International Journal of Difference Equations
ISSN 0973-6069, Volume 11, Number 1, pp. 19–36 (2016)
http://campus.mst.edu/ijde

Remarks on Fuzzy Differential Systems

El Hassan Eljaoui
Said Melliani and **Lalla S. Chadli**
University of Sultan Moulay Slimane
Department of Mathematics
Laboratory of Applied Mathematics & Scientific Calculus
Beni Mellal, P.O. Box 523, Morocco
eljaouihass@gmail.com
melliani@fstbm.ac.ma and chadli@fstbm.ac.ma

Abstract

The fuzzification of a crisp ordinary differential equation (ODE) or a classic ordinary differential system (ODS) can be done in many different ways and forms. This is due to the usage of the Hukuhara or the Minkowski differences instead of the standard (crisp) difference. In this paper, we investigate the fuzzy possible versions of a crisp differential system. Then, we establish a numerical comparison between the obtained solutions regarding their interval of definition.

AMS Subject Classifications: 39A10.
Keywords: Fuzzy differential systems, Hukuhara difference, Minkowski difference, generalized Hukuhara differentiability, fuzzy Laplace transform method.

1 Introduction

In the majority of the recent works dealing with the subject of fuzzy differential equations (FDEs) or fuzzy differential systems (FDSs), the researchers adopted the Hukuhara difference and the (strongly generalized) Hukuhara differentiability. So, a question arises, have they done the best choice? Are there any other alternatives?

According to the authors in [3], it is easy to check that generally, the following fuzzy differential equations are not equivalent:

$$\begin{cases} y'(x) = a(x)y(x) + b(x) \\ y(x_0) = y_0, \end{cases} \qquad \begin{cases} y'(x) - a(x)y(x) = b(x) \\ y(x_0) = y_0 \end{cases}$$

Received November 9, 2015; Accepted March 22, 2016
Communicated by Ewa Schmeidel

and

$$\begin{cases} y'(x) - b(x) = a(x)y(x) \\ y(x_0) = y_0. \end{cases}$$

So, depending on how the three (equivalent) crisp problems are written and then how they are fuzzified, we get three different results. Any of them can be chosen in modeling the real behavior of a dynamical system under uncertainty (see [3]). The latter opinion is shared by T. G. Bhaskar et al in [4], and precisely the relation between the solution's stability and the fuzzification of the derivative is studied. Also, the authors of [4] declared that the nondecreasing of the solution's length of FDEs, as the time increases, is due to the assumption of the existence of the Hukuhara difference.

In the same context, we will investigate in this work, the impact of the fuzzification using the Hukuhara or the Minkowski differences and its consequences on the solutions of FDSs via the following crisp example:

$$\begin{cases} x'(t) = 3x(t) - y(t) + 5\cos(t) \\ y'(t) = x(t) + y(t) + 10\sin(t) \\ x(0) = y(0) = 2. \end{cases} \tag{1.1}$$

The unknown $(x(t), y(t))$ can be interpreted (for $t \geq 0$) as the dynamical evolution of the state of some "system", the function $u(t) = (\cos(t), \sin(t))$ as a sinusoidal control and the trajectory of $(x(t), y(t))$ can be regarded as the response of this system.

To proceed to a fuzzification of (1.1), one can hesitate between the use of the Minkowski or the Hukuhara differences. It is well known that the first type of difference is not invertible, even if it always exists. But, the second one, which is invertible, does not always exist (see [10]).

We use the fuzzy Laplace transform to study the fuzzy versions of the crisp systems (1.1), under the generalized Hukuhara differentiability assumption, and using five approaches:

- The (pure) Hukuhara difference approach, which leads to the FDS

$$\begin{cases} x'(t) + y(t) = 3x(t) + 5\cos(t) \\ y'(t) = x(t) + y(t) + 10\sin(t) \\ x(0,\alpha) = y(0,\alpha) = (\alpha + 1, 3 - \alpha). \end{cases} \tag{1.2}$$

- The four (graduate) Minkowski difference approaches, for which the degree depends on the number of the Minkowski differences used.

Then, the following natural questions, concerning these five fuzzy differential systems, are required, are all the fuzzy versions of this ODS equivalent? Do they have the same solutions? Are these solutions defined on the same interval? Do they have the same behavior as that of the ODS from which they are generated? Moreover, which of the forgoing fuzzy approaches better reflects the classic case? Which of them gives the maximal fuzzy solution and under what conditions?

The aim of this work is to provide answers to each of the previous questions, using the fuzzy Laplace transform (see [1, 6, 7]), which remains one of the most famous and effective methods for solving fuzzy differential equations and systems.

The present paper can be viewed as a continuation of some of our recent works (see [7] and [8]) in the domain of fuzzy differential equations and systems.

The remainder of this paper is organized as follows. Section 2 is reserved for some preliminaries. Section 3 deals with the Hukuhara approach. In Section 4, the four mixed Hukuhara-Minkowski approaches are studied. In the last section, we compare the five approaches, and then we present a concluding remark.

2 Preliminaries

By $P_K(\mathbb{R})$ we denote the family of all nonempty compact convex subsets of $\mathbb{R}$ and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual. Denote

$$E = \Big\{ u : \mathbb{R} \longrightarrow [0,1] \mid u \text{ satisfies (i) – (iv) below} \Big\}$$

where

(i) u is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semi-continuous,

(iv) The closure of the support supp$u = \{x \in \mathbb{R} | u(x) > 0\}$ of u is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$. Then, from (i)–(iv), it follows that the α-level set $[u]^\alpha \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$. It is well known that

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k\,[u]^\alpha .$$

Let $D : E \times E \longrightarrow [0, \infty)$ be a function which is defined by the equation

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d\Big([u]^\alpha, [v]^\alpha\Big),$$

where d is the Hausdorff metric defined in $P_K(\mathbb{R})$. Then, it is easy to see that (E, D) is a complete metric space (see [9]).

Definition 2.1. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(\alpha), \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$, which satisfy the following requirements:

(1) $\underline{u}(\alpha)$ is bounded nondecreasing left continuous in $(0, 1]$, and right continuous at 0;

(2) $\overline{u}(\alpha)$ is bounded nonincreasing left continuous in $(0, 1]$, and right continuous at 0;

(3) $\underline{u}(\alpha) \leq \overline{u}(\alpha)$ for all $0 \leq \alpha \leq 1$.

The length of $u = (\underline{u}, \overline{u})$ is level-wise defined by the formula $\text{len}(u(\alpha)) = \overline{u}(\alpha) - \underline{u}(\alpha)$. For $u, v \in E$, if there exists $w \in E$ such that $u = v + w$, then w is the Hukuhara difference of u and v denoted by $u \ominus v$.

Theorem 2.2 (See [11]). *Let $f(x) = (\underline{f}(x, r), \overline{f}(x, r))$ be a fuzzy valued function on $[a, \infty[$. For any fixed $r \in [0, 1]$, assume $\underline{f}(x, r), \overline{f}(x, r)$ are Riemann integrable on $[a, b]$ for every $b \geq a$, and there are two positive constants $\underline{M}(r)$ and $\overline{M}(r)$ such that $\int_a^b |\underline{f}(x, r)|dx \leq \underline{M}(r)$ and $\int_a^b |\overline{f}(x, r)|dx \leq \overline{M}(r)$ for every $b \geq a$. Then $\int_a^\infty f(x)dx$ is a fuzzy number. Furthermore, we have*

$$\int_a^\infty f(x)dx = \left(\int_a^\infty \underline{f}(x, r)dx, \int_a^\infty \overline{f}(x, r)dx \right).$$

Definition 2.3. We say that a mapping $f : (a, b) \to E$ is strongly generalized differentiable at $x_0 \in (a, b)$ if there exists an element $f'(x_0) \in E$ such that

(i) for all $h > 0$ very small, $f(x_0 + h) \ominus f(x_0)$; $f(x_0) \ominus f(x_0 - h)$ exist and

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

or (ii) for all $h > 0$ very small, $f(x_0) \ominus f(x_0 + h)$; $f(x_0 - h) \ominus f(x_0)$ exist and

$$\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0)$$

or (iii) for all $h > 0$ very small, $f(x_0 + h) \ominus f(x_0)$; $f(x_0 - h) \ominus f(x_0)$ exist and

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0)$$

or (iv) for all $h > 0$ very small, $f(x_0) \ominus f(x_0 + h)$; $f(x_0) \ominus f(x_0 - h)$ exist and

$$\lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0).$$

All the limits are taken in the metric space $(E\,,\,D)$, and at the end points of (a, b), we consider only one-sided derivatives.

The following theorem (see [2]) allows us to consider case (i) or (ii) of Definition 2.3 almost everywhere in the domain of the functions under discussion.

Theorem 2.4. *Let $f : (a, b) \to E$ be strongly generalized differentiable on each point $x \in (a, b)$ in the sense (iii) or (iv). Then $f'(x) \in \mathbb{R}$ for all $x \in (a, b)$.*

Theorem 2.5 (See [5])**.** *Let $f : \mathbb{R} \to E$ such that $f(t) = (\underline{f}(t, r), \overline{f}(t, r))$. Then*

(1) If f is (i)-differentiable, then $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable and

$$f'(t) = (\underline{f}'(t, r), \overline{f}'(t, r)).$$

Therefore $\mathrm{len}(f'(t)) = \overline{f}'(t, r) - \underline{f}'(t, r)$.

(2) If f is (ii)-differentiable, then $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable and

$$f'(t) = (\overline{f}'(t, r), \underline{f}'(t, r)).$$

Hence $\mathrm{len}(f'(t)) = \underline{f}'(t, r) - \overline{f}'(t, r)$.

Definition 2.6 (See [1])**.** Let f be a continuous and fuzzy integrable function. Then the fuzzy Laplace transform of f is defined by

$$\mathbf{L}[f(x)] = \int_0^\infty e^{-px} f(x)dx, \quad p > 0.$$

Denote by $\mathcal{L}(g(x))$ the classical Laplace transform of a crisp function g. Then

$$\mathbf{L}[f(x)] = \left(\mathcal{L}(\underline{f}(x, r)), \mathcal{L}(\overline{f}(x, r))\right).$$

Theorem 2.7 (See [1])**.** *Let f' be an integrable fuzzy-valued function and f be the primitive of f' on $[0, \infty[$.*

(i) If f is (i)-differentiable, then $\mathbf{L}[f'(x)] = p\mathbf{L}[f(x)] \ominus f(0)$.

(ii) If f is (ii)-differentiable, then $\mathbf{L}[f'(x)] = (-f(0)) \ominus (-p)\mathbf{L}[f(x)]$.

3 Hukuhara Difference Approach

Now we consider the crisp differential system

$$\begin{cases} x'(t) = 3x(t) - y(t) + 5\cos(t) \\ y'(t) = x(t) + y(t) + 10\sin(t) \\ x(0) = y(0) = 2 \end{cases} \tag{3.1}$$

which we will fuzzify in five different ways using the Hukuhara and the Minkowski differences together or separately. Then we will solve the obtained fuzzy versions by the fuzzy Laplace transform method for FDSs (for more detail about this algorithm, see our recent work [6]) .

Using the crisp Laplace transform algorithm, we deduce that the solution of the classic problem (3.1) is defined all over $\mathbb{R}$ by

$$\begin{cases} x_c(t) = 5e^{2t} - 3\cos(t) - \sin(t) \\ y_c(t) = 5e^{2t} - 3\cos(t) - 6\sin(t). \end{cases} \tag{3.2}$$

The first way to fuzzify (3.1) is by using the Hukuhara approach as follows:

$$\begin{cases} x'(t) + y(t) = 3x(t) + 5\cos(t) \\ y'(t) = x(t) + y(t) + 10\sin(t) \\ x(0,\alpha) = y(0,\alpha) = (\alpha+1, 3-\alpha). \end{cases} \tag{3.3}$$

- Case I: If x and y are (i)-differentiable, then utilizing the Laplace transform method (see [6]), we obtain

$$\begin{cases} \mathcal{L}[\underline{x}(t,\alpha)] = \dfrac{(\alpha+1)p^2 + 5p + \alpha + 6}{p^3 - 2p^2 + p - 2} \\ \mathcal{L}[\overline{x}(t,\alpha)] = \dfrac{(3-\alpha)p^2 + 5p + 8 - \alpha}{p^3 - 2p^2 + p - 2} \\ \mathcal{L}[\underline{y}(t,\alpha)] = \dfrac{(\alpha+1)p^2 + \alpha + 16}{p^3 - 2p^2 + p - 2} \\ \mathcal{L}[\overline{y}(t,\alpha)] = \dfrac{(3-\alpha)p^2 + 18 - \alpha}{p^3 - 2p^2 + p - 2}. \end{cases} \tag{3.4}$$

By the inverse Laplace transform, we deduce

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{2t} \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{2t} \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{2t} \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{2t}. \end{cases} \tag{3.5}$$

The lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ are respectively given by

$$\begin{cases} \text{len}(x(t,\alpha)) = 2(1-\alpha)e^{2t} \geq 0; \quad t \in \mathbb{R} \\ \text{len}(y(t,\alpha)) = 2(1-\alpha)e^{2t} \geq 0; \quad t \in \mathbb{R} \\ \text{len}(x'(t,\alpha)) = 4(1-\alpha)e^{2t} \geq 0; \quad t \in \mathbb{R} \\ \text{len}(y'(t,\alpha)) = 4(1-\alpha)e^{2t} \geq 0; \quad t \in \mathbb{R}. \end{cases} \tag{3.6}$$

So, this solution is valid all over $\mathbb{R}$.

- Case II: If x is (i)-differentiable and y is (ii)-differentiable, then similarly we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^t\left(\cosh(\sqrt{5}t) + \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^t\left(\cosh(\sqrt{5}t) + \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^t\left(\cosh(\sqrt{5}t) - \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^t\left(\cosh(\sqrt{5}t) - \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right). \end{cases} \tag{3.7}$$

The lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ are respectively given by

$$\begin{cases} \text{len}(x(t,\alpha)) = 2(1-\alpha)e^t\left(\cosh(\sqrt{5}t) + \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \text{len}(y(t,\alpha)) = 2(1-\alpha)e^t\left(\cosh(\sqrt{5}t) - \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \text{len}(x'(t,\alpha)) = 4(1-\alpha)e^t\left(\cosh(\sqrt{5}t) + \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \text{len}(y'(t,\alpha)) = 2(1-\alpha)e^t\left(\cosh(\sqrt{5}t) - \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right). \end{cases} \tag{3.8}$$

One can verify that this solution is valid only over $[-\lambda,\lambda]$, where $\lambda \in]0.430; 0.431[$ and precisely

$$\lambda = \frac{\sqrt{5}}{10}\ln\left(\frac{3\sqrt{5}+5}{3\sqrt{5}-5}\right).$$

- Case III: If x is (ii)-differentiable and y is (i)-differentiable, then we have

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^t\left(\cosh(\sqrt{5}t) - \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^t\left(\cosh(\sqrt{5}t) - \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^t\left(\cosh(\sqrt{5}t) + \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^t\left(\cosh(\sqrt{5}t) + \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right). \end{cases}$$

The study of the lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ respectively shows that this solution is also valid only over $[-\lambda,\lambda]$, where $\lambda = \frac{\sqrt{5}}{10}\ln\left(\frac{3\sqrt{5}+5}{3\sqrt{5}-5}\right)$.

- Case IV: If x and y are (ii)-differentiable, then analogously we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-2t} \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-2t} \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{-2t} \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{-2t}. \end{cases}$$

The lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ are respectively given by

$$\begin{cases} \operatorname{len}(x(t,\alpha)) = 2(1-\alpha)e^{-2t} \geq 0; \quad t \in \mathbb{R} \\ \operatorname{len}(y(t,\alpha)) = 2(1-\alpha)e^{-2t} \geq 0; \quad t \in \mathbb{R} \\ \operatorname{len}(x'(t,\alpha)) = 4(1-\alpha)e^{-2t} \geq 0; \quad t \in \mathbb{R} \\ \operatorname{len}(y'(t,\alpha)) = 2(1-\alpha)e^{-2t} \geq 0; \quad t \in \mathbb{R}. \end{cases}$$

So, this solution is valid all over $\mathbb{R}$.

4 Minkowski Approaches

4.1 Minkowski Approach of Degree 1

The second way to fuzzify (3.1) is by using Hukuhara and Minkowski differences simultaneously as follows:

$$\begin{cases} x'(t) = 3x(t) + (-1)y(t) + 5\cos(t) \\ y'(t) = x(t) + y(t) + 10\sin(t) \\ x(0,\alpha) = y(0,\alpha) = (\alpha+1, 3-\alpha). \end{cases} \tag{4.1}$$

- Case I: If x and y are (i)-differentiable, then utilizing the Laplace transform algorithm, we obtain

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{2t}\left(\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{2t}\left(\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{2t}\cosh(\sqrt{2}t) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{2t}\cosh(\sqrt{2}t). \end{cases}$$

The lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ are respectively given by

$$\begin{cases} \operatorname{len}(x(t,\alpha)) = 2(1-\alpha)e^{2t}\left(\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)\right) \\ \operatorname{len}(y(t,\alpha)) = 2(1-\alpha)e^{2t}\cosh(\sqrt{2}t) \\ \operatorname{len}(x'(t,\alpha)) = 4(1-\alpha)e^{2t}\left(4\cosh(\sqrt{2}t) + 3\sqrt{2}\sinh(\sqrt{2}t)\right) \\ \operatorname{len}(y'(t,\alpha)) = 4(1-\alpha)e^{2t}\left(2\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)\right). \end{cases}$$

One can verify that this solution is valid only over the interval $[-\beta, \infty[$, where $\beta \in]0.623; 0.624[$ and precisely

$$\beta = \frac{\sqrt{2}}{4} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right).$$

- Case II: If x is (i)-differentiable and y is (ii)-differentiable, then similarly we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^t \left(\cosh(\sqrt{3}t) + \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^t \left(\cosh(\sqrt{3}t) + \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^t \left(\cosh(\sqrt{3}t) - \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^t \left(\cosh(\sqrt{5}t) - \sqrt{3}\sinh(\sqrt{3}t)\right). \end{cases}$$

The lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ are respectively given by

$$\begin{cases} \operatorname{len}(x(t,\alpha)) = 2(1-\alpha)e^t \left(\cosh(\sqrt{3}t) + \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \operatorname{len}(y(t,\alpha)) = 2(1-\alpha)e^t \left(\cosh(\sqrt{3}t) - \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \operatorname{len}(x'(t,\alpha)) = 4(1-\alpha)e^t \left(2\cosh(\sqrt{3}t) + \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \operatorname{len}(y'(t,\alpha)) = 4(1-\alpha)e^t \cosh(\sqrt{3}t). \end{cases}$$

One can verify that this solution is valid only over $[-\gamma, \gamma]$, where $\gamma \in]0.380; 0.381[$ and precisely

$$\gamma = \frac{\sqrt{3}}{6} \ln \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right).$$

- Case III: If x is (ii)-differentiable and y is (i)-differentiable, then we have

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-t} \left(\cosh(\sqrt{3}t) - \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-t} \left(\cosh(\sqrt{3}t) - \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{-t} \left(\cosh(\sqrt{3}t) + \sqrt{3}\sinh(\sqrt{3}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{-t} \left(\cosh(\sqrt{3}t) + \sqrt{3}\sinh(\sqrt{3}t)\right). \end{cases}$$

The study of the lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ respectively shows that this solution is also valid only over $[-\gamma, \gamma]$, where $\gamma = \frac{\sqrt{3}}{6} \ln \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right)$.

- Case IV: If x and y are (ii)-differentiable, then analogously we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-2t}\left(\cosh(\sqrt{2}t) - \sqrt{2}\sinh(\sqrt{2}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-2t}\left(\cosh(\sqrt{2}t) - \sqrt{2}\sinh(\sqrt{2}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{-2t}\cosh(\sqrt{2}t) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{-2t}\cosh(\sqrt{2}t). \end{cases}$$

The study of the lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ respectively shows that this solution is valid only on the interval $[-\infty, \beta[$, where

$$\beta = \frac{\sqrt{2}}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

4.2 Minkowski Approach of Degree 2

The third manner to fuzzify (3.1) is by using two Minkowski differences as follows:

$$\begin{cases} x'(t) = 3x(t) + (-1)y(t) + 5\cos(t) \\ y'(t) + (-1)y(t) = x(t) + 10\sin(t) \\ x(0,\alpha) = y(0,\alpha) = (\alpha+1, 3-\alpha). \end{cases} \tag{4.2}$$

- Case I: If x and y are (i)-differentiable, then by the Laplace transform, we obtain

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{t}\left(\cosh(\sqrt{5}t) + \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{t}\left(\cosh(\sqrt{5}t) + \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{t}\left(\cosh(\sqrt{5}t) - \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{t}\left(\cosh(\sqrt{5}t) - \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right). \end{cases}$$

The lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ are respectively given by

$$\begin{cases} \operatorname{len}(x(t,\alpha)) = 2(1-\alpha)e^{t}\left(\cosh(\sqrt{5}t) + \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \operatorname{len}(y(t,\alpha)) = 2(1-\alpha)e^{t}\left(\cosh(\sqrt{5}t) - \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \operatorname{len}(x'(t,\alpha)) = 8(1-\alpha)e^{t}\left(\cosh(\sqrt{5}t) + \frac{2\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \operatorname{len}(y'(t,\alpha)) = \frac{8\sqrt{5}}{5}(1-\alpha)e^{t}\sinh(\sqrt{5}t). \end{cases}$$

One can verify that this solution is valid only over the interval $[0, \infty[$.

- Case II: If x is (i)-differentiable and y is (ii)-differentiable, then similarly we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{2t}(2t+1) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{2t}(2t+1) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{2t}(1-2t) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{2t}(1-2t). \end{cases}$$

The lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ are respectively given by

$$\begin{cases} \text{len}(x(t,\alpha)) = (1-\alpha)e^{2t}(2t+1) \\ \text{len}(y(t,\alpha)) = 2(1-\alpha)e^{2t}(1-2t) \\ \text{len}(x'(t,\alpha)) = 8(1-\alpha)e^{2t}(t+1) \\ \text{len}(y'(t,\alpha)) = 8(1-\alpha)te^{2t}. \end{cases}$$

One can verify that this solution is valid only over $\left[0, \frac{1}{2}\right]$.

- Case III: If x is (ii)-differentiable and y is (i)-differentiable, then we have

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-2t}(1-2t) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-2t}(1-2t) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{-2t}(2t+1) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{-2t}(2t+1). \end{cases}$$

The study of the lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ respectively shows that this solution is valid only over $\left[\frac{-1}{2}, 0\right]$.

- Case IV: If x and y are (ii)-differentiable, then analogously we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-t}\left(\cosh(\sqrt{5}t) - \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-t}\left(\cosh(\sqrt{5}t) - \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)(1+2t)e^{-t}\left(\cosh(\sqrt{5}t) + \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)(1+2t)e^{-t}\left(\cosh(\sqrt{5}t) + \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right). \end{cases}$$

The study of the lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ respectively shows that this solution is valid only on the interval $]-\infty, 0]$.

4.3 Minkowski Approach of Degree 3

The fourth way to fuzzify (3.1) is by using three Minkowski differences as follows:

$$\begin{cases} x'(t) + (-3)x(t) = (-1)y(t) + 5\cos(t) \\ y'(t) + (-1)x(t) = y(t) + 10\sin(t) \\ x(0,\alpha) = y(0,\alpha) = (\alpha+1, 3-\alpha). \end{cases} \quad (4.3)$$

- Case I: If x and y are (i)-differentiable, then using the Laplace transform, we have

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-t}\left(\cosh(\sqrt{3}t) - \frac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-t}\left(\cosh(\sqrt{3}t) - \frac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{-t}\left(\cosh(\sqrt{3}t) + \frac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{-t}\left(\cosh(\sqrt{3}t) + \frac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right). \end{cases}$$

The lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ are respectively given by

$$\begin{cases} \text{len}(x(t,\alpha)) = 2(1-\alpha)e^{-t}\left(\cosh(\sqrt{3}t) - \frac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \text{len}(y(t,\alpha)) = 2(1-\alpha)e^{-t}\left(\cosh(\sqrt{3}t) + \frac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \text{len}(x'(t,\alpha)) = 4(1-\alpha)e^{-t}\left(-\cosh(\sqrt{3}t) + \frac{2\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \text{len}(y'(t,\alpha)) = \frac{4\sqrt{3}}{3}(1-\alpha)e^{-t}\sinh(\sqrt{3}t). \end{cases}$$

One can verify that this solution is valid over the interval $[\delta, \infty[$, where $\delta \in]0.760; 0.761[$ and precisely $\delta = \frac{\sqrt{3}}{6}\ln\left(\frac{2\sqrt{3}+3}{2\sqrt{3}-3}\right)$.

- Case II: If x is (i)-differentiable and y is (ii)-differentiable, then similarly we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-2t}\cosh(\sqrt{2}t) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-2t}\cosh(\sqrt{2}t) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{-2t}\left(\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{-2t}\left(\cosh(\sqrt{2}t) + \sqrt{2}\sinh(\sqrt{2}t)\right). \end{cases}$$

One can verify that no solution exists in this case, since x is not (i)-differentiable.

- Case III: If x is (ii)-differentiable and y is (i)-differentiable, then we have

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{2t}\cosh(\sqrt{2}t) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{2t}\cosh(\sqrt{2}t) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{2t}\left(\cosh(\sqrt{2}t) - \sqrt{2}\sinh(\sqrt{2}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{2t}\left(\cosh(\sqrt{2}t) - \sqrt{2}\sinh(\sqrt{2}t)\right). \end{cases}$$

One can verify that no solution exists in this case, since x is not (ii)-differentiable.

- Case IV: If x and y are (ii)-differentiable, then analogously we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{t}\left(\cosh(\sqrt{3}t) + \dfrac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{t}\left(\cosh(\sqrt{3}t) + \dfrac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{t}\left(\cosh(\sqrt{3}t) - \dfrac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{t}\left(\cosh(\sqrt{3}t) - \dfrac{\sqrt{3}}{3}\sinh(\sqrt{3}t)\right). \end{cases}$$

The study of the lengths of $x(t,\alpha), y(t,\alpha), x'(t,\alpha), y'(t,\alpha)$ respectively shows that this solution is valid only on the interval $]-\infty,-\delta]$, where

$$\delta = \frac{\sqrt{3}}{6}\ln\left(\frac{2\sqrt{3}+3}{2\sqrt{3}-3}\right).$$

4.4 Minkowski Approach of Degree 4

The fifth possible way to fuzzify (3.1) is by using four Minkowski differences:

$$\begin{cases} x'(t) + (-3)x(t) = (-1)y(t) + 5\cos(t) \\ y'(t) + (-1)x(t) + (-1)y(t) = 10\sin(t) \\ x(0,\alpha) = y(0,\alpha) = (\alpha+1, 3-\alpha). \end{cases} \tag{4.4}$$

- Case I: If x and y are (i)-differentiable, then by the Laplace transform, we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-2t} \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-2t} \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{-2t} \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{-2t}. \end{cases}$$

One can verify that no solution exists in this case, since x and y are not (i)-differentiable.

- Case II: If x is (i)-differentiable and y is (ii)-differentiable, then

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{-t}\left(\cosh(\sqrt{5}t) - \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{-t}\left(\cosh(\sqrt{5}t) - \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{-t}\left(\cosh(\sqrt{5}t) + \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{-t}\left(\cosh(\sqrt{5}t) + \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right). \end{cases}$$

  One can verify that no solution exists in this case, since y is not (ii)-differentiable.

- Case III: If x is (ii)-differentiable and y is (i)-differentiable, then

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{t}\left(\cosh(\sqrt{5}t) + \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{t}\left(\cosh(\sqrt{5}t) + \frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{t}\left(\cosh(\sqrt{5}t) - \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right) \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{t}\left(\cosh(\sqrt{5}t) - \frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right). \end{cases}$$

  One can verify that no solution exists in this case, since y is not (i)-differentiable.

- Case IV: If x and y are (ii)-differentiable, then analogously we get

$$\begin{cases} \underline{x}(t,\alpha) = x_c(t) + (\alpha-1)e^{2t} \\ \overline{x}(t,\alpha) = x_c(t) + (1-\alpha)e^{2t} \\ \underline{y}(t,\alpha) = y_c(t) + (\alpha-1)e^{2t} \\ \overline{y}(t,\alpha) = y_c(t) + (1-\alpha)e^{2t}. \end{cases}$$

  One can verify that no solution exists in this case, since x and y are not (ii)-differentiable.

5 Conclusion

5.1 Comparison of the Five Approaches

Table 5.1 gives the definition's interval of the solution obtained using Hukuhara (H) and Minkowski (M) approaches. where $\beta, \gamma, \lambda, \delta$ are given in the previous subsections.

	(H)	(M) 1	(M) 2	(M) 3	(M) 4
Case I	$\mathbb{R}$	$[-\beta, \infty[$	$[0, \infty[$	$[\delta, \infty[$	$\emptyset$
Case II	$[-\lambda, \lambda]$	$[-\gamma, \gamma]$	$\left[0, \frac{1}{2}\right]$	$\emptyset$	$\emptyset$
Case III	$[-\lambda, \lambda]$	$[-\gamma, \gamma]$	$\left[\frac{-1}{2}, 0\right]$	$\emptyset$	$\emptyset$
Case IV	$\mathbb{R}$	$]-\infty, \beta]$	$]-\infty, 0]$	$]-\infty, -\delta]$	$\emptyset$

Table 5.1: Solution's interval by the five approaches.

Remark 5.1. Table 5.1 shows us the following.

(i) For the Hukuhara and the Minkowski 1–3 approaches and in both cases I and IV, the intervals of the solution are all infinite, but in cases II and III these intervals are finite. In each case, the solution's interval under the Hukuhara difference is larger than that one under the Minkowski differences. For each approach, the solution's intervals (consequently the solution) is maximal in both cases I and IV.

(ii) For each approach, the solution's intervals in cases I and IV (respectively II and III) are symmetric with respect to zero.

(iii) In each case, the length of the solution'intervals is decreasing as the number of the Minkowski differences used increases. Moreover, this interval becomes empty (then no solution exists) when we utilize four Minkowski differences in all cases or three Minkowski differences in cases II and III.

(iv) In the Hukuhara approach and under the condition of (i)-differentiability (respectively (ii)-differentiability) of the both unknowns x and y, the solution is defined on $\mathbb{R}$ as in the crisp problem.

5.2 Second Minkowski Approach of Degree 2'

There is another alternative for the Minkowski approach of degree 2 called of degree 2':

$$\begin{cases} x'(t) = 3x(t) + (-1)y(t) + 5\cos(t) \\ y'(t) + (-1)x(t) = y(t) + 10\sin(t) \\ x(0, \alpha) = y(0, \alpha) = (\alpha + 1, 3 - \alpha) \end{cases} \tag{5.1}$$

in which we use $y' + (-1)x$ as first member of the second equation, instead of $y' + (-1)y$. Its solution can be written as follows:

$$\begin{cases} \underline{x}(t, \alpha) = x_c(t) + (\alpha - 1)x_F(t, \alpha) \\ \overline{x}(t, \alpha) = x_c(t) + (1 - \alpha)x_F(t, \alpha) \\ \underline{y}(t, \alpha) = y_c(t) + (\alpha - 1)y_F(t, \alpha) \\ \overline{y}(t, \alpha) = y_c(t) + (1 - \alpha)y_F(t, \alpha), \end{cases}$$

where (x_c, y_c) is the crisp solution given in (3.2), "the fuzzy parts" $x_F(t,\alpha), y_F(t,\alpha)$ and the solution's interval I are described in Table 5.2. Notice that in the Minkowski approach of degree 2 the solution's interval is larger than that corresponding to the Minkowski approach of degree 2'.

	Case I	Case II
$x_F(t,\alpha)$	$(1+2t)e^{2t}$	$e^t\left(\cosh(\sqrt{5}t)+\frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right)$
$y_F(t,\alpha)$	$(1-2t)e^{2t}$	$e^t\left(\cosh(\sqrt{5}t)-\frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right)$
Interval I	$\left[\frac{-1}{2},0\right]$	$[-\lambda,0]$

	Case III	Case IV
$x_F(t,\alpha)$	$e^{-t}\left(\cosh(\sqrt{5}t)-\frac{3\sqrt{5}}{5}\sinh(\sqrt{5}t)\right)$	$(1-2t)e^{-2t}$
$y_F(t,\alpha)$	$e^{-t}\left(\cosh(\sqrt{5}t)+\frac{\sqrt{5}}{5}\sinh(\sqrt{5}t)\right)$	$(1+2t)e^{-2t}$
Interval I	$[0,\lambda]$	$\left[0,\frac{1}{2}\right]$

Table 5.2: Solution's fuzzy parts and interval by Minkowski 2' approach.

5.3 Concluding Remark

The solution of a fuzzy differential system, under the (generalized) Hukuhara differentiability, is maximal when we adopt the Hukuhara difference approach. And the solution's interval become smaller if we introduce more Minkowski differences. This is naturally due to the fact that the (generalized) Hukuhara differentiability's definition is based on the Hukuhara difference and not on the Minkowski one.

For each approach, we get the maximal solution by assuming that the unknowns $x(t)$ and $y(t)$ are fuzzy differentiable with the same type of differentiability. Precisely, in cases I and IV for the Hukuhara approach, and in case I (respectively case IV) for the Minkowski approaches, if we suppose that the variable $t \geq 0$ (respectively $t \leq 0$).

So, the perfect and the natural way to "fuzzify" the differential system (1.1) and many other crisp similar systems is by using the Hukuhara difference, under the assumption that the mappings $x(t)$ and $y(t)$ must be both (i)-differentiable or both (ii)-differentiable.

On the other hand, all our remarks and comments on the definition's domain of the solution of FDSs are done through a particular crisp example, because it is uneasy to compute these solutions in the general case, the determination of their domains is more difficult and their comparison is arduous. But even if our study is based on a very particular crisp system, it seems that similar analysis and conclusions can be made for many other analogous crisp systems.

The previous analysis of the different eventual fuzzy versions of the given original ODS, exhibits a diversity of solutions of FDSs in comparison with the classical problem from which they are generated.

Moreover, we notice that the exigency of the existence of the Hukuhara differences as main assumption for defining the strongly generalized derivative is the principal cause of the decreasing of the fuzzy solution's diameter, according to the number of the Minkowski differences utilized in the formulation of the corresponding FDS.

Besides that, since FDS is generated from ODS, the solution of the latter (ODS) is included in the solutions of each FDS plus undesirable elements that result of the fuzzification strategy.

Indeed, taking $\alpha = 1$ in a fuzzy solution $(x(t, \alpha), y(t, \alpha))$ of a given FDS yields the classical solution of the corresponding crisp ODS and avoids its undesirable part.

Furthermore, it seems that the study of the impact of the fuzzification's choices on the solutions of FDS is much fruitful than what is expected and needs deeper exploration.

For future research, one can use the generalized Hukuhara difference (see [10]) to solve fuzzy differential systems in the generic case.

Acknowledgements

The authors are grateful to the main editor and anonymous referees for their constructive comments and valuable suggestions, which has greatly improved this paper.

References

[1] T. Allahviranloo and M. B. Ahmadi. Fuzzy Laplace transforms. *Soft Computing*, 14:235–243, 2010.

[2] Barnabas Bede and Sorin G Gal. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets and Systems*, 151(3):581–599, 2005.

[3] Barnabas Bede, Imre J Rudas, and Attila L Bencsik. First order linear fuzzy differential equations under generalized differentiability. *Inform. Sci.*, 177(7):1648–1662, 2007.

[4] T.G. Bhaskar, V. Lakshmikantham, and V. Devi. Revisiting fuzzy differential equations. *Nonlinear Analysis*, 58:351–358, 2004.

[5] Y. Chalco-Cano and H. Roman-Flores. On new solutions of fuzzy differential equations. *Chaos Solitons Fractals*, 38(1):112–119, 2008.

[6] E. Eljaoui, S. Melliani, and L.S. Chadli. Solving a system of tow fuzzy linear differential equations by fuzzy laplace transform method. *To appear*.

[7] E. Eljaoui, S. Melliani, and L.S. Chadli. Solving second-order fuzzy differential equations by the fuzzy Laplace transform method. *Adv. Difference Equ.*, 66:14 pp, 2015.

[8] S. Melliani, E. Eljaoui, and L. S. Chadli. Solving fuzzy linear differential equations by a new method. *Ann. Fuzzy Math. Inform.*, 9(2):307–323, 2015.

[9] Madan L. Puri, Ralescu, and Dan A. Fuzzy random variables. *J. Math. Anal. Appl.*, 114 , no. 2,(2):409–422, 1986.

[10] Luciano Stefanini. A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Sets and Systems*, 161(11):1564–1584, 2010.

[11] H.C. Wu. The improper fuzzy Riemann integral and its numerical integration. *Inform. Sci.*, 111:109–137, 1999.