International Journal of Difference Equations
ISSN 0973-6069, Volume 11, Number 2, pp. 123–137 (2016)
http://campus.mst.edu/ijde

Eventual Monotonicity in Nonlinear Difference Equations

A. M. Amleh
Paris-Sorbonne University Abu Dhabi
Sciences and Engineering Department
Abu Dhabi, UAE
amal.amleh@psuad.ac.ae

Emmanouil Drymonis
Providence College
Mathematics and Computer Science
Providence, RI 02918, USA
edrymoni@providence.edu

Abstract

This paper continues our study on the global character of solutions of a certain rational difference equation. We present several new results together with some known techniques which establish the eventually monotonic character of solutions for a class of nonlinear difference equations in a certain range of their parameters. Open problems and conjectures are provided for further investigations.

AMS Subject Classifications: 39A20, 39A22, 39A30.
Keywords: Monotonicity of solutions, rational difference equations, rational systems, Riccati equations.

1 Introduction

Consider the rational difference equation

$$x_{n+1} = \frac{(\alpha x_n + \beta x_n x_{n-1} + \gamma x_{n-1})x_n}{Ax_n + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \ldots \tag{1.1}$$

Received May 6, 2016; Accepted June 27, 2016
Communicated by Gerasimos Ladas

where the initial conditions are positive and the parameters are nonnegative real numbers such that the denominator is always positive. Eq. (1.1) was introduced in [12] where several special cases were investigated and the global character of their solutions was presented.

Eq. (1.1), which contains some interesting and some challenging special cases of second–order rational difference equations, also arises from the rational system in the plane:

$$\left.\begin{aligned} x_{n+1} &= \frac{x_n}{y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \; n = 0, 1, \ldots \quad . \tag{1.2}$$

Systems of rational difference equations are currently under investigation by many scholars in the field. As a result, many important theorems addressing boundedness, global stability and bifurcation analysis are being developed. For some results on systems of difference equations, see [3–8, 10, 13, 17–19]. For further reading on nonlinear difference equations, see [1, 2, 9, 11, 14, 15].

In Section 2, we introduce three new results which provide conditions under which all solutions of a nonlinear difference equation are eventually monotonic. In Section 3, we present a couple of simple techniques that utilize a comparison theorem, Riccati difference equations, and identities to describe the global behavior of all solutions of the equations. In Section 4, we present open problems and conjectures for further investigations. The theorems presented in this paper have the potential to be extended and generalized to address the global character of solutions of a larger family of nonlinear difference equations.

2 Main Results

In this section, we present three new theorems which describe the global behavior of all solutions of certain types of difference equations. Following the theorems, several illustrative examples, which are special cases of Eq. (1.1), are presented. In these examples, all the parameters present are assumed to be strictly positive real numbers.

Theorem 2.1. *Consider the nonlinear difference equation*

$$x_{n+1} = x_n f(x_n, x_{n-1}), \; n = 0, 1, \ldots \tag{2.1}$$

with the following assumptions:

1. $f \in C[(0,\infty) \times (0,\infty), (0,\infty)]$.
2. *Eq.* (2.1) *has a unique positive equilibrium point* $\bar{x}$.
3. $f(x, y)$ *decreases in the second argument.*

4. $f(x,x) > 1 \iff x > \bar{x}$ *and* $f(x,x) < 1 \iff x < \bar{x}$.

Then every positive and non eventually equilibrium solution $\{x_n\}$ *of Eq.* (2.1) *is eventually monotonic and more specifically does exactly one of the following:*

(i) If there exists an $N \geq 0$ *such that*

$$x_N \leq x_{N-1} \text{ and } x_N \leq \bar{x}, \tag{2.2}$$

then $x_{n+1} \leq x_n$ *for all* $n \geq N$ *and* $x_n \to 0$.

(ii) If there exists an $N \geq 0$ *such that*

$$x_N \geq x_{N-1} \text{ and } x_N \geq \bar{x}, \tag{2.3}$$

then $x_{n+1} \geq x_n$ *for all* $n \geq N$ *and* $x_n \to \infty$.

(iii) If there is no $N \geq 0$ *such that neither* (2.2) *nor* (2.3) *can happen, then one of the following two things can happen:*

(a) if $x_{-1} < x_0 < \bar{x}$, *then* $\{x_n\}$ *increases to* $\bar{x}$,

(b) if $\bar{x} < x_0 < x_{-1}$, *then* $\{x_n\}$ *decreases to* $\bar{x}$.

Proof. Let $\{x_n\}$ be a solution of Eq. (2.1) with initial conditions $x_{-1}, x_0 > 0$. We present only the proof of part (i). The proof of part (ii) is similar and will be omitted. Assume that there exists some $N \geq 0$ such that (2.2) holds. Then

$$x_{N+1} = x_N f(x_N, x_{N-1}) \leq x_N f(x_N, x_N) \leq x_N$$

and the result follows by induction.

For part (iii) we establish only part (a). The proof of (b) is similar and will be omitted. So assume that

$$x_{-1} < x_0 < \bar{x}$$

holds. If

$$x_0 < \bar{x} \leq x_1$$

holds, then from (ii) we have a contradiction. Clearly we must have $x_1 < \bar{x}$. If

$$x_1 \leq x_0 < \bar{x}$$

holds, then again we reach a contradiction according to (i). Hence, we must have

$$x_{-1} < x_0 < x_1 < \bar{x}.$$

Similarly,

$$x_{-1} < x_0 < x_1 < x_2 < \bar{x}.$$

Inductively we have that

$$x_{-1} < x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots < \bar{x}$$

and the result follows. □

Theorem 2.2. *Consider the nonlinear difference equation*

$$x_{n+1} = x_n f(x_n, x_{n-1}),\ n = 0, 1, \dots \tag{2.4}$$

with the following assumptions:

1. $f \in C[(0,\infty) \times (0,\infty), (0,\infty)]$.
2. $f(x,y) \geq 1 \iff x \geq \dfrac{sy}{D+y}, \quad s \leq D,$

where s and D are positive real numbers. Then Eq. (2.4) *has no positive equilibrium points and every positive solution is eventually monotonic. More precisely, every solution $\{x_n\}$ of Eq.* (2.4) *does exactly one of the following.*

(a) If there exists an $N \geq 0$ such that $x_N \geq \dfrac{sx_{N-1}}{D+x_{N-1}}$, then $x_{n+1} > x_n$ for all $n > N$ and the solution diverges to infinity.

(b) If there is no $N \geq 0$ such that (a) holds, then $x_{n+1} < x_n$ for all $n \geq 0$ and the solution converges to zero.

Proof. Assume that $\bar{x}$ is a positive equilibrium point of Eq. (2.4). That is,

$$\bar{x} = \bar{x} f(\bar{x}, \bar{x}) \iff f(\bar{x}, \bar{x}) = 1$$

which implies that

$$\bar{x} = \frac{s\bar{x}}{D+\bar{x}} \iff \bar{x} = s - D \leq 0,$$

which is a contradiction. In order to prove (a) assume that there exists an $N \geq 0$ such that $x_N \geq \dfrac{sx_{N-1}}{D+x_{N-1}}$. Then, we have

$$f(x_N, x_{N-1}) \geq 1 \iff x_N f(x_N, x_{N-1}) \geq x_N \iff x_{N+1} \geq x_N.$$

But then it holds

$$x_{N+1} \geq x_N \geq \frac{s}{D} x_N > \frac{sx_N}{D+x_N},$$

or

$$x_{N+1} > \frac{sx_N}{D+x_N} \iff f(x_{N+1}, x_N) > 1 \iff x_{N+1} f(x_{N+1}, x_N) > x_{N+1} \iff$$

$$x_{N+2} > x_{N+1},$$

and the proof follows by induction. If there is no $N \geq 0$ such that (a) holds, then clearly (b) must follow. $\square$

Example 2.3. Consider the nonlinear difference equation

$$x_{n+1} = \frac{(\alpha x_n + x_n x_{n-1})x_n}{x_{n-1}}, \quad n = 0, 1, \dots \quad . \tag{2.5}$$

When $\alpha < 1$, Eq. (2.5) has the unique equilibrium point $\bar{x} = 1 - \alpha$ which is a saddle. Here Theorem 2.1 applies and describes the behavior of all solutions. To see this, the function

$$f(x, y) = \frac{\alpha x + xy}{y}$$

is continuous in its domain and is decreasing in y. Finally

$$f(x, x) \geq 1 \iff x \geq \bar{x}.$$

Hence, solutions either converge to the equilibrium, eventually decrease to zero or increase to infinity.

When $\alpha \geq 1$, Theorem 2.2 applies. To see this,

$$f(x, y) = \frac{\alpha x + xy}{y} \geq 1 \iff x \geq \frac{y}{\alpha + y} = \frac{sy}{D + y},$$

here $s = 1 \leq \alpha = D$, and in this case, solutions eventually increase to infinity or decrease to zero.

Example 2.4. Consider the nonlinear difference equation

$$x_{n+1} = \frac{(\alpha x_n + x_n x_{n-1} + x_{n-1})x_n}{C x_{n-1}}, \quad n = 0, 1, \dots \quad . \tag{2.6}$$

When $\alpha + 1 < C$, Eq. (2.6) has the unique equilibrium point $\bar{x} = C - \alpha - 1$ which is a saddle. In this case, Theorem 2.1 applies and describes the behavior of all solutions. To see this, the function

$$f(x, y) = \frac{\alpha x + xy + y}{Cy}$$

is continuous on its domain and is decreasing in y. Finally

$$f(x, x) \geq 1 \iff x \geq \bar{x}.$$

Hence, solutions either converge to the equilibrium, eventually decrease to zero or increase to infinity.

When $\alpha + 1 \geq C$ and $C > 1$, Theorem 2.2 applies since

$$f(x, y) = \frac{\alpha x + xy + y}{Cy} \geq 1 \iff x \geq \frac{(C - 1)y}{\alpha + y} = \frac{sy}{D + y},$$

here $s = C - 1 \leq \alpha = D$. Hence, solutions eventually increase to infinity or decrease to zero.

Example 2.5. Consider the difference equation

$$x_{n+1} = \frac{(\alpha x_n + x_n x_{n-1})x_n}{Ax_n + x_{n-1}}, \quad n = 0, 1, \dots . \tag{2.7}$$

When $\alpha \geq 1 + A$, Theorem 2.2 applies since

$$f(x, y) = \frac{\alpha x + xy}{Ax + y} \geq 1 \iff x \geq \frac{y}{\alpha - A + y} = \frac{sy}{D + y},$$

where $s = 1 \leq \alpha - A = D$. Hence, every solution eventually either decreases to zero or increases to infinity.

Theorem 2.6. *Consider the nonlinear difference equation*

$$x_{n+1} = x_n f(x_n, x_{n-1}),\ n = 0, 1, \dots \tag{2.8}$$

with the following assumptions:

1. $f \in C[(0, \infty) \times (0, \infty), (0, \infty)]$.
2. $f(x, y) \leq 1 \iff y \geq \dfrac{sx}{E + x}, \quad s \leq E,$

where s and E are positive real numbers. Then Eq. (2.8) *has no positive equilibrium points and every positive solution is eventually monotonic. More precisely every solution $\{x_n\}$ of Eq.* (2.8) *does either one of the following.*

(a) If there exists an $N \geq 0$ such that $x_{N-1} \geq \dfrac{sx_N}{E + x_N}$, then $x_{n+1} < x_n$ for all $n > N$ and the solution converges to zero.

(b) If there is no $N \geq 0$ such that (a) holds, then $x_{n+1} > x_n$ for all $n \geq 0$ and the solution diverges to infinity.

Proof. Assume that $\bar{x}$ is a positive equilibrium point of Eq. (2.8). That is

$$\bar{x} = \bar{x} f(\bar{x}, \bar{x}) \iff f(\bar{x}, \bar{x}) = 1$$

which implies that

$$\bar{x} = \frac{s\bar{x}}{E + \bar{x}} \quad \bar{x} = s - E \leq 0$$

which is a contradiction. In order to prove (a) assume that there exists an $N \geq 0$ such that $x_{N-1} \geq \dfrac{sx_N}{E + x_N}$. Then, we have

$$f(x_N, x_{N-1}) \leq 1 \iff x_N f(x_N, x_{N-1}) \leq x_N \iff x_{N+1} \leq x_N.$$

But then it holds

$$x_N \geq x_{N+1} \geq \frac{s}{E} x_{N+1} > \frac{s x_{N+1}}{E + x_{N+1}},$$

or

$$x_N > \frac{s x_{N+1}}{E + x_{N+1}} \iff f(x_{N+1}, x_N) < 1 \iff x_{N+1} f(x_{N+1}, x_N) < x_{N+1} \iff$$

$$x_{N+2} < x_{N+1},$$

and the proof follows by induction. If there is no $N \geq 0$ such that (a) holds, then clearly (b) must follow. □

Remark 2.7. The difference equations presented in the following two examples are special cases of Eq. (1.1) (and System (1.2)) where the boundedness character was presented in [12] based on the results established in [3]. Hence, all positive solutions of these two difference equations are bounded.

Example 2.8. Consider the difference equation

$$x_{n+1} = \frac{(\alpha x_n + x_{n-1}) x_n}{x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, \dots . \tag{2.9}$$

When $1 + \alpha \leq C$, Theorem 2.6 applies and describes the behavior of all solutions. To see this,

$$f(x, y) = \frac{\alpha x + y}{xy + Cy} \leq 1 \iff y \geq \frac{\alpha x}{C - 1 + x} = \frac{sx}{E + x},$$

where $s = \alpha \leq C - 1 = E$. Since all solutions are bounded, it follows that every solution eventually decreases to zero.

Example 2.9. Consider the difference equation

$$x_{n+1} = \frac{x_n^2}{A x_n + x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, \dots, \tag{2.10}$$

and assume that $A + C \geq 1$ and $A < 1$. Then Theorem 2.6 applies and describes the behavior of all solutions. To see this,

$$f(x, y) = \frac{x}{Ax + xy + Cy} \leq 1 \iff y \geq \frac{(1 - A)x}{C + x} = \frac{sx}{E + x},$$

where $1 - A = s \leq C = E$. Since all solutions are bounded, it follows that every solution eventually decrease to zero.

3 Comparison Principle, Riccati Equations and Identities

In this section, first we present a simple technique that relies on a known comparison principle (see [16]) and helps to establish the global behavior of solutions of certain types of nonlinear difference equations. Then we combine properties of Riccati difference equations with identities satisfied by solutions to describe the long term behavior of all solutions of those difference equations.

To demonstrate the technique, we consider Eq. (1.1) in a certain range of its parameters where the equation has no equilibrium points. For this range of parameters, solutions are eventually monotonic and either increase to infinity or decrease to zero. In order to establish this eventually monotonic behavior, we examine the ratios of consecutive terms in the solution of the difference equation. This is described in the proof of the following lemma.

Lemma 3.1. *Let* $\{x_n\}$ *be a solution of*

$$x_{n+1} = \frac{(\alpha x_n + x_n x_{n-1} + x_{n-1})x_n}{Ax_n + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \ldots, \tag{3.1}$$

and assume that

$$B > 1, \quad C > \alpha, \quad \textit{and} \quad \alpha B + B + C < BC.$$

Then $\{x_n\}$ *eventually decreases to zero.*

Proof. By dividing both sides of Eq. (3.1) by x_n, we have

$$\begin{aligned}\frac{x_{n+1}}{x_n} &= \frac{\alpha x_n + x_n x_{n-1} + x_{n-1}}{Ax_n + Bx_n x_{n-1} + Cx_{n-1}} \\ &\leq \left(\frac{\alpha}{C}\right)\frac{x_n}{x_{n-1}} + \frac{1}{B} + \frac{1}{C}.\end{aligned}$$

Let $z_{n+1} = \dfrac{x_{n+1}}{x_n}$ and $z_0 \leq w_0$ where w_0 is the initial condition of the linear difference equation

$$w_{n+1} = \left(\frac{\alpha}{C}\right) w_n + \frac{1}{B} + \frac{1}{C}.$$

Based on the fact that

$$z_n \leq w_n \quad \text{for all} \quad n \geq 0$$

and

$$\lim_{n\to\infty} w_n = \bar{w} = \frac{C + B}{B(C - \alpha)},$$

it follows that for each $\epsilon > 0$, there exists an $N \geq 0$ such that for all $n \geq N$ we have

$$z_n \leq w_n < \frac{C+B}{B(C-\alpha)} + \epsilon.$$

But

$$\bar{w} = \frac{C+B}{B(C-\alpha)} < 1 \iff \alpha B + B + C < BC.$$

Hence we can choose ϵ such that eventually

$$z_{n+1} = \frac{x_{n+1}}{x_n} < 1 \iff x_{n+1} < x_n$$

from which the results follows since in the given range of the parameters, the difference equation has no equilibrium points. □

The following example uses the above idea to show that every solution of a given difference equation eventually increases and diverges to infinity.

Example 3.2. Consider the difference equation

$$x_{n+1} = \frac{(\alpha x_n + x_n x_{n-1} + x_{n-1})x_n}{Ax_n + Cx_{n-1}}, \quad n = 0, 1, \dots . \tag{3.2}$$

Let $\{x_n\}$ be a solution of Eq. (3.2) and assume that $A + C < \alpha$, then $\{x_n\}$ eventually increases to infinity. To establish this, introduce the change of variables $x_n = \frac{1}{y_n}$. Then Eq. (3.2) becomes

$$y_{n+1} = \frac{(Ay_{n-1} + Cy_n)y_n}{\alpha y_{n-1} + 1 + y_n}, \quad n = 0, 1, \dots . \tag{3.3}$$

Now we have

$$\frac{y_{n+1}}{y_n} \leq \left(\frac{C}{\alpha}\right) \frac{y_n}{y_{n-1}} + \frac{A}{\alpha},$$

from which it follows that, in the given range of the parameters, $\{y_n\}$ eventually decreases to zero (in the absence of an equilibrium points). Hence, $\{x_n\}$ eventually increases to infinity.

Example 3.3. Consider the difference equation

$$x_{n+1} = \frac{(\alpha x_n + \gamma x_{n-1})x_n}{Ax_n + x_{n-1}}, \quad n = 0, 1, \dots . \tag{3.4}$$

When $A + 1 \neq \alpha + \gamma$, Eq. (3.4) has no equilibrium points.

(i) Let $A + 1 > \alpha + \gamma$. Then every solution of Eq. (3.4) eventually decreases to zero.

(ii) Let $A+1 < \alpha+\gamma$. Then every solution of Eq. (3.4) eventually increases to infinity.

To establish (i) and (ii), introduce the change of variables

$$z_n = \frac{x_n}{x_{n-1}},$$

and Eq. (3.4) transforms to the Riccati difference equation

$$z_{n+1} = \frac{\alpha z_n + \gamma}{A z_n + 1}, \quad n = 0, 1, \dots .$$

Hence, every solution converges to

$$\bar{z} = \frac{\alpha - 1 + \sqrt{(1-\alpha)^2 + 4\gamma A}}{2A} > 0.$$

(i) Let $A+1 > \alpha+\gamma$. Then $z_n \to \bar{z} < 1$. Hence, eventually

$$\frac{x_{n+1}}{x_n} < 1 \iff x_{n+1} < x_n.$$

(ii) Let $A+1 < \alpha+\gamma$. Then $z_n \to \bar{z} > 1$. Hence, eventually

$$\frac{x_{n+1}}{x_n} > 1 \iff x_{n+1} > x_n.$$

Lemma 3.4. *Let $\{x_n\}$ be a solution to the difference equation*

$$x_{n+1} = \frac{(\alpha x_n + x_n x_{n-1} + x_{n-1})x_n}{A x_n + x_n x_{n-1}}, \quad n = 0, 1, \dots, \tag{3.5}$$

where $1+\alpha = A$. Then

(i) $x_{-1} < x_0 \Rightarrow x_{2n-1} < x_{2n+1}$ *and* $x_{2n} > x_{2n+2}$, *for all* $n \geq 0$.

(ii) $x_{-1} > x_0 \Rightarrow x_{2n-1} > x_{2n+1}$ *and* $x_{2n} < x_{2n+2}$, *for all* $n \geq 0$.

Moreover, $\{x_n\}$ converges to an equilibrium point.

Proof. When $1+\alpha = A$, Eq. (3.5) has infinitely many equilibrium points that are non–hyperbolic. Moreover, every solution of Eq. (3.5) satisfies the following two identities from which (i) and (ii) are easily established.

$$x_{n+1} - x_{n-1} = \left(\frac{\alpha + x_{n-1}}{A + x_{n-1}}\right)(x_n - x_{n-1}), \quad \text{for all } n \geq 0,$$

and

$$x_{n+1} - x_n = \left(\frac{1}{1 + \alpha + x_{n-1}}\right)(x_{n-1} - x_n), \quad \text{for all } n \geq 0.$$

Next assume, for the sake of contradiction that

$$\lim_{n\to\infty} x_{2n} = m \geq 0 \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = \infty.$$

From Eq. (3.5) we have

$$x_{2n+1} = \frac{(\alpha x_{2n} + x_{2n}x_{2n-1} + x_{2n-1})x_{2n}}{Ax_{2n} + x_{2n}x_{2n-1}} \text{ or } x_{2n+1} = \frac{\alpha x_{2n} + x_{2n}x_{2n-1} + x_{2n-1}}{A + x_{2n-1}}$$

or

$$x_{2n+1} = \frac{\alpha \frac{x_{2n}}{x_{2n-1}} + x_{2n} + 1}{\frac{A}{x_{2n-1}} + 1}$$

and by taking limits we conclude that

$$\infty = m + 1$$

which is a contradiction. Hence,

$$\lim_{n\to\infty} x_{2n+1} = l > 0,$$

and the result follows since Eq. (3.5) has no prime period–two solutions. □

The following is a well-known lemma about Cauchy sequences.

Lemma 3.5. *Let $\{x_n\}$ be a sequence of real numbers and let r be a real number that satisfies $0 < r < 1$. Suppose that $|x_{n+1} - x_n| \leq r|x_n - x_{n-1}|$ for all $n > 1$. Then $\{x_n\}$ is a Cauchy sequence and hence converges.*

Lemma 3.6. *Let $\{x_n\}$ be a solution of Eq.* (3.4) *and assume that*

$$A + 1 = \alpha + \gamma \text{ and } \gamma < 1,$$

then $\{x_n\}$ converges to a finite limit.

Proof. It is easy to verify that $\{x_n\}$ satisfies the following identity

$$x_{n+1} - x_n = \frac{(1-\gamma)x_n}{Ax_n + x_{n-1}}(x_n - x_{n-1}) \tag{3.6}$$

from which the following holds true.

(i) $x_{-1} < x_0 \Rightarrow x_n < x_{n+1}$, for all $n \geq 0$ and

$$\lim_{n\to\infty} x_n = \infty \text{ or } \lim_{n\to\infty} x_n = \bar{x}.$$

(ii) $x_{-1} > x_0 \Rightarrow x_n > x_{n+1}$, for all $n \geq 0$ and

$$\lim_{n\to\infty} x_n = 0 \text{ or } \lim_{n\to\infty} x_n = \bar{x}.$$

For $z_n = \dfrac{x_n}{x_{n-1}} \to \bar{z} = \dfrac{\alpha - 1 + \sqrt{(1-\alpha)^2 + 4\gamma A}}{2A} = 1$. Let $\epsilon < \dfrac{\gamma}{1-\gamma}$, then there exists an N such that

$$\frac{x_n}{x_{n-1}} < 1 + \epsilon, \text{ for all } n \geq N.$$

Then from Identity (3.6):

$$|x_{n+1} - x_n| < (1-\gamma)(1+\epsilon)|x_n - x_{n-1}|, \text{ for all } n \geq N,$$

from which it follows from Lemma 3.5 that $\{x_n\}$ is a Cauchy sequence and thus converges since $0 < (1-\gamma)(1+\epsilon) < 1$. □

4 Open Problems and Conjectures

Conjecture 4.1. The positive equilibrium $\bar{x}$ of each of the difference equations #13, #14, #22, #23, #24, #25, and #27 (see the Appendix) in the range of their parameters when $\bar{x}$ is locally asymptotically stable is also a global attractor.

Conjecture 4.2. For the difference equations #10, #21, #25, and #27 when the positive equilibrium $\bar{x}$ is a saddle point, every positive solution either converges monotonically to $\bar{x}$, or eventually decreases to zero or increases to infinity.

Conjecture 4.3. For the difference equations #14, #23, and #25 in the range of their parameters where each difference equation has infinitely many equilibrium points, every solution converges to a finite limit.

Open Problem 4.4. For the difference equations #22, #23, #25, and #27 in the range of their parameters where there are no equilibrium points, describe the long term behavior of the solutions in terms of the initial conditions.

Appendix

In [12], an Appendix was presented with all the nontrivial cases of Eq. (1.1). Several of these equations have been addressed throughout that paper and the current one. Here we present the special cases in which, in some range of their parameters, the global behavior of their solutions has not been settled. The numbering we refer to is the one presented in [12].

$\#8\ \ x_{n+1} = \dfrac{x_n^2}{x_n x_{n-1} + C x_{n-1}}$	May's Host Parasitoid Model
$\#10\ \ x_{n+1} = \dfrac{(\alpha x_n + x_n x_{n-1}) x_n}{A x_n + x_{n-1}}$	$\alpha < 1 + A$
$\#13\ \ x_{n+1} = \dfrac{(\alpha x_n + x_{n-1}) x_n}{x_n x_{n-1} + C x_{n-1}}$	$\alpha + 1 > C$
$\#14\ \ x_{n+1} = \dfrac{(\alpha x_n + x_n x_{n-1}) x_n}{B x_n x_{n-1} + x_{n-1}}$	$\alpha, B > 1,$ or $\alpha = B = 1$
$\#21\ \ x_{n+1} = \dfrac{(\alpha x_n + x_n x_{n-1} + x_{n-1}) x_n}{A x_n + C x_{n-1}}$	$\alpha + 1 < A + C$
$\#22\ \ x_{n+1} = \dfrac{(\alpha x_n + x_n x_{n-1} + x_{n-1}) x_n}{A x_n + B x_n x_{n-1}}$	Open for study except when $\alpha + 1 < A,\ B < 1,$ $\alpha + 1 = A,\ B = 1$
$\#23\ \ x_{n+1} = \dfrac{(\alpha x_n + x_n x_{n-1} + x_{n-1}) x_n}{B x_n x_{n-1} + C x_{n-1}}$	Open for study except when $\alpha + 1 < C,\ \ B < 1$
$\#24\ \ x_{n+1} = \dfrac{(\alpha x_n + x_{n-1}) x_n}{A x_n + x_n x_{n-1} + C x_{n-1}}$	$\alpha + 1 > A + C$
$\#25\ \ x_{n+1} = \dfrac{(\alpha x_n + x_n x_{n-1}) x_n}{A x_n + B x_n x_{n-1} + x_{n-1}}$	Open for study
$\#27\ \ x_{n+1} = \dfrac{(\alpha x_n + x_n x_{n-1} + x_{n-1}) x_n}{A x_n + B x_n x_{n-1} + C x_{n-1}}$	Open for study except when $\alpha + 1 = A + C,\ B = 1,$ or $B > 1,\ C > \alpha,\ \alpha B + B + C < BC$

References

[1] A.M. Amleh, E. Camouzis, and G. Ladas, *On the dynamics of a rational difference equation, Part 1*, Int. J. Difference Equ., **3** (1) (2008), 1–35.

[2] A.M. Amleh, E. Camouzis, and G. Ladas, *On the dynamics of a rational difference equation, Part 2*, Int. J. Difference Equ., **3** (2) (2008), 195–225.

[3] A.M. Amleh, E. Camouzis, G. Ladas, and M. Radin *Patterns of Boundedness of a Rational System in the Plane*, J. Differ. Equ. Appl., **16** (10) (2010), 1197–1236, 2010.

[4] E. Camouzis, E. Drymonis, and G. Ladas, *On the global character of the system* $x_{n+1} = \frac{\alpha_1}{x_n + y_n}$ and $y_{n+1} = \frac{\gamma_2 y_n}{B_2 x_n + y_n}$, Comm. Appl. Nonlinear Anal., **16** (2) (2009), 51–64.

[5] E. Camouzis, E. Drymonis, and G. Ladas, *Patterns of boundedness of the rational system* $x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + C_1 y_n}$ and $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n}$, Fasc. Math., **44** (2010), 9–18.

[6] E. Camouzis, E. Drymonis, and G. Ladas, *Patterns of boundedness of the rational system* $x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + B_1 x_n + C_1 y_n}$ and $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n}$, Comm. Appl. Nonlinear Anal., **18** (2011), 1–23.

[7] E. Camouzis, E. Drymonis, G. Ladas, and W. Tikjha, *Patterns of boundedness of the rational system* $x_{n+1} = \frac{\alpha_1}{A_1 + B_1 x_n + C_1 y_n}$ and $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + y_n}$, J. Differ. Equ. Appl., **18** (1) (2012), 89–110.

[8] E. Camouzis, M. R. S. Kulenović, G. Ladas, and O. Merino, *Rational Systems in the Plane*, J. Differ. Equ. Appl., **15** (3) (2009), 303–323.

[9] E. Camouzis and G. Ladas, *Dynamics of Third-Order Rational Difference Equations; With Open Problems and Conjectures*, Chapman & Hall/CRC Press, 2008.

[10] E. Camouzis and G. Ladas, *Global results on rational systems in the plane, I*, J. Differ. Equ. Appl., **16** (8) (2010), 975–1013.

[11] E. Camouzis and G. Ladas, *When does local stability imply global attractivity in rational equations?* J. Differ. Equ. Appl., **12** (8) (2006), 863–885.

[12] E. Drymonis, *On the Global Character of the Difference Equation, Part 1*, Comm. Appl. Nonlinear Anal., **21** (4) (2014), 1–15.

[13] M. Garić-Demirović, M. R. S. Kulenović and M. Nurkanović, *Global Behavior of Four Competitive Rational Systems of Difference Equations on the Plane*, Discrete Dyn. Nat. Soc., Volume **2009** (2009), Article ID 153058, 34 pages.

[14] S. Jasarevic and M.R.S. Kulenović, *Basin of Attraction of Equilibrium and Boundary Points of Second–Order Difference Equations*, J. Differ, Equ. Appl., **20** (5-6) (2014), 947–959.

[15] E.A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC Press, 2005.

[16] V.L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[17] M. R. S. Kulenović and O. Merino, *Competitive-exclusion versus competitive-coexistence for systems in the plane*, Discrete Contin. Dyn. Syst. Ser. B, **6** (5) (2006), 1141–1156.

[18] G. Lugo and F. J. Palladino, *On the boundedness character of rational systems in the plane*, J. Differ. Equ. Appl., **17** (12) (2011), 1801–1811.

[19] E. Magnucka-Blandzi and J. Popenda, *On the asymptotic behavior of a rational system of difference equations*, J. Differ. Equ. Appl., **5** (3) (1999), 271–286.