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A System of Difference Equations with Solutions Associated to Fibonacci Numbers

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Abstract

This paper deals with form, the periodicity and the stability of the solutions of the system of difference equations

$$x_{n+1} = \frac{1}{1+y_{n-2}},\ y_{n+1} = \frac{1}{1+x_{n-2}},\ n \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, and y_0 are real numbers.

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1 Introduction

The theory of difference equations developed greatly during the last twenty-five years of the twentieth century. The applications of the theory of difference equations is rapidly increasing to various fields such as numerical analysis, biology, economics, control theory, finite mathematics and computer science. Thus, there is every reason for studying the theory of difference equations as a well deserved discipline.

Recently, there have been a growing interest in the study of solving systems of difference equations, see for example, [1, 3–6, 9, 15–17], and the references cited therein.

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In these researches, authors are interested in investigating the form, periodicity, boundedness, (local and global) stability, and asymptotic behavior of solutions of various systems of difference equations of nonlinear types.

In [14], Tollu et al. investigated the dynamics of the solutions of the two difference equations

$$x_{n+1} = \frac{1}{\pm 1 + x_n}, \quad n \in \mathbb{N}_0,$$

which were then extended by Halim to systems of two nonlinear difference equations

$$x_{n+1} = \frac{1}{1 \pm y_n},\ y_{n+1} = \frac{1}{1 \pm x_n}, n \in \mathbb{N}_0$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions are real numbers in [7].
In paper [12], Stević show that the following systems of difference equations

$$x_{n+1} = \frac{u_n}{1 + v_n}, \quad y_{n+1} = \frac{w_n}{1 + s_n}, n \in \mathbb{N}_0$$

where u_n, v_n, w_n, s_n are some of the sequences x_n or y_n, with real initial values x_0 and y_0, are solvable in fourteen out of sixteen possible cases.

Motivated by these fascinating results, we shall determine the form and investigate the periodicity and global character of the solutions of the systems

$$x_{n+1} = \frac{1}{1 + y_{n-2}}, \quad y_{n+1} = \frac{1}{1 + x_{n-2}}, \quad n \in \mathbb{N}_0. \tag{1.1}$$

where the initial conditions are real numbers.

2 Preliminaries

A well-known recurrence sequence of order two is the widely studied Fibonacci sequence $\{F_n\}_{n=1}^{\infty}$ recursively defined by the recurrence relation

$$F_1 = F_2 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad (n \geq 1). \tag{2.1}$$

As a result of the definition (2.1), it is conventional to define $F_0 = 0$. Various problems involving Fibonacci numbers have been formulated and extensively studied by many authors. Different generalizations and extensions of Fibonacci sequence have also been introduced and thoroughly investigated (see for example [8]). An interesting property of this integer sequence is that the ratio of its successive terms converges to the well-known golden mean (or the golden ratio) $\phi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$. For more fascinating properties of Fibonacci numbers we refer the readers to [11].

Now, in the rest of this section we shall present some basic notations and results on the study of nonlinear difference equation which will be useful in our investigation, for more details, see for example [2].

Let f and g be two continuously differentiable functions:

$$f:\ I^3 \times J^3 \longrightarrow I, \quad g:\ I^3 \times J^3 \longrightarrow J, \quad I, J \subseteq \mathbb{R}$$

and for $n, k \in \mathbb{N}_0$, consider the system of difference equations

$$\begin{cases} x_{n+1} = & f\left(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}\right) \\ y_{n+1} = & g\left(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}\right) \end{cases} \tag{2.2}$$

where $(x_{-2}, x_{-1}, x_0) \in I^3$ and $(y_{-2}, y_{-1}, y_0) \in J^3$.

Define the map

$$H:\ I^3 \times J^3 \longrightarrow I^3 \times J^3$$

by

$$H(W) = (f_0(W), f_1(W), f_2(W), g_0(W), g_1(W), g_2(W))$$

where

$$\begin{gathered} W = (u_0, u_1, u_2, v_0, v_1, v_2)^T, \\ f_0(W) = f(W), f_1(W) = u_0, f_2(W) = u_1, \\ g_0(W) = g(W), g_1(W) = v_0, g_2(W) = v_1. \end{gathered}$$

Let

$$W_n = \left(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}\right)^T.$$

Then, we can easily see that system (2.2) is equivalent to the following system written in vector form

$$W_{n+1} = H(W_n),\ n = 0, 1, \ldots, \tag{2.3}$$

that is

$$\begin{cases} x_{n+1} &=& f\left(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}\right) \\ x_n &=& x_n \\ x_{n-1} &=& x_{n-1} \\ y_{n+1} &=& g\left(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}\right) \\ y_n &=& y_n \\ y_{n-1} &=& y_{n-1}. \end{cases}$$

Definition 2.1 (Equilibrium point)**.** An equilibrium point $(\overline{x}, \overline{y}) \in I \times J$ of system (2.2) is a solution of the system

$$\begin{cases} x = f\left(x, x, x, y, y, y\right), \\ y = g\left(x, x, x, y, y, y\right). \end{cases}$$

Furthermore, an equilibrium point $\overline{W} \in I^3 \times J^3$ of system (2.3) is a solution of the system

$$W = H(W).$$

Definition 2.2 (Stability)**.** Let $\overline{W}$ be an equilibrium point of system (2.3) and $\| \,.\, \|$ be any norm (e.g., the Euclidean norm).

1. The equilibrium point $\overline{W}$ is called stable (or locally stable) if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|W_0 - \overline{W}\| < \delta$ implies $\|W_n - \overline{W}\| < \epsilon$ for $n \geq 0$.

2. The equilibrium point $\overline{W}$ is called asymptotically stable (or locally asymptotically stable) if it is stable and there exists $\gamma > 0$ such that $\|W_0 - \overline{W}\| < \gamma$ implies
$$\|W_n - \overline{W}\| \to 0,\; n \to +\infty.$$

3. The equilibrium point $\overline{W}$ is said to be global attractor (respectively global attractor with basin of attraction a set $G \subseteq I^3 \times J^3$), if for every W_0 (respectively for every $W_0 \in G$)
$$\|W_n - \overline{W}\| \to 0,\; n \to +\infty.$$

4. The equilibrium point $\overline{W}$ is called globally asymptotically stable (respectively globally asymptotically stable relative to G) if it is asymptotically stable, and if for every W_0 (respectively for every $W_0 \in G$),
$$\|W_n - \overline{W}\| \to 0,\; n \to +\infty.$$

5. The equilibrium point $\overline{W}$ is called unstable if it is not stable.

Remark 2.3. Clearly, $(\overline{x}, \overline{y}) \in I \times J$ is an equilibrium point for system (2.2) if and only if $\overline{W} = (\overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y}) \in I^3 \times J^3$ is an equilibrium point of system (2.3).

From here on, by the stability of the equilibrium points of system (2.2), we mean the stability of the corresponding equilibrium points of the equivalent system (2.3).

The linearized system, associated to system (2.3), about the equilibrium point
$$\overline{W} = (\overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y})$$
is given by
$$W_{n+1} = AW_n,\; n = 0, 1, \ldots$$
where A is the Jacobian matrix of the map H at the equilibrium point $\overline{W}$ given by
$$A = \begin{pmatrix}
\frac{\partial f_0}{\partial u_0}(\overline{W}) & \frac{\partial f_0}{\partial u_1}(\overline{W}) & \frac{\partial f_0}{\partial u_2}(\overline{W}) & \frac{\partial f_0}{\partial v_0}(\overline{W}) & \frac{\partial f_0}{\partial v_1}(\overline{W}) & \frac{\partial f_0}{\partial v_2}(\overline{W}) \\
\frac{\partial f_1}{\partial u_0}(\overline{W}) & \frac{\partial f_1}{\partial u_1}(\overline{W}) & \frac{\partial f_1}{\partial u_2}(\overline{W}) & \frac{\partial f_1}{\partial v_0}(\overline{W}) & \frac{\partial f_1}{\partial v_1}(\overline{W}) & \frac{\partial f_1}{\partial v_2}(\overline{W}) \\
\frac{\partial f_2}{\partial u_0}(\overline{W}) & \frac{\partial f_2}{\partial u_1}(\overline{W}) & \frac{\partial f_2}{\partial u_2}(\overline{W}) & \frac{\partial f_2}{\partial v_0}(\overline{W}) & \frac{\partial f_2}{\partial v_1}(\overline{W}) & \frac{\partial f_2}{\partial v_2}(\overline{W}) \\
\frac{\partial g_0}{\partial u_0}(\overline{W}) & \frac{\partial g_0}{\partial u_1}(\overline{W}) & \frac{\partial g_0}{\partial u_2}(\overline{W}) & \frac{\partial g_0}{\partial v_0}(\overline{W}) & \frac{\partial g_0}{\partial v_1}(\overline{W}) & \frac{\partial g_0}{\partial v_2}(\overline{W}) \\
\frac{\partial g_1}{\partial u_0}(\overline{W}) & \frac{\partial g_1}{\partial u_1}(\overline{W}) & \frac{\partial g_1}{\partial u_2}(\overline{W}) & \frac{\partial g_1}{\partial v_0}(\overline{W}) & \frac{\partial g_1}{\partial v_1}(\overline{W}) & \frac{\partial g_1}{\partial v_2}(\overline{W}) \\
\frac{\partial g_2}{\partial u_0}(\overline{W}) & \frac{\partial g_2}{\partial u_1}(\overline{W}) & \frac{\partial g_2}{\partial u_2}(\overline{W}) & \frac{\partial g_2}{\partial v_0}(\overline{W}) & \frac{\partial g_2}{\partial v_1}(\overline{W}) & \frac{\partial g_2}{\partial v_2}(\overline{W})
\end{pmatrix}.$$

Theorem 2.4 (See [10]). *If all the eigenvalues of the Jacobian matrix A lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{W}$ of system* (2.3) *is asymptotically stable. On the other hand, if at least one eigenvalue of the Jacobian matrix A have absolute value greater than one, then the equilibrium point $\overline{W}$ of system* (2.3) *is unstable.*

Now, we are in the position to investigate the form and behavior of solutions of the system (1.1) and this is the content of the next section.

3 Main Result

3.1 Form of the Solutions

In this section, we give the explicit form of solutions of the system of difference equations

$$x_{n+1} = \frac{1}{1+y_{n-2}}, \quad y_{n+1} = \frac{1}{1+x_{n-2}} \tag{3.1}$$

where the initial values are arbitrary real numbers with the restriction that

$$x_{-2}, y_{-2}, x_{-1}, y_{-1}, x_0, y_0 \notin \left\{ -\frac{F_{n+1}}{F_n};\ n = 1, 2, \dots \right\}.$$

The following theorem describes the form of the solutions of system (3.1).

Theorem 3.1. *Let $\{x_n, y_n\}_{n\geq -k}$ be a solution of* (3.1). *Then, for $n = 0, 1, \dots,$*

$$x_{6n+i} = \frac{F_{2n+1} + F_{2n}y_{i-3}}{F_{2n+2} + F_{2n+1}y_{i-3}}, \quad i = 1, 2, 3,$$
$$y_{6n+i} = \frac{F_{2n+1} + F_{2n}x_{i-3}}{F_{2n+2} + F_{2n+1}x_{i-3}}, \quad i = 1, 2, 3,$$
$$x_{6n+i} = \frac{F_{2n+2} + F_{2n+1}x_{i-6}}{F_{2n+3} + F_{2n+2}x_{i-6}}, \quad i = 4, 5, 6,$$
$$y_{6n+i} = \frac{F_{2n+2} + F_{2n+1}y_{i-6}}{F_{2n+3} + F_{2n+2}y_{i-6}}, \quad i = 4, 5, 6.$$

Proof. From (3.1) we have

$$\begin{aligned} x_1 &= \frac{1}{1+y_{-2}},\ x_2 = \frac{1}{1+y_{-1}}, x_3 = \frac{1}{1+y_0}, \\ y_1 &= \frac{1}{1+x_{-2}},\ y_2 = \frac{1}{1+x_{-1}}, y_3 = \frac{1}{1+x_0}, \end{aligned}$$

and

$$\begin{aligned} x_4 &= \frac{1+x_{-2}}{2+x_{-2}},\ x_5 = \frac{1+x_{-1}}{2+x_{-1}}, x_6 = \frac{1+x_0}{2+x_0}, \\ y_4 &= \frac{1+y_{-2}}{2+y_{-2}},\ y_5 = \frac{1+y_{-1}}{2+y_{-1}}, y_6 = \frac{1+y_0}{2+y_0}. \end{aligned}$$

So, the result hold for $n = 0$. Suppose now that $n \geq 1$ and that our assumption holds for $n-1$. That is,

$$x_{6(n-1)+i} = \frac{F_{2n-1} + F_{2n-2}y_{i-3}}{F_{2n} + F_{2n-1}y_{i-3}},\ i = 1, 2, 3, \tag{3.2}$$

$$y_{6(n-1)+i} = \frac{F_{2n-1} + F_{2n-2}x_{i-3}}{F_{2n} + F_{2n-1}x_{i-3}},\ i = 1, 2, 3, \tag{3.3}$$

$$x_{6(n-1)+i} = \frac{F_{2n} + F_{2n-1}x_{i-6}}{F_{2n+1} + F_{2n}x_{i-6}},\ \ i = 4, 5, 6, \tag{3.4}$$

$$y_{6(n-1)+i} = \frac{F_{2n} + F_{2n-1}y_{i-6}}{F_{2n+1} + F_{2n}y_{i-6}},\ \ i = 4, 5, 6. \tag{3.5}$$

For $i = 1, 2, 3$, it follows from (3.1), (3.2), and (3.3) that

$$\begin{aligned}
x_{6n+i} &= \frac{1}{1 + y_{6n-3+i}} \\
&= \frac{1}{1 + \frac{1}{1+x_{6(n-1)+i}}} \\
&= \frac{\frac{(F_{2n}+F_{2n-1})+(F_{2n-1}+F_{2n-2})y_{i-3}}{F_{2n}+F_{2n-1}y_{i-3}}}{\frac{2F_{2n}+F_{2n-1}+2F_{2n-1}y_{i-3}+F_{2n-2}y_{i-3}}{F_{2n}+F_{2n-1}y_{i-3}}} \\
&= \frac{F_{2n+1} + F_{2n}y_{i-3}}{F_{2n+1} + F_{2n} + (F_{2n-1} + F_{2n})y_{i-3}} \\
&= \frac{F_{2n+1} + F_{2n}y_{i-3}}{F_{2n+2} + F_{2n+1}y_{i-3}},
\end{aligned}$$

and

$$\begin{aligned}
y_{6n+i} &= \frac{1}{1 + x_{6n-3+i}} \\
&= \frac{1}{1 + \frac{1}{1+y_{6(n-1)+i}}} \\
&= \frac{\frac{(F_{2n}+F_{2n-1})+(F_{2n-1}+F_{2n-2})x_{i-3}}{F_{2n}+F_{2n-1}x_{i-3}}}{\frac{2F_{2n}+F_{2n-1}+2F_{2n-1}x_{i-3}+F_{2n-2}x_{i-3}}{F_{2n}+F_{2n-1}x_{i-3}}} \\
&= \frac{F_{2n+1} + F_{2n}x_{i-3}}{F_{2n+1} + F_{2n} + (F_{2n-1} + F_{2n})x_{i-3}} \\
&= \frac{F_{2n+1} + F_{2n}x_{i-3}}{F_{2n+2} + F_{2n+1}x_{i-3}}.
\end{aligned}$$

Similarly, for $i = 4, 5, 6$, from (3.1), (3.4), and (3.5), we get

$$\begin{aligned}
x_{6n+i} &= \frac{1}{1 + y_{6n-(1+k)+i}} \\
&= \frac{1}{1 + \frac{1}{1+x_{6(n-1)+i}}} \\
&= \frac{\frac{F_{2n+1}+F_{2n}+F_{2n}x_{i-6}+F_{2n-1}x_{i-6}}{F_{2n+1}+F_{2n}x_{i-6}}}{\frac{2F_{2n+1}+F_{2n}+2F_{2n}x_{i-6}+F_{2n-1}x_{i-6}}{F_{2n+1}+F_{2n}x_{i-6}}} \\
&= \frac{F_{2n+2} + F_{2n+1}x_{i-6}}{F_{2n+1} + F_{2n+2} + F_{2n}x_{i-6} + F_{2n+1}x_{i-6}} \\
&= \frac{F_{2n+2} + F_{2n+1}x_{i-6}}{F_{2n+3} + F_{2n+2}x_{i-6}},
\end{aligned}$$

and

$$\begin{aligned}
y_{6n+i} &= \frac{1}{1 + x_{6n-3+i}} \\
&= \frac{1}{1 + \frac{1}{1+y_{6(n-1)+i}}} \\
&= \frac{\frac{F_{2n+1}+F_{2n}+F_{2n}y_{i-6}+F_{2n-1}y_{i-6}}{F_{2n+1}+F_{2n}y_{i-6}}}{\frac{2F_{2n+1}+F_{2n}+2F_{2n}y_{i-6}+F_{2n-1}y_{i-6}}{F_{2n+1}+F_{2n}y_{i-6}}} \\
&= \frac{F_{2n+2} + F_{2n+1}y_{i-6}}{F_{2n+1} + F_{2n+2} + F_{2n}y_{i-6} + F_{2n+1}y_{i-6}} \\
&= \frac{F_{2n+2} + F_{2n+1}y_{i-6}}{F_{2n+3} + F_{2n+2}y_{i-6}}.
\end{aligned}$$

This completes the proof. $\square$

3.2 Global Stability of Positive Solutions

In this section we study the asymptotic behavior of positive solutions of the system (3.1).

Let $I = J = (0, +\infty)$ and consider the functions

$$f : I^3 \times J^3 \longrightarrow I, \; g : I^3 \times J^3 \longrightarrow J$$

defined by

$$\begin{aligned}
f(u_0, u_1, u_2, v_0, v_1, v_2) &= \frac{1}{1 + v_2}, \\
g(u_0, u_1, u_2, v_0, v_1, v_2) &= \frac{1}{1 + u_2}.
\end{aligned}$$

Lemma 3.2. *System* (3.1) *has a unique positive equilibrium point in $I \times J$, namely*

$$E := \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right).$$

Proof. Clearly the system

$$\overline{x} = \frac{1}{1+\overline{y}}, \ \overline{y} = \frac{1}{1+\overline{x}},$$

has a unique solution in $I \times J$ which is

$$(\overline{x}, \overline{y}) = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right).$$

The proof is complete. □

Theorem 3.3. *The equilibrium point E is locally asymptotically stable.*

Proof. The linearized system about the equilibrium point

$$\overline{E} = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right) \in I^3 \times J^3$$

is given by

$$X_{n+1} = AX_n, \ X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2})^T \tag{3.6}$$

where A is 6×6 matrix and given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-3+\sqrt{5}}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-3+\sqrt{5}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We can find the eigenvalues of the matrix A from the characteristic polynomial

$$P(\lambda) = \det(A - \lambda I_6) = \lambda^6 - \left(\frac{-3+\sqrt{5}}{2} \right)^2 = 0.$$

Now, consider the two functions defined by

$$a(\lambda) = \lambda^6, \quad b(\lambda) = \left(\frac{-3+\sqrt{5}}{2} \right)^2 < 1.$$

We have

$$|b(\lambda)| < |a(\lambda)|\,, \forall\lambda : |\lambda| = 1.$$

Thus, by Rouche's Theorem, all zeros of $P(\lambda) = a(\lambda) - b(\lambda) = 0$ lie in $|\lambda| < 1$. So, by Theorem (2.4), we get that E is locally asymptotically stable. □

Theorem 3.4. *The equilibrium point E is globally asymptotically stable.*

Proof. Let $\{x_n, y_n\}_{n\geq -k}$ be a solution of (3.1). By Theorem (3.3) we need only to prove that E is global attractor, that is

$$\lim_{n\to\infty}(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) = \overline{E},$$

or equivalently

$$\lim_{n\to\infty}(x_n, y_n) = E.$$

To do this, we prove that for $i = 1, \ldots, 6$ we have

$$\lim_{n\to+\infty} x_{6n+i} = \lim_{n\to+\infty} y_{6n+i} = \frac{-1+\sqrt{5}}{2}.$$

For $i = 1, 2, 3$, it follows from Theorem (3.1) that

$$\begin{aligned}\lim_{n\to+\infty} x_{6n+i} &= \lim_{n\to+\infty} \frac{F_{2n+1} + F_{2n}y_{i-3}}{F_{2n+2} + F_{2n+1}y_{i-3}} \\ &= \lim_{n\to+\infty} \frac{1 + \frac{F_{2n}}{F_{2n+1}}y_{i-3}}{\frac{F_{2n+2}}{F_{2n+1}} + y_{i-3}},\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\lim_{n\to+\infty} y_{6n+i} &= \lim_{n\to+\infty} \frac{F_{2n+1} + F_{2n}x_{i-3}}{F_{2n+2} + F_{2n+1}x_{i-3}} \\ &= \lim_{n\to+\infty} \frac{1 + \frac{F_{2n}}{F_{2n+1}}x_{i-3}}{\frac{F_{2n+2}}{F_{2n+1}} + x_{i-3}}.\end{aligned} \tag{3.8}$$

Using Binet's formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ n \in \mathbb{N}_0 \tag{3.9}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$, we get

$$\lim_{n\to+\infty} \frac{F_{2n}}{F_{2n+1}} = \lim_{n\to+\infty} \frac{\alpha^{2n} \times \frac{1-\left(\frac{\beta}{\alpha}\right)^{2n}}{\alpha-\beta}}{\alpha^{2n+1} \times \frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}}{\alpha-\beta}} = \frac{1}{\alpha}, \tag{3.10}$$

similarly we get

$$\lim_{n\to+\infty}\frac{F_{2n+2}}{F_{2n+1}}=\alpha. \tag{3.11}$$

Thus, from (3.7)–(3.11), we get

$$\lim_{n\to+\infty}x_{6n+i}=\frac{1+\frac{1}{\alpha}y_{i-3}}{\alpha+y_{i-3}}=\frac{1}{\alpha}=\frac{-1+\sqrt{5}}{2},$$

$$\lim_{n\to+\infty}y_{6n+i}=\frac{1+\frac{1}{\alpha}x_{i-3}}{\alpha+x_{i-3}}=\frac{1}{\alpha}=\frac{-1+\sqrt{5}}{2}.$$

By the same arguments, we get, for $i=4,5,6$:

$$\lim_{n\to+\infty}x_{6n+i}=\lim_{n\to+\infty}y_{6n+i}=\frac{-1+\sqrt{5}}{2}.$$

This completes the proof. $\square$

Remark 3.5. If $x_{i_0-3}=\overline{x}=\dfrac{-1+\sqrt{5}}{2}$ $\left(\text{respectively } y_{i_0-3}=\overline{y}=\dfrac{-1+\sqrt{5}}{2}\right)$ for some $1\le i_0\le 3$, then for $n=0,1,\ldots$,

$$y_{3n+i_0}=\frac{-1+\sqrt{5}}{2}\quad\left(\text{respectively } x_{3n+i_0}=\frac{-1+\sqrt{5}}{2}\right).$$

Using the fact that

$$\overline{x}=\overline{y}=\frac{1}{1+\overline{x}}=\frac{1}{1+\overline{y}}$$

and Theorem (3.1), we get

$$\begin{aligned}y_{3n+i_0}&=\frac{F_{2n+1}+F_{2n}\overline{x}}{F_{2n+2}+F_{2n+1}\overline{x}}=\frac{F_{2n+1}+\frac{F_{2n}}{1+\overline{x}}}{F_{2n+2}+F_{2n+1}\overline{x}}=\frac{\frac{F_{2n+1}+F_{2n}+\overline{x}F_{2n+1}}{1+\overline{x}}}{F_{2n+2}+F_{2n+1}\overline{x}}\\&=\frac{\frac{F_{2n+2}+\overline{x}F_{2n+1}}{1+\overline{x}}}{F_{2n+2}+F_{2n+1}\overline{x}}=\frac{1}{1+\overline{x}}=\overline{x},\end{aligned}$$

and

$$\begin{aligned}x_{3n+i_0}&=\frac{F_{2n+1}+F_{2n}\overline{y}}{F_{2n+2}+F_{2n+1}\overline{y}}=\frac{F_{2n+1}+\frac{F_{2n}}{1+\overline{y}}}{F_{2n+2}+F_{2n+1}\overline{y}}=\frac{\frac{F_{2n+1}+F_{2n}+\overline{y}F_{2n+1}}{1+\overline{y}}}{F_{2n+2}+F_{2n+1}\overline{y}}\\&=\frac{\frac{F_{2n+2}+\overline{y}F_{2n+1}}{1+\overline{y}}}{F_{2n+2}+F_{2n+1}\overline{y}}=\frac{1}{1+\overline{y}}=\overline{y}.\end{aligned}$$

Similarly, if $x_{i_0-6} = \overline{x} = \dfrac{-1+\sqrt{5}}{2}$ $\left(\text{respectively } y_{i_0-6} = \overline{y} = \dfrac{-1+\sqrt{5}}{2}\right)$ for some $4 \le i_0 \le 6$, then for $n = 0, 1, \ldots,$

$$y_{3n+i_0} = \frac{-1+\sqrt{5}}{2} \left(\text{respectively } x_{3n+i_0} = \frac{-1+\sqrt{5}}{2}\right).$$

Example 3.6. We illustrate results of this section, we consider the following numerical example. Assume $x_{-2} = 2$, $x_{-1} = 1.5$, $x_0 = 0.05$, $y_{-2} = 1$, $y_{-1} = 0.5$ and $y_0 = 35$ (see Fig. (3.1)). In figure (3.1) we see that the sequences $(x_n)_{n=1}^{\infty}$ and $(x_n)_{n=1}^{\infty}$ of solution of equations (3.1) with given initial conditions converges to the equilibrium point $\left(\dfrac{-1+\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$.

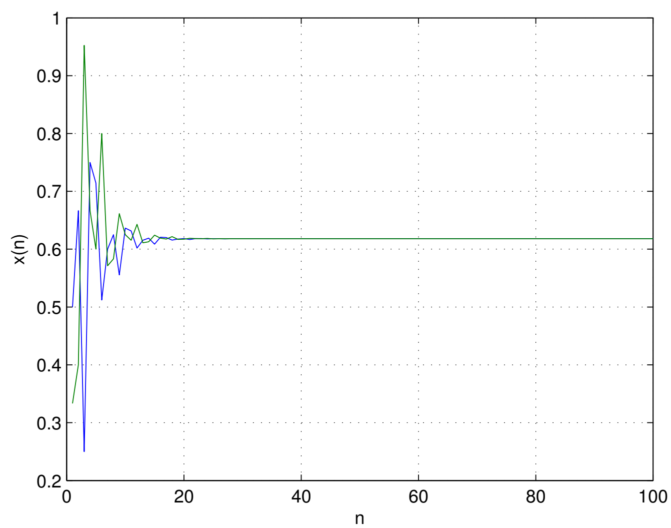

Figure 3.1: The sequences $(x_n)_{n=1}^{\infty}$ (blue) and $(y_n)_{n=1}^{\infty}$ (green) of solution of the equation (3.1) with initial conditions $x_{-2} = 2$, $x_{-1} = 1.5$, $x_0 = 0.05$, $y_{-2} = 1$, $y_{-1} = 0.5$ and $y_0 = 35$.

4 Conclusion

In this study, we mainly obtained the relationship between the solutions of system of difference equations (3.1) and Fibonacci numbers. We also presented that the solutions of system in (3.1) actually converge to equilibrium point. The results in this paper can be extended to the following system of difference equations

$$x_{n+1} = \frac{1}{\pm 1 \pm y_{n-k}}, \; y_{n+1} = \frac{1}{\pm 1 \pm x_{n-k}}, \; n, \; k \in \mathbb{N}_0,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0$ are real numbers.

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