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Evolution Semigroups and Exponential Stability of Periodic Difference Evolution Equations

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Abstract

We prove that the discrete system $\zeta_{n+1} = \mathcal{A}_n\zeta_n$ is uniformly exponentially stable if and only if the unique solution of the Cauchy problem

$$\begin{cases} \zeta_{n+1} & = \mathcal{A}_n\zeta_n + z(n+1), \quad n \in \mathbb{Z}_+ \\ \zeta_0 & = 0, \end{cases}$$

is bounded for any natural number n and any almost p-periodic sequence $z(n)$ with $z(0) = 0$. Here, $\mathcal{A}_n$ is a sequence of bounded linear operators on Banach space X.

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1 Introduction

The investigation of the difference equations $\zeta_{n+1} = \mathcal{A}_n\zeta_n$ or $\zeta_{n+1} = \mathcal{A}_n\zeta_n + f_n$ leads to the idea of discrete evolution family. The main interest is the asymptotic behavior of

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the solutions and different types of stabilities in the study of such systems. There are a number of spectral criteria for the characterizations of stability of such systems.

New difficulties appear in the study of nonautonomous systems, especially because the part of the solution generated by the forced term (f_n), i.e., $\sum_{k=\nu}^{n} \mathbb{U}(n,k)f_k$, is not a convolution in the classical sense. These difficulties may be passed by using the so-called evolution semigroups. The evolution semigroups were exhaustively studied in [5]. Clark et al. [7] developed this efficient method to the study of continuous case. So far, there are few related results regarding the investigation of discrete systems.

Recently, Buşe et al. [3] considered the uniform exponential stability of discrete nonautonomous systems on the space of sequences denoted by $C_{00}(\mathbb{Z}_+, X)$ and Wang et al. [8] gave a result on the uniform exponential stability of discrete evolution families on the space of periodic sequences denoted by $\mathbb{P}_0^p(\mathbb{Z}_+, X)$. The objective of this paper is to extend results obtained in [8] to the space of almost periodic sequences denoted by $\mathbb{AP}_0(\mathbb{Z}_+, X)$. Similar results of this type for the continuous case may be found in the paper of Buşe and Jitianu [2] and the references cited therein.

2 Notation and Preliminaries

Let X be a real or complex Banach space and $\mathcal{B}(X)$ be the Banach algebra of all linear and bounded operators acting on X. We denote by $\|\cdot\|$ the norms of operators and vectors. Denote by $\mathbb{R}_+$ the set of real numbers and by $\mathbb{Z}_+$ the set of all nonnegative integers. Let $\mathcal{B}(\mathbb{Z}_+, X)$ be the space of X-valued bounded sequences with the supremum norm, and $\mathbb{P}_0^p(\mathbb{Z}_+, X)$ be the space of p-periodic (with $p \geq 2$) sequences $z(n)$ with $z(0) = 0$. Then $\mathbb{P}_0^p(\mathbb{Z}_+, X)$ is a closed subspace of $\mathcal{B}(\mathbb{Z}_+, X)$. Throughout this paper, $\mathcal{A} \in \mathcal{B}(X)$, $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$, and $r(\mathcal{A}) := \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$ denotes the spectral radius of $\mathcal{A}$. It is well known that $r(\mathcal{A}) = \lim_{n\to\infty} \|\mathcal{A}^n\|^{\frac{1}{n}}$. The resolvent set of $\mathcal{A}$ is defined as $\rho(\mathcal{A}) := \mathbb{C}\backslash\sigma(\mathcal{A})$, i.e., the set of all $\lambda \in \mathbb{C}$ for which $\mathcal{A} - \lambda I$ is an invertible operator in $\mathcal{B}(X)$. We give some results in the framework of general Banach space and spaces of sequences as defined above. Recall that $\mathcal{A}$ is power bounded if there exists a positive constant M such that $\|\mathcal{A}^n\| \leq M$ for all $n \in \mathbb{Z}_+$.

The family $\mathcal{U} := \{\mathbb{U}(n,m) : n, m \in \mathbb{Z}_+, n \geq m\}$ of bounded linear operators is called p-periodic discrete evolution family, for a fixed integer $p \in \{2, 3, \ldots\}$, if it satisfies the following properties:

- $\mathbb{U}(n,n) = I$, for all $n \in \mathbb{Z}_+$.
- $\mathbb{U}(n,m)\mathbb{U}(m,r) = \mathbb{U}(n,r)$, for all $n \geq m \geq r$, $n, m, r \in \mathbb{Z}_+$.
- $\mathbb{U}(n+p, m+p) = \mathbb{U}(n,m)$, for all $n \geq m$, $n, m \in \mathbb{Z}_+$.

It is known that any p-periodic evolution family $\mathcal{U}$ is exponentially bounded, that is, there exist $\omega \in \mathbb{R}$ and $M_\omega \geq 0$ such that

$$\|\mathbb{U}(n,m)\| \leq M_\omega e^{\omega(n-m)}, \text{ for all } n \geq m \in \mathbb{Z}_+. \tag{2.1}$$

When a family $\mathcal{U}$ is exponentially bounded, its growth bound, $\omega_0(\mathcal{U})$, is the infimum of all $\omega \in \mathbb{R}$ for which there exists $M_\omega \geq 1$ such that the relation (2.1) is fulfilled. It is well known that the family $\mathcal{U}$ is uniformly exponentially stable if $\omega_0(\mathcal{U})$ is negative, or equivalently, there exist $M > 0$ and $\omega > 0$ such that $\|\mathbb{U}(n,m)\| \leq Me^{-\omega(n-m)}$, for all $n \geq m \in \mathbb{Z}_+$. As a consequence of (2.1), the following remark is stated.

Remark 2.1. The discrete evolution family $\mathcal{U}$ is uniformly exponentially stable if and only if $r(\mathbb{U}(p,0)) < 1$.

The map $\mathbb{U}(p,0)$ is also called the Poincaré map or monodromy operator of the evolution family $\mathcal{U}$.

Consider the discrete Cauchy problem:

$$\begin{cases} \zeta_{n+1} &= \mathcal{A}_n\zeta_n + e^{i\theta(n+1)}z(n+1), \quad n \in \mathbb{Z}_+ \\ \zeta_0 &= 0, \end{cases} \tag{$\mathcal{A}_n, \theta, 0$}$$

where the sequence $(\mathcal{A}_n)$ is p-periodic, i.e., $\mathcal{A}(n+p) = \mathcal{A}_n$ for all $n \in \mathbb{Z}_+$ and a fixed $p \in \{2,3,\ldots\}$.

Let

$$\mathbb{U}(n,k) := \begin{cases} \mathcal{A}_{n-1}\mathcal{A}_{n-2}\cdots\mathcal{A}_k, & \text{if } k \leq n-1, \\ I, & \text{if } k = n. \end{cases}$$

Then, the family $\{\mathbb{U}(n,k)\}_{n\geq k\geq 0}$ is a discrete p-periodic evolution family. For the uniform exponential stability of the Cauchy problem $(\mathcal{A}_n, 0)$ in discrete and continuous autonomous cases, over finite dimensional spaces; see [4,9].

Let us divide n by p, i.e., $n = lp + r$ for some $l \in \mathbb{Z}_+$, where $r \in \{0,1,\ldots,p-1\}$. We consider the following sets which will be useful along this work.

$$\mathcal{A}_j := \{1+jp, 2+jp, \ldots, (j+1)p-1\}, \quad \text{for all } j \in \mathbb{Z}_+.$$

If $r \in \{1,2,\ldots,p-1\}$, then define

$$B_l := \{lp+1, lp+2, \ldots, lp+r\}$$

and

$$C := \{0, p, 2p, \ldots, lp\}.$$

It is clear that

$$\bigcup_{j=0}^{l-1} A_j \bigcup B_l \bigcup C = \{0,1,2,\ldots,n\}. \tag{2.2}$$

With the help of partition (2.2), we construct the set $\mathcal{W}$ which consists of the p-periodic X-valued sequences of the form

$$z(k) = \begin{cases} (k-jp)[(1+j)p-k]\mathbb{U}(k-jp,0)x, & \text{if } k \in \mathcal{A}_j, \\ k(p-k)\mathbb{U}(k,0)x, & \text{if } k \in B_l, \\ 0, & \text{if } k \in C, \end{cases} \tag{2.3}$$

where $x \in X$.

In [8], with the help of the set $\mathcal{W}$, the following theorem was obtained.

Theorem 2.2 (See [8]). *Let* $\mathcal{U} := \{\mathbb{U}(n,m) : n,m \in \mathbb{Z}_+, n \geq m\}$ *be a discrete evolution family on* X. *If the sequence*

$$\zeta_n = \sum_{k=0}^{n} e^{i\theta k}\mathbb{U}(n,k)z(k)$$

is bounded for each real number θ *and each* p*-periodic sequence* $z(n) \in \mathcal{W}$*, then* $\mathcal{U}$ *is uniformly exponentially stable.*

3 Uniform Exponential Stability of Discrete Evolution Family on Space $\mathbb{AP}_0(\mathbb{Z}_+, X)$

A sequence $g : \mathbb{Z}_+ \to X$ is said to be almost periodic sequence if for any $\epsilon > 0$ there exists an integer $H(\epsilon) > 0$ such that any discrete interval of length $H(\epsilon)$ contains an integer τ satisfying

$$\|g(n+\tau) - g(n)\| \leq \epsilon \quad \text{for all } n \in \mathbb{Z}_+.$$

The integer number τ is called ϵ-translation number of $g(n)$. The set of all almost periodic sequences will be denoted by $\mathbb{AP}(\mathbb{Z}_+, X)$. For further details about almost periodic functions, we refer the reader to the books [1, 6]. Obviously, $\mathbb{AP}(\mathbb{Z}_+, X)$ is a subset of $\mathcal{B}(\mathbb{Z}_+, X)$. Denote by $\mathcal{A}_0(\mathbb{Z}_+, X)$ the set of all sequences $\{g(n)\}_{n\in\mathbb{Z}_+}$ for which there exist an $N_g \in \mathbb{Z}_+$ with $N_g > 0$ and an $h_g \in \mathbb{P}_0^p(\mathbb{Z}_+, X)$ such that

$$g(n) = \begin{cases} 0, & \text{if } 0 \leq n < N_g, \\ h_g(n), & \text{if } n \geq N_g. \end{cases}$$

Let $\mathbb{AP}_0(\mathbb{Z}_+, X) := \overline{\text{span}}\{\mathcal{A}_0(\mathbb{Z}_+, X)\}$. This space is closed in $\mathcal{B}(\mathbb{Z}_+, X)$.

Let us define the evolution semigroup $\mathbb{T} := \{\mathcal{T}(n), n \in \mathbb{Z}_+\}$ associated to $\mathcal{U}$ on $\mathbb{AP}_0(\mathbb{Z}_+, X)$ as

$$(\mathcal{T}(r)g)(n) = \begin{cases} \mathbb{U}(r, n-r)g(n-r), & \text{if } n \geq r \geq 0, \\ 0, & \text{if } 0 \leq n \leq r. \end{cases}$$

We state the following lemma for the above evolution semigroup.

Lemma 3.1. *Let* $\mathcal{U} := \{\mathbb{U}(n,m)\}_{n\geq m\in\mathbb{Z}_+}$ *be a discrete evolution family of bounded linear operators on* $\mathbb{AP}_0(\mathbb{Z}_+,X)$*. Then the sequence* $\mathcal{T}(r)g$*, namely,*

$$(\mathcal{T}(r)g)(n) = \begin{cases} \mathbb{U}(r,n-r)g(n-r), & \text{if } n \geq r \geq 0, \\ 0, & \text{if } 0 \leq n \leq r, \end{cases}$$

belongs to $\mathbb{AP}_0(\mathbb{Z}_+,X)$*.*

Proof. We show first that $\mathcal{T}(r)g \in \mathcal{A}_0(\mathbb{Z}_+,X)$ for any $g \in \mathcal{A}_0(\mathbb{Z}_+,X)$. Since $g \in \mathcal{A}_0(\mathbb{Z}_+,X)$, there exist an $N_g \in \mathbb{Z}_+$ with $N_g > 0$ and an $h_g \in \mathbb{P}_0^p(\mathbb{Z}_+,X)$ such that

$$g(n) = \begin{cases} 0, & \text{if } 0 \leq n < N_g, \\ h_g(n), & \text{if } n \geq N_g. \end{cases}$$

Let $N_{\mathcal{T}(r)g} := r + N_g$ and $h_{\mathcal{T}(r)g}(\cdot) := \mathbb{U}(r,\cdot - r)h_g(\cdot - r)$. Obviously, $h_{\mathcal{T}(r)g}$ is a p-periodic sequence. Next, we claim that

$$(\mathcal{T}(r)g)(n) = \begin{cases} 0, & \text{if } 0 \leq n < N_{\mathcal{T}(r)g}, \\ h_{\mathcal{T}(r)g}(n), & \text{if } n \geq N_{\mathcal{T}(r)g}. \end{cases}$$

If $n \leq N_{\mathcal{T}(r)g} = r + N_g$, then $n - r < N_g$ and $g(n-r) = 0$. Thus,

$$(\mathcal{T}(r)g)(n) = \mathbb{U}(r,n-r)g(n-r) = 0.$$

If $n \geq N_{\mathcal{T}(r)g} = r + N_g$, then $n - r \geq N_g$ and $g(n-r) = h_g(n-r)$. Hence,

$$\begin{aligned} (\mathcal{T}(r)g)(n) &= \mathbb{U}(r,n-r)g(n-r) \\ &= \mathbb{U}(r,n-r)h_g(n-r) \\ &= h_{\mathcal{T}(r)g}(n). \end{aligned}$$

Therefore, $\mathcal{T}(r)g \in \mathcal{A}_0(\mathbb{Z}_+,X)$. Let us choose $\epsilon > 0$ and $f, g \in \mathcal{A}_0(\mathbb{Z}_+,X)$ such that

$$\|f - g\|_{\mathcal{B}(\mathbb{Z}_+,X)} < \epsilon.$$

Since $\mathcal{T}(r)g \in \mathcal{A}_0(\mathbb{Z}_+,X)$, we have, for some $M \geq 1$ and $\nu \in \mathbb{R}$,

$$\begin{aligned} \|\mathcal{T}(r)f - \mathcal{T}(r)g\|_{\mathcal{B}(\mathbb{Z}_+,X)} &= \sup_{n\geq r} \|\mathbb{U}(r,n-r)[f(n-r) - g(n-r)]\| \\ &\leq Me^{\nu r} \sup_{n\geq r} \|f(n-r) - g(n-r)\| \\ &\leq Me^{\nu r}\epsilon. \end{aligned}$$

This completes the proof. □

Similar analysis to that in the continuous case, the "infinitesimal generator" of the discrete semigroups denoted by $\mathbb{G}$ is defined as $\mathbb{G} := \mathcal{T}(1) - I$. For discrete semigroups, the Taylor formula of order one is

$$\mathcal{T}(n)g - g = \sum_{k=0}^{n-1} \mathcal{T}(k)\mathbb{G}g, \quad \forall n \in \mathbb{Z}_+, n \geq 1, \forall g \in X. \tag{3.1}$$

Lemma 3.2. *Assume that* $\mathbb{G}_{\mathbb{T}}$ *is the infinitesimal generator of the discrete evolution semigroup* $(\mathcal{T}(r)g)(n)$ *and let* $g, f \in \mathbb{AP}_0(\mathbb{Z}_+, X)$ *be such that* $(\mathbb{G}_{\mathbb{T}})g = -f$*. Then* $g(n) = \sum_{k=0}^{n-1} \mathbb{U}(n,k)f(k)$ *for all* $n \in \mathbb{Z}_+$.

Proof. Using the Taylor formula (3.1), we have

$$\mathcal{T}(n)g - g = \sum_{m=0}^{n-1} \mathcal{T}(m)\mathbb{G}_{\mathbb{T}}g = -\sum_{m=0}^{n-1} \mathcal{T}(m)f.$$

Hence, for every $n \in \mathbb{Z}_+$,

$$\begin{aligned} g(n) &= (\mathcal{T}(n)g)(n) + \sum_{m=0}^{n-1} (\mathcal{T}(m)f)(n) \\ &= 0 + \sum_{m=0}^{n-1} \mathbb{U}(m, n-m)f(n-m) \\ &= \sum_{k=0}^{n} \mathbb{U}(n,k)f(k). \end{aligned}$$

The proof is complete. □

The following theorem gives the uniform exponential stability of discrete evolution family on $\mathbb{AP}_0(\mathbb{Z}_+, X)$.

Theorem 3.3. *Let* $\mathcal{U} := \{\mathbb{U}(n,m) : n, m \in \mathbb{Z}_+, n \geq m\}$ *be a discrete semigroup on* X*. The following four statements are equivalent:*

- *(1)* $\mathcal{U}$ *is uniformly exponentially stable.*
- *(2) The evolution semigroup* $\mathbb{T}$ *associated to* $\mathcal{U}$ *on* $\mathbb{AP}_0(\mathbb{Z}_+, X)$ *is uniformly exponentially stable.*
- *(3) The series* $\sum_{k=0}^{n} \mathbb{U}(n,k)g(k) \in \mathbb{AP}_0(\mathbb{Z}_+, X)$*, for each* $g \in \mathbb{AP}_0(\mathbb{Z}_+, X)$.
- *(4)* $\sup_{n \in \mathbb{Z}_+} \left\| \sum_{k=0}^{n} \mathbb{U}(n,k)g(k) \right\| < \infty$*, for all* $g \in \mathbb{AP}_0(\mathbb{Z}_+, X)$.

Proof. (1)⇒ (2): Let $\mathcal{U}$ be uniformly exponentially stable. Then there exist two positive constants N and ν such that

$$\|\mathbb{U}(n,m)\| \leq Ne^{-\nu(n-m)}, \text{ for all } n \geq m \in \mathbb{Z}_+.$$

Equivalently,

$$\|\mathbb{U}(n-m,0)\| \le Ne^{-\nu(n-m)}, \text{ for } n-m \ge 0, n, m \in \mathbb{Z}_+,$$

i.e.,

$$\|\mathbb{U}(r,0)\| \le Ne^{-\nu r}, \text{ for } r \in \mathbb{Z}_+.$$

Let $g \in \mathbb{AP}_0(\mathbb{Z}_+, X)$. Then

$$\begin{aligned} \|\mathcal{T}(r)g\| &= \sup_{n\ge r} \|\mathbb{U}(r, n-r)g(n-r)\| \\ &\le Ne^{-\nu r}\|g\|, \end{aligned}$$

which implies that $\mathbb{T}$ is uniformly exponentially stable.

(2) $\Rightarrow$ (3): Assume that $\mathbb{T}$ is uniformly exponentially stable. By Remark 2.1, $r(\mathcal{T}(1)) < 1$, which implies that $1 \in \rho(\mathcal{T}(1))$, i.e., $\mathcal{T}(1) - I$ is invertible. Hence, for each $g \in \mathbb{AP}_0(\mathbb{Z}_+, X)$, there exists an $f \in \mathbb{AP}_0(\mathbb{Z}_+, X)$ such that $(\mathcal{T}(1) - I)g = -f$. On the other hand, it follows from Lemma 3.2 that, for every $n \in \mathbb{Z}_+$, $g(n) = \sum_{k=0}^{n} \mathbb{U}(n,k)f(k)$.

(3)$\Rightarrow$ (4): Using the definition, it is obvious.

(4)$\Rightarrow$ (1): This is a direct consequence of Theorem 2.2. The proof is complete. □

Consider the Cauchy problem

$$\begin{cases} \zeta_{n+1} &= \mathcal{A}_n\zeta_n + g(n+1), \quad n \in \mathbb{Z}_+, \\ \zeta_0 &= 0. \end{cases} \qquad (\mathcal{A}_n, 0)$$

The solution of this Cauchy problem is

$$\zeta_n = \sum_{k=0}^{n} \mathbb{U}(n,k)g(k).$$

On the basis of Theorem 3.3, we can obtain the following result.

Corollary 3.4. *The system $\zeta_{n+1} = \mathcal{A}_n\zeta_n$ is uniformly exponentially stable if and only if the unique solution of the Cauchy problem $(\mathcal{A}_n, 0)$ is bounded for any real number θ and any almost periodic sequence $g(n)$ with $g(0) = 0$.*

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