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Global Stability and Periodic Solutions for a Second-Order Quadratic Rational Difference Equation

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Abstract

We investigate the boundedness and persistence of solutions, the global stability of the positive fixed point and the occurrence of periodic solutions for the quadratic rational difference equation

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C}$$

with nonnegative parameters and initial values. We obtain sufficient conditions that imply the global asymptotic stability of a positive fixed point. We also obtain necessary and sufficient conditions for the occurrence of solutions of prime period 2 when $\gamma, aA + B > 0$. Global convergence of solutions of planar systems of rational equations are studied by folding these systems to equations of the above type.

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1 Introduction

We investigate the dynamics of the second-order equation

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{1.1}$$

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where

$$0 \leq a < 1, \quad \alpha, \beta, \gamma, A, B \geq 0, \quad \alpha + \beta + \gamma, A + B, C > 0. \tag{1.2}$$

Equation (1.1) is a quadratic-fractional equation since it can be written as

$$x_{n+1} = \frac{aAx_n^2 + aBx_n x_{n-1} + (aC + \alpha)x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{1.3}$$

and (1.3) is a special case of the equation

$$x_{n+1} = \frac{px_n^2 + qx_n x_{n-1} + \delta x_{n-1}^2 + c_1 x_n + c_2 x_{n-1} + c_3}{Ax_n + Bx_{n-1} + C} \tag{1.4}$$

which includes rational equations that are the sum of linear equation and a linear fractional equation mentioned in [5]:

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C}.$$

When $a = 0$, (1.1) reduces to linear fractional case that has been studied extensively; see, e.g., [12] and references therein. More recently, second-order linear fractional equations have appeared in [1, 2].

The study of rational equations with quadratic terms is less extensive, but recently they have been of increasing interest; see, e.g., [5–7, 9, 11, 16]. Some instances where quadratic terms appear in the denominator have been studied in [3, 15].

In this paper we show that when (1.2) holds then (1.1) typically does not have periodic solutions of period greater than 2. Further, we show that if solutions of period 2 do not occur then the solutions of (1.1) converge to the unique positive fixed point. When $aA + B, \gamma > 0$ we obtain necessary and sufficient conditions for the occurrence of periodic solutions and in particular prove that such solutions may appear if and only if the positive fixed point is a saddle.

In the final section we apply the results obtained for (1.1) to rational planar systems of type

$$x_{n+1} = ax_n + by_n + c \tag{1.5}$$

$$y_{n+1} = \frac{a'x_n + b'y_n + c'}{a''x_n + b''y_n + c''} \tag{1.6}$$

by folding the system into an equation of type (1.1). Our use of this method (discussed below) is not standard in the published literature and leads to new results. We derive sufficient conditions on the parameters of (1.5)–(1.6) that imply global convergence of orbits initiated in the positive quadrant to a unique fixed point. In a new twist, we see that this behavior occurs even if some of the parameters in (1.5)–(1.6) are negative. In such a case the positive quadrant is not invariant but avoidance of singularities by the aforementioned orbits is assured by the existence of a corresponding solution for (1.1) which does not have any singularities under assumptions (1.2). We briefly discuss applications to biological models of species populations and pose conjectures for future research.

2 Existence and Boundedness of Solutions

When (1.2) holds we may assume that $C = 1$ in (1.1) without loss of generality by dividing the numerator and denominator of the fractional part by C and relabeling the parameters. Thus we consider

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + 1}. \tag{2.1}$$

Note that the underlying function

$$f(u, v) = au + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}$$

is continuous on $[0, \infty) \times [0, \infty)$. The next result gives sufficient conditions for the positive solutions of (2.1) to be uniformly bounded from above and below by positive bounds.

Lemma 2.1. *Let* (1.2) *hold and assume further that*

$$\alpha = 0 \text{ if } A = 0 \quad \text{and} \quad \beta = 0 \text{ if } B = 0. \tag{2.2}$$

Then the following are true:

(a) Every solution $\{x_n\}$ *of* (2.1) *with nonnegative initial values is uniformly bounded from above, i.e., there is a number* $M > 0$ *such that* $x_n \leq M$ *for all* n *sufficiently large.*

(b) If $\gamma > 0$ *then there is* $L \in (0, M)$ *such that* $L \leq x_n \leq M$ *for all large* n*. Moreover,* $[L, M]$ *is an invariant interval for* (2.1).

Proof. (a) Let

$$\rho_1 = \begin{cases} \alpha/A \text{ if } A > 0 \\ 0 \quad\;\; \text{if } A = 0 \end{cases} \qquad \rho_2 = \begin{cases} \beta/B \text{ if } B > 0 \\ 0 \quad\;\; \text{if } B = 0 \end{cases}$$

By (1.2), $\delta = \rho_1 + \rho_2 + \gamma > 0$ and for all $n \geq 0$

$$x_{n+1} = ax_n + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + 1} \leq ax_n + \rho_1 + \rho_2 + \gamma = ax_n + \delta.$$

Thus

$$\begin{aligned} x_1 &\leq ax_0 + \delta \\ x_2 &\leq ax_1 + \delta \leq a^2 x_0 + \delta(1 + a). \end{aligned}$$

Proceeding this way inductively, we obtain

$$\begin{aligned} x_n &\le a^n x_0 + \delta(1 + a + \ldots + a^{n-1}) \\ &= a^n x_0 + \delta \frac{1-a^n}{1-a} \\ &= \frac{\delta}{1-a} + a^n \left[x_0 - \frac{\delta}{1-a} \right]. \end{aligned}$$

For every $\varepsilon > 0$ there is a positive integer N such that if $n \ge N$ then the right hand side above is less than $\delta/(1-a) + \varepsilon$. In particular, if $\varepsilon = a/(1-a)$ then for all $n \ge N$,

$$x_n \le \frac{\delta}{1-a} + \frac{a}{1-a} = \frac{\delta + a}{1-a} := M.$$

(b) Suppose that $\gamma > 0$. Then for all $n > N$

$$x_n \ge \frac{\gamma}{(A+B)M+1} := L.$$

To verify that $L < M$ we observe that

$$M \ge (1-a)M = a + \delta \ge a + \gamma > a + L \ge L.$$

Finally, we establish that $f(u,v) \in [L,M]$ for all $u,v \in [L,M]$. If $u,v \in [L,M]$ then

$$f(u,v) \le aM + \delta = \frac{a\delta + a^2}{1-a} + \delta = \frac{\delta + a^2}{1-a} \le \frac{\delta + a}{1-a} = M.$$

Further,

$$f(u,v) \ge \frac{\gamma}{(A+B)M+1} = L \text{ for all } 0 \le u,v \le M$$

and the proof is complete. □

We emphasize that conditions (2.2) allow $A > 0$ with $\alpha = 0$ and $B > 0$ with $\beta = 0$. More instances of invariant intervals for the special case $a = 0$ can be found in [12].

Remark 2.2. If $a \ge 1$ then the solutions of (2.1) may not be uniformly bounded. In fact, all nontrivial solutions of (2.1) are unbounded since $x_{n+1} \ge ax_n$ for all n if $a > 1$. When $a = 1$ solutions may still be unbounded as is readily seen in the following, first-order special case:

$$x_{n+1} = x_n + \frac{\alpha x_n}{Ax_n + 1}.$$

3 Existence and Local Stability of Unique Positive Fixed Points

The fixed point of (2.1) must satisfy the following equation:

$$x = ax + \frac{\alpha x + \beta x + \gamma}{Ax + Bx + 1}.$$

Combining and rearranging terms yields

$$(1-a)(A+B)x^2 - [\alpha + \beta - (1-a)]\, x - \gamma = 0$$

i.e., the fixed points must be the roots of the quadratic equation

$$S(t) = d_1 t^2 - d_2 t - d_3 \tag{3.1}$$

where

$$d_1 = (1-a)(A+B), \;\; d_2 = \alpha + \beta - (1-a), \;\; d_3 = \gamma.$$

If (1.2) holds then $d_1 > 0$ and $d_3 \geq 0$. There are two more cases to consider.

Case 1: If $d_2 = 0$ then (3.1) has two roots given by

$$t_{\pm} = \pm\sqrt{\frac{d_3}{d_1}}.$$

Thus if $\gamma > 0$ then the unique positive fixed point of (2.1) is

$$\bar{x} = \sqrt{\frac{\gamma}{(1-a)(A+B)}}.$$

Case 2: When $d_2 \neq 0$ then the roots of (3.1) are given by

$$t_{\pm} = \frac{\alpha + \beta - (1-a) \pm \sqrt{[\alpha + \beta - (1-a)]^2 + 4(1-a)(A+B)\gamma}}{2(1-a)(A+B)}.$$

In particular, if $\gamma > 0$ then the unique positive fixed point of (2.1) is

$$\bar{x} = \frac{\alpha + \beta - (1-a) + \sqrt{[\alpha + \beta - (1-a)]^2 + 4(1-a)(A+B)\gamma}}{2(1-a)(A+B)}. \tag{3.2}$$

The above discussions imply the following.

Lemma 3.1. *If* (1.2) *holds and* $\gamma > 0$ *then* (2.1) *has a positive fixed point* $\bar{x}$ *that is uniquely given by* (3.2).

We now consider the local stability of $\bar{x}$ under the hypotheses of the above lemma. The characteristic equation associated with the linearization of (2.1) at the point $\bar{x}$ is given by

$$\lambda^2 - f_u(\bar{x},\bar{x})\lambda - f_v(\bar{x},\bar{x}) = 0 \tag{3.3}$$

where

$$f(u,v) = au + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}$$

with

$$f_u = a + \frac{\alpha(Au + Bv + 1) - A(\alpha u + \beta v + \gamma)}{(Au + Bv + 1)^2} = a + \frac{(B\alpha - A\beta)v + \alpha - A\gamma}{(Au + Bv + 1)^2}$$

and

$$f_v = \frac{\beta(Au + Bv + 1) - B(\alpha u + \beta v + \gamma)}{(Au + Bv + 1)^2} = \frac{(A\beta - B\alpha)u + \beta - B\gamma}{(Au + Bv + 1)^2}.$$

Define

$$\begin{aligned} f_u(\bar{x},\bar{x}) &= a + \frac{\alpha - (1-a)A\bar{x}}{(A+B)\bar{x} + 1} := p \\ f_v(\bar{x},\bar{x}) &= \frac{\beta - (1-a)B\bar{x}}{(A+B)\bar{x} + 1} := q \end{aligned}$$

and note that the fixed point $\bar{x}$ is locally asymptotically stable if both roots of (3.3), namely,

$$\lambda_1 = \frac{p - \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad \lambda_2 = \frac{p + \sqrt{p^2 + 4q}}{2}$$

are inside the unit disk of the complex plain. Both roots are complex if and only if $p^2 + 4q < 0$ or $q < -(p/2)^2$. In this case, $|\lambda_1| = |\lambda_2| = -q$ so both roots have modulus less than 1 if and only if $q > -1$ or equivalently, $q + 1 > 0$, i.e.,

$$\begin{aligned} \beta - (1-a)B\bar{x} + (A+B)\bar{x} + 1 &> 0 \\ (A + aB)\bar{x} + \beta + 1 &> 0. \end{aligned}$$

This is clearly true if (1.2) holds. So if (1.2) holds and $\gamma > 0$ and if $-1 < q < -p^2/4$ then $\bar{x}$ is locally asymptotically stable with complex roots or eigenvalues.

Now suppose that $q \geq -p^2/4$ and the eigenvalues are real. First, observe that $p + q < 1$, or equivalently

$$\begin{aligned} [(2a-1)A + aB]\bar{x} + \alpha - (A+B)\bar{x} - (1-a) + \beta - (1-a)B\bar{x} &< 0 \\ 2(1-a)(A+B)\bar{x} &> \alpha + \beta - (1-a) \end{aligned} \tag{3.4}$$

which is true if (1.2) holds and $\gamma > 0$; see (3.2). Next, note that $p < 2$. To see this, $p - 2 < 0$ if and only if

$$\alpha - (1-a)A\bar{x} - (2-a)[(A+B)\bar{x}+1] < 0. \tag{3.5}$$

Since by (3.4)

$$\begin{aligned}(2-a)(A+B)\bar{x} &= 2(1-a)(A+B)\bar{x} + a(A+B)\bar{x} \\ &> \alpha + \beta - (1-a) + a(A+B)\bar{x}\end{aligned}$$

it follows that

$$\begin{aligned}&\alpha - (1-a)A\bar{x} - (2-a)[(A+B)\bar{x}+1] \\ &\quad = -(1-a)A\bar{x} - (2-a) + \alpha - (2-a)(A+B)\bar{x} \\ &\quad < -(1-a)A\bar{x} - (2-a) - \beta + (1-a) - a(A+B)\bar{x} \\ &\quad = -(1-a)A\bar{x} - 1 - \beta - a(A+B)\bar{x} \\ &\quad < 0.\end{aligned}$$

This proves that (3.5) is true. Finally, $p > -2$, since this is equivalent to

$$\alpha - (1-a)A\bar{x} > -(2+a)[(A+B)\bar{x}+1]$$

or

$$(1+2a)A\bar{x} + (2+a)B\bar{x} > -\alpha - (2+a)$$

which is true if 1.2 holds and $\gamma > 0$.

Now, a routine calculation shows that $\lambda_2 < 1$ if and only if $q < 1 - p$, which is indeed the case by the above. Next, $\lambda_2 > -1$ if and only if

$$p + \sqrt{p^2 + 4q} > -2. \tag{3.6}$$

If $p > -2$ then (3.6) holds trivially. On the other hand, if $p \le -2$ or $p + 2 \le 0$ then

$$\begin{aligned}(2+a)[(A+B)\bar{x}+1] + \alpha - (1-a)A\bar{x} \le 0 \\ (1+2a)A\bar{x} + (2+a)(B\bar{x}+1) + \alpha \le 0\end{aligned}$$

which is not possible if (1.2) holds. It follows that $|\lambda_2| < 1$ if (1.2) holds and $\gamma > 0$.

Next, consider λ_1 and note that $\lambda_1 < 1$ if and only if $p - \sqrt{p^2 + 4q} < 2$. This is clearly true if $p < 2$ which is in fact the case and we conclude that $\lambda_1 < 1$ if (1.2) holds and $\gamma > 0$.

Next, $\lambda_1 > -1$ if and only if

$$p - \sqrt{p^2 + 4q} > -2.$$

This requires that $p > -2$ which is true if (1.2) holds and $\gamma > 0$. Now the above inequality reduces to $p + 1 > q$ or

$$\begin{aligned}\beta - (1-a)B\bar{x} - a[(A+B)\bar{x}+1] - \alpha + (1-a)A\bar{x} &< (A+B)\bar{x}+1 \\ \beta - \alpha - (1+a) &< 2(aA+B)\bar{x}. \qquad (3.7)\end{aligned}$$

We also note that if the reverse of the above inequality holds, i.e.,

$$2(Aa+B)\bar{x} < \beta - \alpha - (1+a). \qquad (3.8)$$

then the above calculation show that $\lambda_1 < -1$ while $|\lambda_2| < 1$. Therefore in this case $\bar{x}$ is a saddle point. If $\beta - \alpha - (1+a) \le 0$ then (3.8) does not hold and $\bar{x}$ is locally asymptotically stable.

The preceding calculations in particular prove the following.

Lemma 3.2. *Let* (1.2) *hold and* $\gamma > 0$*. Then the positive fixed point* $\bar{x}$ *of* (2.1) *is locally asymptotically stable if and only if* (3.7) *holds and a saddle point if and only if the reverse inequality, i.e.,* (3.8)*, holds.*

Since $\bar{x}$ is nonhyperbolic if neither (3.7) nor (3.8) holds, Lemma 3.2 gives a complete picture of the local stability of $\bar{x}$ under its stated hypotheses.

3.1 Global Stability and Convergence of Solutions

In this section, we discuss global convergence to the positive fixed point and start by quoting the following familiar result from [10].

Lemma 3.3. *Let I be an open interval of real numbers and suppose that $f \in C(I^m, \mathbb{R})$ is nondecreasing in each coordinate. Let $\bar{x} \in I$ be a fixed point of the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-m+1}) \qquad (3.9)$$

and assume that the function $h(t) = f(t, \ldots, t)$ satisfies the conditions

$$h(t) > t \text{ if } t < \bar{x} \quad \text{and} \quad h(t) < t \text{ if } t > \bar{x}, \quad t \in I. \qquad (3.10)$$

Then I is an invariant interval of (3.9) *and $\bar{x}$ attracts all solutions with initial values in I.*

We now use the preceding result to obtain sufficient conditions for the global attractivity of the positive fixed point.

Theorem 3.4. *(a) Assume that* (1.2) *holds with $\gamma > 0$ and suppose that $f(u, v)$ is nondecreasing in both arguments. Then* (2.1) *has a unique fixed point $\bar{x} > 0$ that is asymptotically stable and attracts all positive solutions of* (2.1).

(b) Assume that (1.2) *holds with* $\gamma > 0$ *and*

$$B\alpha \leq A\beta \leq B\alpha + 2aB, \quad A\gamma \leq a + \alpha, \quad B\gamma \leq \beta. \tag{3.11}$$

Then (2.1) *has a unique fixed point* $\bar{x} > 0$ *that is asymptotically stable and attracts all positive solutions of* (2.1).

Proof. (a) The existence and uniqueness of $\bar{x} > 0$ follows from Lemma 3.1. Next, the function h in (3.10) takes the form

$$h(t) = at + \frac{(\alpha + \beta)t + \gamma}{(A + B)t + 1}.$$

Note that the fixed point $\bar{x}$ of (2.1) is a solution of the equation $h(t) = t$ so we verify that conditions (3.10) hold. For $t > 0$ the function h may be written as

$$h(t) = \phi(t)t, \quad \text{where } \phi(t) = a + \frac{\alpha + \beta + \gamma/t}{(A + B)t + 1}.$$

Note that $\phi(\bar{x}) = h(\bar{x})/\bar{x} = 1$. Further,

$$\phi'(t) = \frac{-[(A + B)t + 1]\gamma/t^2 - (A + B)[\alpha + \beta + \gamma/t]}{[(A + B)t + 1]^2}$$

so ϕ is decreasing (strictly) for all $t > 0$. Therefore,

$$t < \bar{x} \text{ implies } h(t) = \phi(t)t > \phi(\bar{x})t = t,$$
$$t > \bar{x} \text{ implies } h(t) = \phi(t)t < \phi(\bar{x})t = t.$$

Now by Lemma 3.3 $\bar{x}$ attracts all positive solutions of (2.1). In particular, $\bar{x}$ is not a saddle point so by Lemma 3.2 it is asymptotically stable.

(b) We show that if the inequalities (3.11) hold then the function

$$f(u, v) = au + \frac{\alpha u + \beta v + \gamma}{Au + Bv + 1}$$

is nondecreasing in each of its two coordinates u, v. This is demonstrated by computing the partial derivatives f_u and f_v to show that $f_u \geq 0$ and $f_v \geq 0$. By direct calculation $f_u \geq 0$ iff

$$a(Au + Bv)^2 + 2aAu + (2aB + B\alpha - A\beta)v + a + \alpha - A\gamma \geq 0.$$

The above inequality holds for all $u, v > 0$ if

$$2aB + B\alpha - A\beta \geq 0, \quad A\gamma \leq a + \alpha. \tag{3.12}$$

Similarly, $f_v \geq 0$ iff

$$(A\beta - B\alpha)u + \beta - B\gamma \geq 0$$

which is true for all $u, v > 0$ if

$$A\beta - B\alpha \geq 0, \quad B\gamma \leq \beta. \tag{3.13}$$

By the inequalities (3.12) and (3.13), conditions (3.11) are sufficient for f to be nondecreasing in each of its coordinates. The rest follows from (a). □

The next two results from [12] are stated as lemmas for convenience.

Lemma 3.5. *Let $[a, b]$ be an interval of real numbers and assume that $f : [a, b]\times[a, b] \to [a, b]$ is a continuous function satisfying the following properties:*

(a) $f(x, y)$ is nonincreasing in $x \in [a, b]$ for each $y \in [a, b]$ and $f(x, y)$ is nondecreasing in $y \in [a, b]$ for each $x \in [a, b]$;

(b) Equation (3.14) *has no prime period two solution in $[a, b]$.*

Then the difference equation

$$x_{n+1} = f(x_n, x_{n-1}) \tag{3.14}$$

has a unique fixed point $\bar{x} \in [a, b]$ and every solution converges to $\bar{x}$.

Lemma 3.6. *Let $[a, b]$ be an interval of real numbers and assume that $f : [a, b]\times[a, b] \to [a, b]$ is a continuous function satisfying the following properties:*

(a) $f(x, y)$ is nondecreasing in $x \in [a, b]$ for each $y \in [a, b]$ and $f(x, y)$ is nonincreasing in $y \in [a, b]$ for each $x \in [a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$f(m, M) = m \text{ and } f(M, m) = M$$

it follows that $m = M$.

Then (3.14) *has a unique fixed point $\bar{x} \in [a, b]$ and every solution converges to $\bar{x}$.*

Theorem 3.7. *(a) Let* (1.2) *hold with $\gamma > 0$ and further assume that $\alpha = 0$ if $A = 0$. If $f(u, v)$ is nondecreasing in u and nonincreasing in v, then* (2.1) *has a positive fixed point $\bar{x}$ that attracts every solution with nonnegative initial values.*

(b) If (1.2) *holds with $\gamma, A, B > 0$ and*

$$\frac{\alpha}{A} \geq \gamma \geq \frac{\beta}{B} \tag{3.15}$$

then (2.1) *has a unique positive fixed point $\bar{x}$ that is asymptotically stable and attracts all solutions of* (2.1).

Proof. (a) Note that by hypothesis $f_v \leq 0$ and this implies that $\beta = 0$ if $B = 0$. Now Lemma 2.1 implies that for arbitrary positive initial values there are real numbers $L_0, M_0 > 0$ and a positive integer N such that $x_n \in [L_0, M_0]$ for $n \geq N$. Therefore, to prove the global attractivity of $\bar{x}$ we need only show that the hypotheses of Lemma 3.6 are satisfied with $[a, b] = [L_0, M_0]$.

Next, consider the system

$$f(m, M) = m \text{ and } f(M, m) = M.$$

Clearly, $m = M = \bar{x}$ is a solution to the above system. If we assume that $m \neq M$, then the above system will have a positive solution if $m, M > 0$ and satisfy the following equations:

$$m = am + \frac{\alpha m + \beta M + \gamma}{Am + BM + 1} \tag{3.16a}$$

$$M = aM + \frac{\alpha M + \beta m + \gamma}{AM + Bm + 1}. \tag{3.16b}$$

From (3.16a) we get

$$(1 - a)(Am^2 + BMm + m) = \alpha m + \beta M + \gamma. \tag{3.17}$$

Similarly, from (3.16b) we get

$$(1 - a)(AM^2 + BMm + M) = \alpha M + \beta m + \gamma. \tag{3.18}$$

Taking the difference of both sides of the above two equations in (3.17) and (3.18) yields

$$\begin{aligned} &(1 - a)[A(M^2 - m^2) + (M - m)] = \alpha(M - m) + \beta(m - M) \\ &(1 - a)(M - m)(A(m + M) + 1) = (M - m)(\alpha - \beta). \end{aligned}$$

When $A = \alpha = 0$, then the last expression implies that the system $f(m, M) = m \quad f(M, m) = M$ has no positive solution besides $M = m = \bar{x}$ and we are done. We next assume that $A > 0$. Since $M \neq m$ we get

$$(1 - a)A(m + M) = \alpha - \beta - (1 - a). \tag{3.19}$$

From (3.19) we infer that $\alpha - \beta - (1 - a) > 0$, or stated differently, when $\alpha - \beta - (1 - a) \leq 0$, then the above system has no positive solution besides $m = M = \bar{x}$. Next, we sum the equations in (3.17) and (3.18) to get

$$(1 - a)A(m^2 + M^2) + 2(1 - a)BMm = (\alpha + \beta - (1 - a))(M + m) + 2\gamma.$$

Adding and subtracting $2A(1 - a)Mm$ from the left hand side of the above yields

$$(1 - a)A(m + M)^2 + 2(1 - a)(B - A)Mm = (\alpha + \beta - (1 - a))(M + m) + 2\gamma.$$

Thus

$$\begin{aligned}2(1-a)(B-A)Mm &= (M+m)\left[(\alpha+\beta-(1-a)-(1-a)A(M+m)\right]+2\gamma\\ &= (M+m)\left[(\alpha+\beta-(1-a)-\alpha+\beta+(1-a)\right]+2\gamma\\ &= \frac{2\beta(\alpha-\beta-(1-a))}{(1-a)A}+2\gamma\end{aligned}$$

i.e.,

$$(1-a)(B-A)Mm = \frac{\beta[\alpha-\beta-(1-a)]}{(1-a)A}+\gamma$$

from which we infer that $B-A>0$, since the right hand side of (3.20) is positive. Stated differently, this implies that when $B<A$, the above system has no positive solution besides $m=M=\bar{x}$.

Now, let

$$m+M=\frac{\alpha-\beta-(1-a)}{(1-a)A}:=P$$

and

$$Mm=\frac{\beta(\alpha-\beta-(1-a))}{(1-a)^2A(B-A)}+\frac{\gamma}{(1-a)(B-A)}:=Q.$$

Then $m=P-M$ and $M(P-M)=Q$. Similarly, $M=P-m$ and $m(P-m)=Q$. Thus M and m must be the roots of the quadratic equation

$$S(t)=t^2-Pt+Q.$$

Therefore, for the roots of $S(t)$ to be real, we require that $P^2-4Q>0$, i.e.,

$$\frac{[\alpha-\beta-(1-a)]^2}{(1-a)^2A^2}-\frac{4\beta[\alpha-\beta-(1-a)]}{(1-a)^2A(B-A)}-\frac{4\gamma}{(1-a)(B-A)}>0$$

which is equivalent to

$$\frac{4\gamma(1-a)}{B-A}<\frac{\alpha-\beta-(1-a)}{A}\left[\frac{\alpha-\beta-(1-a)}{A}-\frac{4\beta}{B-A}\right]. \tag{3.20}$$

Now

$$\frac{\alpha-\beta-(1-a)}{A}-\frac{4\beta}{B-A}=\frac{(B-A)\left[\alpha-\beta-(1-a)\right]-4A\beta}{A(B-A)}$$

and

$$\begin{aligned}&(B-A)[\alpha-\beta-(1-a)]-4A\beta\\ &\quad=(B-A)[\alpha-\beta-(1-a)]-4A\beta\\ &\qquad+A[\alpha-\beta-(1-a)]-A[\alpha-\beta-(1-a)]\\ &\quad=(A+B)[\alpha-\beta-(1-a)]-2A[\alpha+\beta-(1-a)].\end{aligned}$$

Thus the inequality in (3.20) becomes

$$\frac{4\gamma(1-a)}{B-A} < \frac{\alpha-\beta-(1-a)}{A}\left[\frac{\alpha-\beta-(1-a)}{A} - \frac{4\beta}{B-A}\right]$$
$$= \frac{\alpha-\beta-(1-a)}{A^2(B-A)}\left[(A+B)[\alpha-\beta-(1-a)] - 2A[\alpha+\beta-(1-a)]\right].$$

Multiplying both sides by $(B-A)(A+B)$ yields

$$\begin{aligned}&4\gamma(1-a)(A+B)\\ &< \frac{(A+B)^2}{A^2}[\alpha-\beta-(1-a)]^2 - \frac{2(A+B)}{A}[\alpha+\beta-(1-a)][\alpha-\beta-(1-a)].\end{aligned}$$

Adding $[\alpha+\beta-(1-a)]^2$ to both sides we get

$$\begin{aligned}&[\alpha+\beta-(1-a)]^2 + 4\gamma(1-a)(A+B)\\ &\quad< [\alpha+\beta-(1-a)]^2 - \frac{2(A+B)}{A}[\alpha+\beta-(1-a)][\alpha-\beta-(1-a)]\\ &\qquad+ \frac{(A+B)^2}{A^2}[\alpha-\beta-(1-a)]^2\\ &\quad= [\alpha+\beta-(1-a) - \frac{A+B}{A}(\alpha-\beta-(1-a))]^2\\ &\quad< [\alpha+\beta-(1-a) - (\alpha-\beta-(1-a)]^2 = 4\beta^2\end{aligned}$$

which implies that

$$[\alpha+\beta-(1-a)]^2 + 4\gamma(1-a)(A+B) - 4\beta^2 < 0. \tag{3.21}$$

But since for the above system to have a solution, $\alpha-\beta-(1-a) > 0$, then $\alpha-(1-a) > \beta$. This implies that the inequality in (3.21) is false (i.e., the roots of $S(t)$ cannot be real), as

$$\begin{aligned}[\alpha+\beta-(1-a)]^2 + 4\gamma(1-a)(A+B) - 4\beta^2 &> (2\beta)^2 + 4\gamma(1-a)(A+B) - 4\beta^2\\ &= 4\gamma(1-a)(A+B) > 0.\end{aligned}$$

Thus the system $f(m, M) = m, f(M, m) = M$ has no positive solution where $m \neq M$. Lemma 2.1 implies that for arbitrary positive initial values, there is an integer N such that $x_n \in [L, M]$ for $n > N$, so with $[a, b] = [L, M]$ and x_N and x_{N+1} as initial values, x_n must converge to $\bar{x}$ by Lemma 3.6.

(b) The condition in (3.15) are sufficient to ensure that $f_u \geq 0$ and $f_v \leq 0$ for all $u, v \geq 0$, and the result follows from (a). □

4 Periodic Solutions

We consider some conditions that lead to the occurrence of periodic solutions of (2.1). In this section, we explicitly assume that $Aa+B>0$. By assumption in (1.2), $Aa+B=0$ implies that $a=B=0$, which reduces (2.1) to the second-order rational equation

$$x_{n+1}=\frac{\alpha x_n+\beta x_{n-1}+\gamma}{Ax_n+1} \tag{4.1}$$

which has been studied in [12, page 167]. In particular, it was shown that when

$$\beta=\alpha+1$$

then every solution of (4.1) converges to a period-two solution.

4.1 Prime Period Two Solutions

The following gives necessary and sufficient conditions for the existence of a positive period 2 solution when $aA+B>0$.

Theorem 4.1. *Assume that* (1.2) *holds with* $\gamma, Aa+B>0$. *Then* (2.1) *has a positive prime period two solution if and only if the following conditions are satisfied:*
(i) $\beta-\alpha-(1+a)>0$*;*
(ii) $A-B>0$;
(iii)

$$\frac{4\gamma}{(1+a)(A-B)} < \left[\frac{\beta-\alpha-(1+a)}{Aa+B}\right]\left[\frac{\beta-\alpha-(1+a)}{Aa+B}-\frac{4\left[Aa(\beta-1)+B(\alpha+a)\right]}{(1+a)(A-B)(Aa+B)}\right] \tag{4.2}$$

Proof. Equation (2.1) has a positive prime period two solution if there exist real numbers $m, M>0$, with $m\neq M$, such that

$$m=aM+\frac{\alpha M+\beta m+\gamma}{AM+Bm+1} \quad\text{and}\quad M=am+\frac{\alpha m+\beta M+\gamma}{Am+BM+1}. \tag{4.3}$$

From (4.3) we obtain

$$(m-aM)(AM+Bm+1)=\alpha M+\beta m+\gamma \tag{4.4}$$

$$(M-am)(Am+BM+1)=\alpha m+\beta M+\gamma. \tag{4.5}$$

Taking the difference of right and left hand sides of (4.4) and (4.5) and rearranging the terms yields

$$(Aa+B)(m-M)(m+M)=(m-M)(\beta-\alpha-(1+a))$$

or

$$m + M = \frac{\beta - \alpha - (1 + a)}{Aa + B}. \tag{4.6}$$

Since $Aa + B > 0$, we infer from (4.6) that $\beta - \alpha - (1 + a) > 0$ is a necessary condition for existence of positive period two solutions.

Similarly, adding the right and left hand sides of (4.4) and (4.5) and rearranging the terms yields

$$2(A - aB)Mm = (\alpha + \beta - (1 - a))(M + m) + (Aa - B)(m^2 + M^2) + 2\gamma.$$

Adding an subtracting $2(Aa - B)Mm$ from the right hand side of the above yields

$$2(1 + a)(A - B)Mm = (m + M)\left[(\alpha + \beta - (1 - a)) + (Aa - B)(m + M)\right] + 2\gamma.$$

Inserting from (4.6) the expression for $m + M$ inside the square bracket yields

$$\begin{aligned} &2(1 + a)(A - B)Mm \\ &\quad = (M + m)\left[(\alpha + \beta - (1 - a)) + \frac{Aa - B}{Aa + B}(\beta - \alpha - (1 + a))\right] + 2\gamma \\ &\quad = \frac{2(M + m)}{Aa + B}\left[Aa(\beta - 1) + B(\alpha + a)\right] + 2\gamma. \end{aligned}$$

Thus

$$(1 + a)(A - B)Mm = \left[\frac{\beta - \alpha - (1 + a)}{(Aa + B)^2}\right]\left[Aa(\beta - 1) + B(\alpha + a)\right] + \gamma. \tag{4.7}$$

Since from (4.6) we have $\beta - \alpha - (1 + a) > 0$, then $\beta - 1 > 0$. Thus the right hand side of (4.7) is positive and therefore, $A - B > 0$ is another necessary condition for existence of positive period two solutions and

$$Mm = \frac{1}{(1 + a)(A - B)}\left[\left[\frac{\beta - \alpha - (1 + a)}{(Aa + B)^2}\right]\left[Aa(\beta - 1) + B(\alpha + a)\right] + \gamma\right]. \tag{4.8}$$

Let

$$P = \frac{\beta - \alpha - (1 + a)}{Aa + B}$$

and

$$K = \frac{1}{(1 + a)(A - B)}\left[\left[\frac{\beta - \alpha - (1 + a)}{(Aa + B)^2}\right]\left[Aa(\beta - 1) + B(\alpha + a)\right] + \gamma\right]$$

with $P, K > 0$. From (4.6) we obtain

$$m = \frac{\beta - \alpha - (1 + a)}{Aa + B} - M = P - M.$$

Inserting the above into (4.8) yields

$$M(P-M)=K \quad \text{or} \quad M^2-PM+K=0. \tag{4.9}$$

Similarly, an identical expression can be obtained for m, i.e.,

$$m^2-Pm+K=0. \tag{4.10}$$

Thus M and m must be real and distinct positive roots of the quadratic equation

$$Q(t)=t^2-Pt+K$$

with

$$t=\frac{P\pm\sqrt{P^2-4K}}{2}$$

which will be the case if and only if

$$P^2-4K>0$$

which is equivalent to (4.2). □

We also establish the following general result on nonexistence of period 2 solutions.

Theorem 4.2. *Let D be a subset of real numbers and assume that $f: D\times D\to D$ is nondecreasing in $x\in D$ for each $y\in D$ and nonincreasing in $y\in D$ for each $x\in D$. Then the difference equation* (3.14) *has no prime period two solution.*

Proof. Assume that the above difference equation has prime period two solution. Then there exist real numbers m and M, such that

$$f(m,M)=M \quad \text{and} \quad f(M,m)=m.$$

When $m=M$, we are done. So assume that $m\neq M$. If $m<M$, then by the hypothesis

$$f(m,M)\leq f(M,M)\leq f(M,m)$$

which implies that $M\leq m$, which is a contradiction.

Similarly, if $m>M$, then by the hypothesis

$$f(M,m)\leq f(M,M)\leq f(m,M)$$

which implies that $m\leq M$, which is also a contradiction. □

Our final result of this section establishes the connection between existence of prime period two solutions and the stability of the fixed point.

Theorem 4.3. *Let* (1.2) *hold with $\gamma, Aa+B>0$. Then* (2.1) *has a positive prime period two solution if and only it $\bar{x}$ is a saddle.*

Proof. First, when $\alpha + \beta - (1-a) = 0$, then the fixed point $\bar{x}$ is given by

$$\bar{x} = \sqrt{\frac{\gamma}{(1-a)(A+B)}}.$$

This implies that

$$\beta - \alpha - (1+a) = -2(a+\alpha) < 0$$

and $\bar{x}$ must be stable so (2.1) has no prime period two solution.

Now assume that $\alpha + \beta - (1-a) \neq 0$. Then the fixed point is given by

$$\bar{x} = \frac{\alpha + \beta - (1-a) + \sqrt{(\alpha + \beta - (1-a))^2 + 4(1-a)(A+B)\gamma}}{2(1-a)(A+B)}.$$

By Lemma 3.2, $\bar{x}$ is a saddle if and only if

$$\bar{x} < \frac{\beta - \alpha - (1+a)}{2(Aa+B)}$$

which implies that $\beta - \alpha - (1+a) > 0$.

Now

$$\bar{x} < \frac{\beta - \alpha - (1+a)}{2(Aa+B)}$$

iff

$$\frac{\alpha + \beta - (1-a) + \sqrt{(\alpha + \beta - (1-a))^2 + 4(1-a)(A+B)\gamma}}{2(1-a)(A+B)} < \frac{\beta - \alpha - (1+a)}{2(Aa+B)}$$

iff

$$\frac{\sqrt{(\alpha + \beta - (1-a))^2 + 4(1-a)(A+B)\gamma}}{(1-a)(A+B)} < \frac{\beta - \alpha - (1+a)}{(Aa+B)} - \frac{\alpha + \beta - (1-a)}{(1-a)(A+B)}$$

iff

$$\sqrt{(\alpha + \beta - (1-a))^2 + 4(1-a)(A+B)\gamma} < \frac{(1-a)(A+B)(\beta - \alpha - (1+a))}{Aa+B} - [\alpha + \beta - (1-a)]$$

iff

$$\begin{aligned}
&(\alpha + \beta - (1-a))^2 + 4(1-a)(A+B)\gamma \\
&\quad < \frac{(1-a)^2(A+B)^2(\beta - \alpha - (1+a))^2}{(Aa+B)^2} \\
&\qquad - \frac{2(1-a)(A+B)(\alpha + \beta - (1-a))(\beta - \alpha - (1+a))}{Aa+B} \\
&\qquad + (\alpha + \beta - (1-a))^2
\end{aligned}$$

iff

$$4(1-a)(A+B)\gamma < \frac{(1-a)^2(A+B)^2(\beta-\alpha-(1+a))^2}{(Aa+B)^2} - \frac{2(1-a)(A+B)(\alpha+\beta-(1-a))(\beta-\alpha-(1+a))}{Aa+B}$$

iff

$$\begin{aligned}4\gamma &< \frac{(1-a)(A+B)(\beta-\alpha-(1+a)^2)}{(Aa+B)^2} - \frac{2(\alpha+\beta-(1-a))(\beta-\alpha-(1+a))}{Aa+B}\\ &= \frac{(\beta-\alpha-(1+a))}{Aa+B}\left[\frac{(1-a)(A+B)(\beta-\alpha-(1+a))}{Aa+B} - 2(\alpha+\beta-(1-a))\right]\\ &= \frac{(\beta-\alpha-(1+a))}{Aa+B}\\ &\quad\times\left[\frac{(1-a)(A+B)(\beta-\alpha-(1+a)) - 2(Aa+B)(\alpha+\beta-(1-a))}{Aa+B}\right].\end{aligned}$$

Adding and subtracting $(1+a)(A-B)[\beta-\alpha-(1+a)]$ to the numerator of the second fraction in previous equation yields

$$\begin{aligned}&(1-a)(A+B)(\beta-\alpha-(1+a)) - 2(Aa+B)(\alpha+\beta-(1-a))\\ &\qquad = (1+a)(A-B)[\beta-\alpha-(1+a)] - 4Aa(\beta-1) - 4B(\alpha+a).\end{aligned}$$

Thus we have

$$\begin{aligned}4\gamma &< \frac{\beta-\alpha-(1+a)}{Aa+B}\\ &\qquad\times\frac{(1+a)(A-B)(\beta-\alpha-(1+a)) - 4\left[Aa(\beta-1)+B(\alpha+a)\right]}{Aa+B}.\end{aligned}$$

Note that since $\gamma > 0$, it must be the case that the right hand side of the last expression is positive, which implies that $A - B > 0$. Dividing both sides of the above expression by $(1+a)(A-B)$ then yields:

$$\begin{aligned}&\frac{4\gamma}{(1+a)(A-B)}\\ &\qquad < \frac{\beta-\alpha-(1+a)}{Aa+B}\left[\frac{(\beta-\alpha-(1+a))}{Aa+B} - \frac{4\left[Aa(\beta-1)+B(\alpha+a)\right]}{(Aa+B)(1+a)(A-B)}\right]\end{aligned}$$

and the proof is complete, since the conditions of Theorem 4.1 are satisfied. □

We end our discussion in this section with the following immediate consequence of the results already established.

Corollary 4.4. *Let* (1.2) *hold with* $\gamma, Aa + B > 0$*, and suppose that* $f(u,v)$ *is nondecreasing in* u *and either nondecreasing or nonincreasing in* v*.*

(a) *Equation* (2.1) *has no periodic solution of period greater than two.*

(b) *If* (2.1) *has no prime period two solution then all solutions of* (2.1) *converge to the positive fixed point* $\bar{x}$*.*

The above results give partial answers to two conjectures in [12] for the special case $a = 0$.

5 Applications

Certain planar systems of rational difference equations may be folded, or reduced, to the second-order equation in (2.1). For a general treatment of folding and its applications, we refer to [17]. In the special case of system (1.5)–(1.6), i.e.,

$$\begin{aligned} x_{n+1} &= ax_n + by_n + c \\ y_{n+1} &= \frac{a'x_n + b'y_n + c'}{a''x_n + b''y_n + c''} \end{aligned}$$

using routine calculations, when $b \neq 0$ we can eliminate one of the two variables and fold the system into the second-order equation [1]

$$x_{n+2} = ax_{n+1} + \frac{(bb' + cb'')x_{n+1} + (bD'_{ab} + cD''_{ab})x_n + bD'_{cb} + cD''_{cb}}{b''x_{n+1} + D''_{ab}x_n + D''_{cb}} \tag{5.1}$$

with

$$y_n = \frac{x_{n+1} - ax_n - c}{b} \tag{5.2}$$

where

$$D'_{ab} = a'b - ab', \quad D''_{ab} = a''b - ab'', \quad D'_{cb} = bc' - b'c, \quad D''_{cb} = bc'' - b''c. \tag{5.3}$$

If $D''_{cb} \neq 0$, i.e., $bc'' \neq b''c$ then we may normalize (5.1) to obtain

$$x_{n+2} = ax_{n+1} + \frac{\alpha x_{n+1} + \beta x_n + \gamma}{Ax_{n+1} + Bx_n + 1} \tag{5.4}$$

with

$$\alpha = \frac{1}{D''_{cb}}\left[bb' + cb''\right], \quad \beta = \frac{1}{D''_{cb}}\left[bD'_{ab} + cD''_{ab}\right], \quad \gamma = \frac{1}{D''_{cb}}\left[bD'_{cb} + cD''_{cb}\right] \tag{5.5a}$$

$$A = \frac{b''}{D''_{cb}}, \quad B = \frac{D''_{ab}}{D''_{cb}}. \tag{5.5b}$$

[1]Notice that when $b = 0$, the first equation of the above system is uncoupled and can be solved directly.

It is routine to check that the orbits of the system (1.5)–(1.6) correspond to the solutions of (5.4) in the sense that if $\{x_n\}$ is a solution of (5.4) with given initial values x_0 and x_1 and $\{y_n\}$ is given by (5.2) for $n \geq 0$, then $\{(x_n, y_n)\}$ is an orbit of (1.5)–(1.6). Conversely, if $\{(x_n, y_n)\}$ is an orbit of (1.5)–(1.6) from an initial point (x_0, y_0) and $x_1 = ax_0 + by_0 + c$, then $\{x_n\}$ is a solution of (5.4).

The results established in the previous sections can help identify similar behavior in the planar system (1.5)–(1.6), some of which we state below.

Corollary 5.1. *Let the parameters of* (1.5)–(1.6) *satisfy*

$$0 \leq a < 1, \quad b > 0, \quad c, b'' \geq 0, \quad bb' > -cb'' \tag{5.6a}$$

$$a'b > \max\{ab', -(1-a)b'\}, \quad a''b > ab'', \quad bc' > b'c, \quad bc'' > b''c. \tag{5.6b}$$

Then the system (1.5)–(1.6) *has a unique fixed point* $(\bar{x}, \bar{y}) \in (0, \infty)^2$ *where* $\bar{x}$ *is given by* (3.2) *and*

$$\bar{y} = \frac{(1-a)\bar{x} - c}{b} \tag{5.7}$$

Proof. The fixed points of (1.5)–(1.6) satisfy the equations:

$$\bar{x} = a\bar{x} + b\bar{y} + c \tag{5.8a}$$

$$\bar{y} = \frac{a'\bar{x} + b'\bar{y} + c'}{a''\bar{x} + b''\bar{y} + c''}. \tag{5.8b}$$

From (5.7) and (5.8b) we obtain

$$\begin{aligned}
\bar{y} &= \frac{a'\bar{x} + b'\frac{(1-a)\bar{x}-c}{b} + c'}{a''\bar{x} + b''\frac{(1-a)\bar{x}-c}{b} + c''} \\
&= \frac{[a'b + b'(1-a)]\bar{x} + bc' - b'c}{[a''b + b''(1-a)]\bar{x} + bc'' - b''c}.
\end{aligned}$$

Thus the conditions in (5.6) imply that the parameters in the folding (5.4) as defined by (5.5) satisfy (1.2) and that $\bar{x}$ given by (3.2) and $\bar{y}$ given by (5.7) are strictly positive. □

Corollary 5.2. *Let* (5.6) *hold and suppose that the parameters of* (1.5)–(1.6) *satisfy either all of the inequalities in (i) or the inequality in (ii) below:*

(i)

$$\begin{gathered}
D''_{ab}(bb' + cb'') \leq b''(bD'_{ab} + cD''_{ab}) \leq D''_{ab}(bb' + cb'') + 2aD''_{ab}D''_{cb} \\
b''(bD'_{cb} + cD''_{cb}) \leq D''_{cb}[bb' + cb'' + aD''_{cb}] \\
D''_{ab}(bD'_{cb} + cD''_{cb}) \leq D''_{cb}(bD'_{ab} + cD''_{ab}).
\end{gathered}$$

(ii)

$$\frac{1}{b''}(bb' + cb'') \geq \frac{1}{D''_{cb}}(bD'_{cb} + cD''_{cb}) \geq \frac{1}{D''_{ab}}(bD'_{ab} + cD''_{ab}).$$

Then all solutions of (1.5)–(1.6) *from initial values* $(x_0, y_0) \in [0, \infty)^2$ *converge to* $(\bar{x}, \bar{y}) \in (0, \infty)^2$.

Proof. The conditions in (5.6) are sufficient to ensure that the parameters in the folding (5.4) as defined by (5.5) satisfy (1.2) and that if $x_0, y_0 \geq 0$, then $x_0, x_1 = ax_0 + by_0 + c \geq 0$. Conditions in (i) and (ii) satisfy the hypotheses of Theorems 3.4 and 3.7 from which the result follows. □

Example 5.3. Consider the following system

$$x_{n+1} = 0.2x_n + y_n + 1 \tag{5.9a}$$

$$y_{n+1} = \frac{1.2x_n + y_n + 2}{x_n + y_n + 2} \tag{5.9b}$$

Routine calculations show that the parameters of (5.9) satisfy the conditions (i) in Corollary 5.2, which implies that all solutions of (5.9) from nonnegative initial values converge to the fixed point in the positive quadrant. We note that the system in (5.9) folds into

$$y_n = x_{n+1} - 0.2x_n - 1$$

$$x_{n+2} = 0.2x_{n+1} + \frac{2x_{n+1} + 1.8x_n + 2}{x_{n+1} + 0.8x_n + 1}.$$

Corollary 5.4. *Let* (5.6) *hold. Then* (1.5)–(1.6) *has a prime period two solution in* $[0, \infty)^2$ *if and only if*

$$2a''\bar{x} < a'b + a''c - ac'' - (1 + a)(b'' + c''). \tag{5.10}$$

Proof. The conditions in (5.6) are sufficient to ensure that the parameters in the folding (5.4) as defined by (5.5) satisfy (1.2) and that if $x_0, y_0 \geq 0$, then $x_0, x_1 = ax_0 + by_0 + c \geq 0$. Straightforward algebraic calculations show that the condition in (3.8) can be expressed with respect to parameters of (1.5)–(1.6) as (5.10). By Lemma 3.2 the fixed point $(\bar{x}, \bar{y})$ is a saddle and the proof follows from Theorem 4.3. □

Example 5.5. Consider the system

$$x_{n+1} = 0.01x_n + y_n + 0.1 \tag{5.11a}$$

$$y_{n+1} = \frac{5x_n + 2y_n + 1}{0.1x_n + y_n + 1}. \tag{5.11b}$$

Routine calculations show that the system in (5.11) satisfies the conditions in Corollary 5.4. Therefore, (5.11) has a prime period two solution. We note that the system in (5.11) folds into

$$y_n = x_{n+1} - 0.01x_n - 0.1$$

$$x_{n+2} = 0.01x_{n+1} + \frac{2.1x_{n+1} + 4.989x_n + 0.89}{x_{n+1} + 0.09x_n + 0.9}.$$

Remark 5.6. We note that the above results hold for (1.5)–(1.6) even if some parameters are negative. Of course, in this case the positive quadrant is not invariant; however, orbits of the system avoid singularities in the plane if (1.5)–(1.6) folds into (1.1) and the parameters of this scalar equation satisfy (1.2).

Example 5.7. Consider the following system

$$x_{n+1} = 0.6x_n + 0.1y_n + 0.2 \tag{5.12a}$$
$$y_{n+1} = \frac{-0.1x_n - 0.1y_n + 0.1}{x_n + 0.05y_n + 2}. \tag{5.12b}$$

By routine calculations we see that the parameters in (5.12) satisfy (ii) in Corollary 5.2. Thus all solutions from nonnegative initial values (x_0, y_0) converge to the positive fixed point in the first quadrant. We note that the system in (5.12) folds into

$$y_n = 10x_{n+1} - 6x_n - 2$$
$$x_{n+2} = 0.6x_{n+1} + \frac{0.019x_n + 0.041}{0.05x_{n+1} + 0.07x_n + 0.19}.$$

where the equation for x_{n+2} does not contain negative parameters.

Special cases of (1.5)–(1.6) have applications in biological models of species populations. An example can be found in [14] that considers a model of stage-structured species population with Beverton–Holt recruitment function given by

$$x_{n+1} = \alpha_1 x_n + \alpha_2 y_n$$
$$y_{n+1} = \frac{bx_n}{1 + c_1 x_n + c_2 y_n}$$

where α_1, α_2 are the survival rates of adult and juvenile members of the species, and c_1 and c_2 are competition coefficients. Using a similar approach as in the current paper, it is shown that for low level of competition, the solutions of the above system converge to a unique positive equilibrium. However, sufficiently large values of c_2 can have a destabilizing effect on the equilibrium and the population will exhibit period-two oscillations.

We conclude with the interesting fact that, beyond the parameter ranges of this paper, allowing negative parameters in the system (1.5)–(1.6) yields a richer variety of dynamic behavior for this system. The results in this section complement those in [13] that show that the rational system in (1.5)–(1.6) can have coexisting periodic orbits of all possible periods as well as stable aperiodic orbits if some parameters are negative.

Example 5.8. To illustrate possible occurrence of complex behavior with negative parameters, consider the following version of (1.5)–(1.6)

$$x_{n+1} = x_n + 2y_n - 2$$
$$y_{n+1} = \frac{0.25x_n + 0.5y_n + 1}{3x_n + 6y_n - 6}$$

It is shown in [13] that this system has periodic orbits of all periods (depending on initial points) and Li–Yorke type chaos occurs.

The results in the current paper complement those shown in [13] to which we refer the reader for more detail. For additional results on planar rational systems exhibiting complex behavior we refer to [17].

6 Concluding Remarks and Future Considerations

We studied the dynamics of a second-order quadratic fractional difference equation with nonnegative parameters and initial values. We obtained several sufficient conditions for the global stability of the positive fixed point. In addition, when (1.2) holds and $\gamma > 0$, we obtained necessary and sufficient conditions for the occurrence of periodic solutions and in particular proved that such solutions may appear if and only if the positive fixed point is a saddle.

A natural extension for future research involves addition of a linear delay term to the above equation, i.e., the study of

$$x_{n+1} = ax_n + bx_{n-1} + \frac{\alpha x_n + \beta x_{n-1} + \gamma}{Ax_n + Bx_{n-1} + C} \tag{6.1}$$

We close with the following conjectures:

Conjecture 6.1. Let (1.2) hold and further assume that $b \geq 0$ and $a + b < 1$. Then the equation in (6.1) does not have any prime periodic solutions of period greater than 2.

Conjecture 6.2. Let (1.2) hold and further assume that $b \geq 0$ and $a + b < 1$. If (6.1) has no prime period two solution, then every solution of (6.1) converges to a unique positive fixed point.

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