(n-2)!(n-2)^{\nu k+1}} \cdot \frac{(n-2)^{\nu k+1}}{n-1}.\end{aligned} \tag{2.9}$$

Using Lemma 2.10 with $z = \nu k + 1$ and n replaced by $n-2$, we have

$$\lim_{n\to\infty} \frac{(\nu k+1+n-2)(\nu k+1+n-3)\cdots(\nu k+1)}{(n-2)!(n-2)^{\nu k+1}} = \frac{1}{\Gamma(\nu k+1)},$$

and

$$\lim_{n\to\infty} \frac{(n-2)^{\nu k+1}}{n-1} = \infty, \quad \text{for} \quad k \geq 1,$$
$$\lim_{n\to\infty} \frac{(n-2)^{\nu k+1}}{n-1} = 1, \quad \text{for} \quad k = 0.$$

Using (2.9), we complete the proof. □

Theorem 2.12. *Assume* $0 < b_2 \leq c(t) < 1$, $t \in \mathbb{N}_{a+1}$, $0 < \nu < 1$. *Further assume* $x(t)$ *is a solution of the Caputo nabla fractional difference equation*

$$\nabla^{\nu}_{a^*} x(t) = c(t)x(t), \quad t \in \mathbb{N}_{a+1} \tag{2.10}$$

satisfying $x(a) > 0$. *Then*

$$x(t) \geq \frac{x(a)}{2} E_{b_2,\nu,0}(t,a),$$

for $t \in \mathbb{N}_{a+1}$.

Proof. From Lemma 2.9, we have

$$\nabla_{a^*}^{\nu} E_{b_2,\nu,0}(t,a) = b_2 E_{b_2,\nu,0}(t,a)$$

and $E_{b_2,\nu,0}(a,a) = 1$.

In Theorem 2.4, take $c_2(t) = b_2$. Then $x(t)$ and

$$y(t) = \frac{x(a)}{2} E_{b_2,\nu,0}(t,a)$$

satisfy

$$\nabla_{a^*}^{\nu} x(t) = c(t)x(t), \tag{2.11}$$

and

$$\nabla_{a^*}^{\nu} y(t) = b_2 y(t), \tag{2.12}$$

respectively, for $t \in \mathbb{N}_{a+1}$ and

$$x(a) > \frac{x(a)}{2} E_{b_2,\nu,0}(a,a) = y(a).$$

From Theorem 2.4, we get that

$$x(t) \geq \frac{x(a)}{2} E_{b_2,\nu,0}(t,a),$$

for $t \in \mathbb{N}_a$. This completes the proof. □

From Lemma 2.11 and the definition of $E_{b_2,\nu,0}(t,a)$, we get the following theorem.

Theorem 2.13. *For* $0 < b_2 < 1$*, we have*

$$\lim_{t\to\infty} E_{b_2,\nu,0}(t,a) = +\infty.$$

From Theorem 2.12 and Theorem 2.13, we have that the following result holds.

Theorem A. Assume $0 < \nu < 1$ and there exists a constant b_2 such that $0 < b_2 \leq c(t) < 1$. Then the solutions of the equation (1.1) with $x(a) > 0$ satisfy

$$\lim_{t\to\infty} x(t) = +\infty.$$

3 Asymptotic Behavior, $c(t) \leq b_1 < 0$,

Lemma 3.1. *Assume* $f : \mathbb{N}_a \to \mathbb{R}$*,* $0 < \nu < 1$*. Then*

$$\nabla_a^{-(1-\nu)} \nabla f(t) = \nabla \nabla_a^{-(1-\nu)} f(t) - f(a) H_{-\nu}(t,a). \tag{3.1}$$

Proof. Using integration by parts and $H_{-\nu}(t,t)=0$, we see that

$$\nabla_a^{-(1-\nu)}\nabla f(t)=\int_a^t H_{-\nu}(t,\rho(s))\nabla f(s)\nabla s \tag{3.2}$$
$$=H_{-\nu}(t,s)f(s)|_{s=a}^{t}+\int_a^t H_{-\nu-1}(t,\rho(s))f(s)\nabla s$$
$$=-H_{-\nu}(t,a)f(a)+\int_a^t H_{-\nu-1}(t,\rho(s))f(s)\nabla s.$$

Using the composition rule $\nabla_a^{\nu}\nabla_a^{-\mu}f(t)=\nabla_a^{\nu-\mu}f(t)$, for $\nu,\mu>0$, we have

$$\begin{aligned}\nabla\nabla_a^{-(1-\nu)}f(t) &= \nabla_a^{\nu}f(t)\\ &= \int_a^t H_{-\nu-1}(t,\rho(s))f(s)\nabla s.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), it follows that (3.1) holds. □

From Lemma 3.1, it is easy to get the following corollary which will be useful later.

Corollary 3.2. *For $0<\nu<1$, the following equality holds:*

$$\nabla_a^{-\nu}\nabla f(t)=\nabla\nabla_a^{-\nu}f(t)-H_{\nu-1}(t,a)f(a). \tag{3.4}$$

for $t\in\mathbb{N}_a$.

Lemma 3.3. *Assume that $0<\nu<1$ and $x(t)$ is a solution of the fractional equation*

$$\nabla_{a^*}^{\nu}x(t)=c(t)x(t), \qquad t\in\mathbb{N}_{a+1} \tag{3.5}$$

satisfying $x(a)>0$, Then $x(t)$ satisfies the integral equation

$$x(t)=\int_a^t H_{\nu-1}(t,\rho(s))c(s)x(s)\nabla s+x(a)$$

$$=\sum_{s=a+1}^{t}\frac{(t-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)}c(s)x(s)+x(a).$$

Proof. Using Lemma 3.1 and the composition rule: $\nabla_a^{\alpha}\nabla_a^{-\beta}f(t)=\nabla_a^{\alpha-\beta}f(t)$, for $\alpha,\beta>0$ in [6, Chapter 3], we get

$$\begin{aligned}\nabla_{a*}^{\nu}x(t) &= \nabla_a^{-(1-\nu)}\nabla x(t)\\ &= \nabla\nabla_a^{-(1-\nu)}x(t)-x(a)H_{-\nu}(t,a)\\ &= \nabla_a^{\nu}x(t)-x(a)H_{-\nu}(t,a).\end{aligned}$$

From (3.5), we have

$$\nabla_a^{\nu} x(t) = c(t)x(t) + x(a)H_{-\nu}(t,a).$$

Applying the operator $\nabla_a^{-\nu}$ to each side we obtain

$$\nabla_a^{-\nu}\nabla_a^{\nu} x(t) = \nabla_a^{-\nu} c(t)x(t) + x(a)\nabla_a^{-\nu} H_{-\nu}(t,a),$$

which can be written in the form

$$\nabla_a^{-\nu}\nabla\nabla_a^{-(1-\nu)} x(t) = \nabla_a^{-\nu} c(t)x(t) + x(a)\nabla_a^{-\nu} H_{-\nu}(t,a).$$

Using Corollary 3.2, we get that

$$\begin{aligned}\nabla\nabla_a^{-\nu}\nabla_a^{-(1-\nu)} x(t) - \frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)}\nabla_a^{-(1-\nu)} x(t)|_{t=a} \\ = \nabla_a^{-\nu} c(t)x(t) + x(a)\nabla_a^{-\nu} H_{-\nu}(t,a).\end{aligned}$$

Using

$$\nabla_a^{-(1-\nu)} x(t)|_{t=a} = \int_a^a H_{-\nu}(a,\rho(s))x(s)\nabla s = 0,$$

we see that

$$\nabla\nabla_a^{-\nu}\nabla_a^{-(1-\nu)} x(t) = \nabla_a^{-\nu} c(t)x(t) + x(a)\nabla_a^{-\nu} H_{-\nu}(t,a).$$

Using the composition rule, [6, Chapter 3]

$$\nabla_a^{-\nu}\nabla_a^{-(1-\nu)} x(t) = \nabla_a^{-1} x(t) \text{ and } \nabla\nabla_a^{-1} x(t) = x(t),$$

we get that

$$x(t) = \nabla_a^{-\nu} c(t)x(t) + x(a)\nabla_a^{-\nu} H_{-\nu}(t,a).$$

Using the power rules $\nabla_a^{-\nu} H_{-\nu}(t,a) = H_0(t,a) = 1$ (see [6, Chapter 3]), we have

$$\begin{aligned} x(t) &= \nabla_a^{-\nu} c(t)x(t) + x(a) \\ &= \int_a^t H_{\nu-1}(t,\rho(s))c(s)x(s)\nabla s + x(a) \\ &= \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s))c(s)x(s) + x(a) \\ &= \sum_{s=a+1}^{t} \frac{(t-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)} c(s)x(s) + x(a), \end{aligned} \tag{3.6}$$

which completes the proof. □

The following lemma is from [3].

Lemma 3.4. *Assume* $1 < \nu < 1$, $|b| < 1$. *Then the Mittag–Leffler function*

$$E_{b,\nu,\nu-1}(t, \rho(a)) = \sum_{k=0}^{\infty} b^k H_{\nu k+\nu-1}(t, \rho(a))$$

is the unique solution of the IVP

$$\nabla^{\nu}_{\rho(a)} x(t) = bx(t), \qquad t \in \mathbb{N}_{a+1} \tag{3.7}$$
$$x(a) = \frac{1}{1-b}.$$

Lemma 3.5. *Assume* $0 < \nu < 1$, $|b| < 1$. *Then any solution of the equation*

$$\nabla^{\nu}_{\rho(a)} x(t) = bx(t), \qquad t \in \mathbb{N}_{a+1} \tag{3.8}$$

satisfying $x(a) > 0$ *is positive on* $\mathbb{N}_a$.

Proof. From Lemma 2.7, we have for $t = a + k$

$$\begin{aligned}
\nabla^{\nu}_{\rho(a)} x(t) &= \int_{\rho(a)}^{t} H_{-\nu-1}(t, \rho(s)) x(s) \nabla s \\
&= \sum_{s=a}^{a+k} H_{-\nu-1}(a+k, s-1) x(s) \\
&= x(a+k) - \nu x(a+k-1) - \frac{\nu(-\nu+1)}{2} x(a+k-2) \\
&\quad - \cdots - \frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!} x(a).
\end{aligned}$$

Using (3.8), we have that

$$\begin{aligned}
&(1-b)x(a+k) \\
&= \nu x(a+k-1) + \frac{\nu(-\nu+1)}{2} x(a+k-2) \\
&\quad + \cdots + \frac{\nu(-\nu+1)\cdots(-\nu+k-1)}{k!} x(a).
\end{aligned} \tag{3.9}$$

We will prove $x(a+k) > 0$ for $k \in \mathbb{N}_0$ by using the principle of strong induction. Since $x(a) > 0$ we have that the base case holds. Now assume that $x(a+i) > 0$, for $i = 0, 1, \cdots, k-1$. Since

$$\frac{\nu(-\nu+1)\cdots(-\nu+i-1)}{i!} > 0$$

for $i = 2, 3, \cdots k-1$, from (3.9), we have $x(a+k) > 0$. This completes the proof. □

Lemma 3.6. *Assume that* $0 < \nu < 1$, $-1 < b < 0$. *Then*

$$\lim_{t\to\infty} E_{b,\nu,0}(t,a) = 0$$

Proof. From Lemma 3.4 and Lemma 3.5, we have $E_{b,\nu,\nu-1}(t,\rho(a)) > 0$, for $t \in \mathbb{N}_{a+1}$. So we have

$$\begin{aligned}
\nabla E_{b,\nu,0}(t,a) &= \sum_{k=0}^{\infty} b^k \nabla H_{\nu k}(t,a) \\
&= \sum_{k=0}^{\infty} b^k H_{\nu k-1}(t,a) = \sum_{k=1}^{\infty} b^k H_{\nu k-1}(t,a) \\
&= b\sum_{k=1}^{\infty} b^{k-1} H_{\nu k-1}(t,a) = b\sum_{j=0}^{\infty} b^j H_{\nu j+\nu-1}(t,a) \\
&= bE_{b,\nu,\nu-1}(t,a) = bE_{b,\nu,\nu-1}(t-1,\rho(a)) < 0,
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$, where we use $H_{-1}(t,a) = 0$. Therefore, $E_{b,\nu,0}(t,a)$ is decreasing for $t \in \mathbb{N}_{a+1}$. From Lemma 2.3, we have $E_{b,\nu,0}(t,a) > 0$ for $t \in \mathbb{N}_{a+1}$. Suppose that

$$\lim_{t\to\infty} E_{b,\nu,0}(t,a) = A \geq 0.$$

In the following, we will prove $A = 0$. If not, $A > 0$. Let $x(t) := E_{b,\nu,0}(t,a) > 0$. From Lemma 3.3, we have

$$\begin{aligned}
x(t) &= \int_a^t H_{\nu-1}(t,\rho(s)) b x(s) \nabla s + x(a) \\
&= b[x(t) + \nu x(t-1) + \frac{\nu(\nu+1)}{2!} x(t-2) \\
&\quad + \cdots + H_{\nu-1}(t,a) x(a+1)] + x(a).
\end{aligned}$$

For fixed $k_0 > 0$, for large t, we have (since $b < 0$)

$$\begin{aligned}
x(t) \leq b\Big[x(t) + \nu x(t-1) + \frac{\nu(\nu+1)}{2!} x(t-2) \\
+ \cdots + \frac{\nu(\nu+1)\cdots(\nu+k_0-1)}{k_0!} x(t-k_0)\Big] + x(a).
\end{aligned}$$

Letting $t \to \infty$, we get that

$$0 < A \leq bA\Big[1 + \nu + \frac{\nu(\nu+1)}{2!} + \cdots + \frac{\nu(\nu+1)\cdots(\nu+k_0-1)}{k_0!}\Big] + x(a). \quad (3.10)$$

Since (using mathematical induction in the first step)

$$\begin{aligned}
&1+\nu+\frac{\nu(\nu+1)}{2!}+\cdots+\frac{\nu(\nu+1)\cdots(\nu+k_0-1)}{k_0!}\\
&=\frac{(\nu+1)(\nu+2)\cdots(\nu+k_0)}{k_0!}\\
&=\frac{(\nu+1)(\nu+2)\cdots(\nu+1+k_0-1)}{(k_0-1)!(k_0-1)^{\nu+1}}\frac{(k_0-1)^{\nu+1}}{k_0}\\
&\to+\infty,
\end{aligned}$$

as $k_0 \to \infty$, where we used (see Lemma 2.10)

$$\frac{1}{\Gamma(\nu+1)}=\lim_{k_0\to\infty}\frac{(\nu+1)(\nu+2)\cdots(\nu+1+k_0-1)}{(k_0-1)!(k_0-1)^{\nu+1}}.$$

So in (3.10), for sufficiently large k_0, the right side of (3.10) is negative, but the left side of (3.10) is positive, which is a contradiction. So $A = 0$. This completes the proof. □

Theorem 3.7. *Assume* $c(t) \le b_1 < 0$, $0 < \nu < 1$, *and* $x(t)$ *is any solution of the Caputo nabla fractional difference equation*

$$\nabla^{\nu}_{a^*}x(t) = c(t)x(t), \quad t \in \mathbb{N}_{a+1} \tag{3.11}$$

satisfying $x(a) > 0$. *Then*

$$x(t) \le 2x(a)E_{b_1,\nu,0}(t,a),$$

for $t \in \mathbb{N}_a$.

Proof. Assume that $b_1 > -1$. Otherwise we can choose $0 > b_1' > -1$, $b_1' > b_1$ and replace b_1 by b_1'. From Lemma 2.9, we have

$$\nabla^{\nu}_{a^*}E_{b_1,\nu,0}(t,a) = b_1E_{b_1,\nu,0}(t,a)$$

and $E_{b_1,\nu,0}(a,a) = H_0(a,a) = 1$.

In Theorem 2.4, take $c_2(t) = b_1$. Then $x(t)$ and $y(t) = 2x(a)E_{b_1,\nu,0}(t,a)$ satisfy

$$\nabla^{\nu}_{a^*}x(t) = c(t)x(t), \tag{3.12}$$

and

$$\nabla^{\nu}_{a^*}y(t) = b_1y(t), \tag{3.13}$$

respectively, for $t \in \mathbb{N}_{a+1}$ and

$$x(a) < 2x(a) = 2x(a)E_{b_1,\nu,0}(a,a) = y(a).$$

From Theorem 2.4, we get that

$$x(t) \le 2x(a)E_{b_1,\nu,0}(t,a),$$

for $t \in \mathbb{N}_a$. This completes the proof.

□

From Theorem 3.7 and Lemma 3.6, we get the following result.

Theorem B. Assume $0 < \nu < 1$ and there exists a constant b_1 such that $c(t) \leq b_1 < 0$. Then the solutions of the equation (1.1) with $x(a) > 0$ satisfy

$$\lim_{t\to\infty} x(t) = 0.$$

4 Asymptotic Behavior with Initial Value, $x(a) < 0$

Consider solutions of the following ν-th order Caputo nabla fractional difference equation

$$\nabla_{a^*}^{\nu} x(t) = c(t)x(t), \qquad t \in \mathbb{N}_{a+1}, \tag{4.1}$$

satisfying $x(a) < 0$.

By making the transformation $x(t) = -y(t)$ and using Theorem A and Theorem B, we get the following results.

Theorem C. Assume $0 < \nu < 1$ and there exists a constant b_2 such that $0 < b_2 \leq c(t) < 1$, $t \in \mathbb{N}_{a+1}$. Then the solutions of the equation (4.1) with $x(a) < 0$ satisfy

$$\lim_{t\to\infty} x(t) = -\infty.$$

Theorem D. Assume $0 < \nu < 1$ and there exists a constant b_1 such that $c(t) \leq b_1 < 0$, $t \in \mathbb{N}_{a+1}$. Then the solutions of the equation (4.1) with $x(a) < 0$ satisfy

$$\lim_{t\to\infty} x(t) = 0.$$

Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 11271380) and Guangdong Province Key Laboratory of Computational Science.

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