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Leslie–Gower Competition Model with Survival Rate in an Almost Automorphic Environment

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Abstract

The paper studies the dynamic of the Leslie–Gower competition model with nonconstant survival rate in an almost automorphic environment. An existence result is established. Further, an illustrative example is given.

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1 Introduction

The main motivation of this paper comes from a couple of sources. In Bohner and Warth [1], the Beverton–Holt dynamic equation was introduced and studied. The results in [1] extend those obtained in the case of the Beverton-Holt model in $\mathbb{Z}_+$.

In Diagana [6], the existence of solutions to a Beverton–Holt dynamic equation in an almost automorphic environment with a nonconstant survival rate, was obtained under some reasonable assumptions.

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In Sacker [9], the global stability of the Leslie–Gower competition model without survival rate in a periodic environment was fully studied for d-species in competition.

In Chow and Hsieh [2], the stability of multi-dimensional discrete-time competitive Beverton–Holt equations with constant coefficients and without survival rates was studied.

The main purpose of this paper consists of studying the dynamic of the Leslie–Gower competition model with nonconstant survival rates. Let $x(t)$ (respectively, $y(t)$) be the total size of the specie S_1 (respectively, the total size of the specie S_2) at generation t. We consider the case of a Leslie–Gower model in which the recruiting functions are of Beverton–Holt types, which we will call a Beverton–Holt competition model with survival rate. More precisely, we are interested in studying the existence of solutions to the Beverton–Holt competition model given by

$$\begin{cases} x(t+1) = \gamma_{11}(t)x(t) + \gamma_{12}(t)y(t) + f(t, x(t), y(t)), & t \in \mathbb{Z}, \\ \\ y(t+1) = \gamma_{21}(t)x(t) + \gamma_{22}(t)y(t) + g(t, x(t), y(t)), & t \in \mathbb{Z}, \end{cases} \tag{1.1}$$

with

$$f(t, x(t), y(t)) = \frac{(1-\gamma_{11}(t))\mu_1 K_1(t)x(t)}{(1-\gamma_{11}(t))K_1(t) + (\mu_1 - 1 + \gamma_{11}(t))x(t) + p(t)y(t)}$$

and

$$g(t, x(t), y(t)) = \frac{(1-\gamma_{22}(t))\mu_2 K_2(t)y(t)}{(1-\gamma_{22}(t))K_2(t) + q(t)x(t) + (\mu_2 - 1 + \gamma_{22}(t))y(t)},$$

where $\gamma_{11}, \gamma_{22} \in (0,1)$ are the survival rates, K_i for $i = 1, 2$ are the carrying capacities, and $\mu_i > 1$ for $i = 1, 2$ are the growth rates respectively for the species S_1 and S_2, and $p, q \geq 0$ and $\gamma_{12}, \gamma_{21} \in (0,1)$ are the coefficients of interspecific competition.

To study the existence of solutions to (1.1), we study a broader model involving a general vector function as a "recruitment function", that is,

$$X(t+1) = A(t)X(t) + F(t, X(t)),\ t \in \mathbb{Z}, \tag{1.2}$$

where

$$X(t) = \begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ \\ c(t) & d(t) \end{pmatrix}, \quad \text{and} \quad F(t, X(t)) = \begin{pmatrix} f_1(t, X(t)) \\ \\ f_2(t, X(t)) \end{pmatrix}$$

with

$$f_1(t, X(t)) = \frac{a_1(t)x(t)}{b_1(t) + c_1(t)x(t) + p(t)y(t)},$$

$$f_2(t, X(t)) = \frac{a_2(t)y(t)}{b_2(t) + q(t)x(t) + c_2(t)y(t)},$$

and $a_i(t) \geq 0$, $b_i(t) > 0$, $\inf_{t\in\mathbb{Z}} b_i(t) \geq m_i$ for some $m_i > 0$, $c_i(t) \geq 0$ for $i = 1, 2$, and $p(t), q(t) \geq 0$ for all $t \in \mathbb{Z}$.

The qualitative theory of difference equations with almost periodic equations (respectively, with almost automorphic coefficients) is a topic of a great interest as almost periodicity (respectively, almost automorphy) are more likely to accurately describe many phenomena occurring in population dynamics than the classical periodicity, see, e.g., Henson *et al.* [8] for instance (see also [7]). In Diagana [5], almost automorphic sequences were studied and utilized to obtain the existence of almost automorphic solution to some nonautonomous difference equations. In this paper we make use of the Banach fixed point principle to establish the existence and uniqueness of an almost automorphic solution to (1.2) under the condition that the coefficients involved in it are almost automorphic and satisfy some additional conditions. Next, we make extensive use of the previous results to establish the existence of almost automorphic solutions to the Beverton–Holt competition model given in (1.1).

The paper is organized as follows: Section 2 is devoted to basic results needed in the sequel. In particular, basic definition on the concept of almost automorphy of sequences as introduced in [5, 10] will be discussed. In Section 3, we prove the main result. In Section 4, we study the case of the Beverton–Holt competition model. In Section 5, an example is given to illustrate our abstract results.

2 Preliminaries

The notations and definitions of this section are similar to those of Diagana [4, 5] but for the sake of clarity we reproduce them here. In this paper, $(\mathbb{X}, d)$, $(\mathbb{R}, |\cdot|)$, $\mathbb{R}_+$, $\mathbb{Z}_+$, and $\mathbb{Z}$ stand respectively for a metric space, the field of real numbers equipped with its natural absolute value, the set of nonnegative real numbers, the set of all nonnegative integers, and the set of all integers.

The notation $l^\infty(\mathbb{Z})$ stands for the metric space of all bounded $\mathbb{X}$-valued sequences equipped with the metric d_∞ defined for each $x = \{x(t)\}_{t\in\mathbb{Z}}, y = \{y(t)\}_{t\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, by

$$d_\infty(x, y) := \sup_{t\in\mathbb{Z}} d(x(t), y(t)).$$

Definition 2.1 (See [4, 5])**.** An $\mathbb{X}$-valued sequence $x = \{x(t)\}_{t\in\mathbb{Z}}$ is said to be almost automorphic if for every sequence $\{h'(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ there exists a subsequence $\{h(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ such that

$$\lim_{n\to\infty} x(t + h(n)) = y(t)$$

is well defined for each $t \in \mathbb{Z}$, and

$$\lim_{n\to\infty} y(t - h(n)) = x(t)$$

for each $t \in \mathbb{Z}$. Equivalently,

$$\lim_{n\to\infty} d(x(t+h(n)), y(t)) = 0$$

is well defined for each $t \in \mathbb{Z}$, and

$$\lim_{n\to\infty} d(y(t-h(n)), x(t)) = 0$$

for each $t \in \mathbb{Z}$.

The collection of all almost automorphic $\mathbb{X}$-valued sequences on $\mathbb{Z}$ will be denoted by $AA(\mathbb{Z})$. This is a complete metric space when it is equipped with the metric d_∞ defined above.

Definition 2.2 (See [4, 5])**.** A sequence of functions $F : \mathbb{Z} \times \mathbb{X} \mapsto \mathbb{X}$, $(t, u) \mapsto F(t, u)$ is called almost automorphic if for every sequence $\{h'(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ there exists a subsequence $\{h(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ such that

$$\lim_{n\to\infty} F(t+h(n), x) = G(t, x)$$

is well defined for each $t \in \mathbb{Z}$ and

$$\lim_{n\to\infty} G(t-h(n), x) = F(t, x)$$

for each $t \in \mathbb{Z}$ and $x \in B$ where $B \subset \mathbb{X}$ is an arbitrary bounded subset. Equivalently,

$$\lim_{n\to\infty} d(F(t+h(n), x), G(t, x)) = 0$$

is well defined for each $t \in \mathbb{Z}$ and

$$\lim_{n\to\infty} d(G(t-h(n), x), F(t, x)) = 0$$

for each $t \in \mathbb{Z}$ and $x \in B$ where $B \subset \mathbb{X}$ is an arbitrary bounded subset.

Theorem 2.3 (See [5, Theorem 4.42, pp. 129–130])**.** *Suppose that $f : \mathbb{Z} \times \mathbb{X} \to \mathbb{X}$, $(t, u) \mapsto f(t, u)$ is almost automorphic in $t \in \mathbb{Z}$ uniformly in $u \in B$ where $B \subset \mathbb{X}$ is an arbitrary bounded subset. If in addition, f is Lipschitz in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{Z}$, that is, there exists, $L > 0$ such that*

$$d(f(t, u), f(t, v)) \leq Ld(u, v) \ \textit{for all} \ \ u, v \in \mathbb{X}, t \in \mathbb{Z},$$

then for every $\mathbb{X}$-valued almost automorphic sequence $x = \{x(t)\}_{t\in\mathbb{Z}}$, the $\mathbb{X}$-valued sequence $F(t) = f(t, x(t))$ is almost automorphic.

Theorem 2.4 (See [5, Theorem 4.43, pp. 131–132])**.** *If $f : \mathbb{Z} \times \mathbb{Y} \to \mathbb{X}$, $(t, u) \mapsto f(t, u)$ is almost automorphic in $t \in \mathbb{Z}$ uniformly in $u \in B$ where $B \subset \mathbb{Y}$ is an arbitrary bounded subset of the metric space $\mathbb{Y}$. If in addition, $x \mapsto f(t, x)$ is uniformly continuous on each bounded subset K of $\mathbb{Y}$ uniformly in $t \in \mathbb{Z}$, then for every $\mathbb{Y}$-valued almost automorphic sequence $x = \{x(t)\}_{t\in\mathbb{Z}}$, the $\mathbb{X}$-valued sequence $F(t) = f(t, x(t))$ is almost automorphic.*

3 Main Result

Let d, d_2 be the metrics defined as follows: for all $x = (a, b) \in \mathbb{R} \times \mathbb{R}$ and $y = (c, d) \in \mathbb{R} \times \mathbb{R}$,

$$d(a, b) := |a - b|$$

and

$$d_2(x, y) := \sqrt{(a-c)^2 + (b-d)^2}.$$

The main result of this paper requires the following technical lemma.

Lemma 3.1. *Consider the function $F : \mathbb{Z} \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by*

$$F(t, (x, y)) := \frac{a(t)x}{b(t) + c(t)x + d(t)y}$$

where the functions $a : \mathbb{Z} \mapsto \mathbb{R}_+$, $b : \mathbb{Z} \mapsto \mathbb{R}_+$, $c : \mathbb{Z} \mapsto \mathbb{R}_+$, and $d : \mathbb{Z} \mapsto \mathbb{R}_+$ are almost automorphic. If there exists $m_0 > 0$ such that $\inf_{t\in\mathbb{Z}} b(t) \geq m_0$ and if $t \mapsto (x(t), y(t))$ is almost automorphic, then $t \mapsto F(t, (x(t), y(t)))$ is almost automorphic.

Proof. The fact that $(x, y) \mapsto F(t, (x, y))$ is uniformly continuous on each bounded subset K of $\mathbb{R}_+ \times \mathbb{R}_+$ uniformly in $t \in \mathbb{Z}$ is clear. It remains to show that $t \mapsto F(t, (x, y))$ is almost automorphic uniformly in $(x, y) \in B$ where $B \subset \mathbb{R}_+ \times \mathbb{R}_+$ is an arbitrary bounded subset. For that, let $L > 0$ be the diameter of B. Now using the fact that $t \mapsto a(t)$ is almost automorphic, it follows that for every sequence $\{h'(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ there exists a subsequence $\{h(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ such that

$$\lim_{n\to\infty} a(t + h(n)) = a'(t)$$

is well defined for each $t \in \mathbb{Z}$, and

$$\lim_{n\to\infty} a'(t - h(n)) = a(t)$$

for each $t \in \mathbb{Z}$. Using the fact that $t \mapsto b(t)$ is almost automorphic, it follows that there exists a subsequence $\{h_1(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ of $\{h'(n)\}_{n\in\mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} b(t + h_1(n)) = b'(t) \tag{3.1}$$

is well defined for each $t \in \mathbb{Z}$, and

$$\lim_{n\to\infty} b'(t - h_1(n)) = b(t) \tag{3.2}$$

for each $t \in \mathbb{Z}$. An important consequence of (3.1) is the fact that $b'(t) \geq m_0$ for all $t \in \mathbb{Z}$. Using the fact that $t \mapsto c(t)$ is almost automorphic, it follows that there exists a subsequence $\{h_2(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ of $\{h_1(n)\}_{n\in\mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} c(t + h_2(n)) = c'(t)$$

is well defined for each $t \in \mathbb{Z}$, and

$$\lim_{n\to\infty} c'(t - h_2(n)) = c(t)$$

for each $t \in \mathbb{Z}$. Similarly, using the fact that $t \mapsto d(t)$ is almost automorphic, it follows that there exists a subsequence $\{s(n)\}_{n\in\mathbb{Z}_+} \subset \mathbb{Z}$ of $\{h_2(n)\}_{n\in\mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} d(t + s(n)) = d'(t)$$

is well defined for each $t \in \mathbb{Z}$, and

$$\lim_{n\to\infty} d'(t - s(n)) = d(t)$$

for each $t \in \mathbb{Z}$. Set

$$\Phi(t,(x,y)) := \frac{a'(t)x}{b'(t) + c'(t)x + d'(t)y},$$

where $a' : \mathbb{Z} \mapsto [0,\infty)$, $b' : \mathbb{Z} \mapsto (0,\infty)$, $c' : \mathbb{Z} \mapsto [0,\infty)$, and $d' : \mathbb{Z} \mapsto [0,\infty)$ are the above-mentioned bounded functions. Now

$$\begin{aligned}
& d(F(t+s(n),(x,y)),\Phi(t,(x,y))) \\
&\quad = \Big|F(t+s(n),(x,y)) - \Phi(t,(x,y))\Big| \\
&\quad = \left|\frac{a(t+s(n))x}{b(t+s(n)) + c(t+s(n))x + d(t+s(n))y} - \frac{a'(t)x}{b'(t) + c'(t)x + d'(t)y}\right| \\
&\quad = \left|\frac{A_n(t)x + B_n(t)x^2 + C_n(t)xy}{(b(t+s(n)) + c(t+s(n))x + d(t+s(n))y)(b'(t) + c'(t)x + d'(t)y)}\right| \\
&\quad \le \left|\frac{A_n(t)x + B_n(t)x^2 + C_n(t)xy}{b(t+s(n))b'(t)}\right| \\
&\quad \le m_0^{-2}\Big|A_n(t)x + B_n(t)x^2 + C_n(t)xy\Big| \\
&\quad \le m_0^{-2}\Big(|A_n(t)|L + |B_n(t)|L^2 + |C_n(t)|L^2\Big),
\end{aligned}$$

where

$$\begin{aligned}
A_n(t) &:= a(t+s(n))b'(t) - a'(t)b(t+s(n)), \\
B_n(t) &:= a(t+s(n))c'(t) - a'(t)c(t+s(n)),
\end{aligned}$$

and

$$C_n(t) := a(t+s(n))d'(t) - a'(t)d(t+s(n)).$$

Now

$$\begin{aligned}|A_n(t)| &= |a(t+s(n))b'(t) - a'(t)b(t+s(n))| \\ &= |(a(t+s(n)) - a'(t))b'(t) + a'(t)b'(t) - a'(t)b(t+s(n))| \\ &= |a(t+s(n)) - a'(t)| \,.\, |b'(t)| + |a'(t)| \,.\, |b'(t) - b(t+s(n))|\end{aligned}$$

and hence

$$\lim_{n\to\infty} |A_n(t)| = 0$$

for each $t \in \mathbb{Z}$. Similarly,

$$\lim_{n\to\infty} |B_n(t)| = 0; \quad \lim_{n\to\infty} |C_n(t)| = 0$$

for each $t \in \mathbb{Z}$. Hence

$$\lim_{n\to\infty} F(t+s(n),(x,y)) = \Phi(t,(x,y))$$

is well defined for each $t \in \mathbb{Z}$ and for all $(x,y) \in B$. Similarly,

$$\lim_{n\to\infty} \Phi(t-s(n),(x,y)) = F(t,(x,y))$$

for each $t \in \mathbb{Z}$ and for all $(x,y) \in B$. Since $(x,y) \mapsto F(t,(x,y))$ is uniformly continuous for all $t \in \mathbb{Z}$ and $t \mapsto F(t,(x,y))$ is almost automorphic for all $(x,y) \in B$ where $B \subset \mathbb{R}_+ \times \mathbb{R}_+$ is an arbitrary bounded subset it follows from Theorem 2.4 that if $t \mapsto (x(t),y(t))$ is almost automorphic, then $t \mapsto F(t,(x(t),y(t)))$ is almost automorphic. This completes the proof. □

Using similar arguments as above, we obtain the following technical lemma.

Lemma 3.2. *Consider the function* $G : \mathbb{Z} \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ *defined by*

$$G(t,(x,y)) = \frac{a(t)y}{b(t)+c(t)x+d(t)y}$$

where the functions $a : \mathbb{Z} \mapsto \mathbb{R}_+$, $b : \mathbb{Z} \mapsto \mathbb{R}_+$, $c : \mathbb{Z} \mapsto \mathbb{R}_+$, *and* $d : \mathbb{Z} \mapsto \mathbb{R}_+$ *are almost automorphic. If there exists* $n_0 > 0$ *such that* $\inf_{t\in\mathbb{Z}} b(t) \geq n_0$ *and if* $t \mapsto (x(t),y(t))$ *is almost automorphic, then* $t \mapsto G(t,(x(t),y(t)))$ *is almost automorphic.*

Combining both Lemma 3.1 and Lemma 3.2, we obtain the following Lemma.

Lemma 3.3. *Let* $X = \begin{pmatrix} x \\ y \end{pmatrix}$. *Consider*

$$F(t,X) = \begin{pmatrix} f_1(t,X) \\ f_2(t,X) \end{pmatrix}$$

with

$$f_1(t,X) = \frac{a_1(t)x}{b_1(t)+c_1(t)x+p(t)y},$$

$$f_2(t,X) = \frac{a_2(t)y}{b_2(t)+q(t)x+c_2(t)y},$$

and $a_i(t) \geq 0$, $b_i(t) > 0$, $\inf_{t\in\mathbb{Z}} b_i(t) \geq m_i$ *for* $i = 1,2$, $c_i(t) \geq 0$, *and* $p(t), q(t) \geq 0$ *for all* $t \in \mathbb{Z}$ *for some* $m_1, m_2 > 0$. *Suppose* a_i, b_i, *and* c_i *for* $i = 1,2$, *and* p, q *are all almost automorphic. If* $t \mapsto X(t)$ *is almost automorphic, then* $t \mapsto F(t, X(t))$ *is almost automorphic.*

Set $X(t,s) := \prod_{r=s}^{t-1} A(r)$ for all $t, s \in \mathbb{Z}$, $t \geq s$, where

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

In the rest of the paper, we suppose there exists a double-sequence $\theta : \mathbb{Z}\times\mathbb{Z} \mapsto (0,\infty)$, $(t,s) \mapsto \theta(t,s)$ such that

$$d_2\Big(X(t,s)u, X(t,s)v\Big) \leq \theta(t,s) d_2\Big(u,v\Big)$$

for all $t, s \in \mathbb{Z}$, $t \geq s$, $u, v \in \mathbb{X}$, and such that

$$\theta_0 := \sup_{t\in\mathbb{Z}} \left(\sum_{s=-\infty}^{t-1} \theta(t, s+1) \right) < \infty.$$

Theorem 3.4. *Under previous assumptions on* $X(t,s)$, *suppose* $a_i : \mathbb{Z} \mapsto [0,\infty)$, $b_i : \mathbb{Z} \mapsto (0,\infty)$, $c_i : \mathbb{Z} \mapsto [0,\infty)$, *and* $p, q : \mathbb{Z} \mapsto [0,\infty)$ *are almost automorphic for* $i = 1,2$ *such that there exist* $\alpha_i > 0$ *and* $m_i > 0$ *with* $\inf_{t\in\mathbb{Z}} b_i(t) \geq m_i$ *for* $i = 1,2$ *and* $\inf_{t\in\mathbb{Z}} c_i(t) \geq \alpha_i$. *Then* (1.2) *has a unique almost automorphic solution whenever* $\sqrt{L} < \theta_0^{-1}$, *where*

$$L = 2\max\left[3\Big(\frac{A_1}{m_1}\Big)^2, 3\Big(\frac{A_2}{m_2}\Big)^2, 2\Big(\frac{P}{m_1\alpha_1}\Big)^2, 2\Big(\frac{Q}{m_2\alpha_2}\Big)^2\right]$$

with $A_i = \sup_{t\in\mathbb{Z}} a_i(t)$, $P = \sup_{t\in\mathbb{Z}} p(t)$, $Q = \sup_{t\in\mathbb{Z}} q(t)$ *for* $i = 1,2$.

Proof. Let $X \in AA(\mathbb{Z})$. Define the function Ψ by setting

$$\Psi X(t) := \sum_{r=-\infty}^{t-1} X(t, r+1) F(r, X(r)).$$

Using Lemma 3.3 it follows that the function $s \mapsto F(s, X(s))$ is almost automorphic. Now since $s \mapsto A(s)$ is almost automorphic too, it follows that the operator Ψ maps $AA(\mathbb{Z})$ into itself. To complete the proof, we have to check that F is a Lipschitzian function in the following sense: there exists $L > 0$ such that

$$d_2(F(s, X_1(s)), F(s, X_2(s))) \leq L d_2(X_1(s), X_2(s)),$$

for all $s \in \mathbb{R}$ and all $X_1 = (x_1, y_1) \in AA(\mathbb{Z})$ and $Y_1 = (y_1, y_2) \in AA(\mathbb{Z})$,

$$\begin{aligned} d_2^2(F(s, X_1(s)), F(s, X_2(s))) &= \Big(f_1(s, X_1(s)) - f_1(s, X_2(s))\Big)^2 \\ &+ \Big(f_2(s, X_1(s)) - f_2(s, X_2(s))\Big)^2 \\ &= I_1 + I_2 \end{aligned}$$

where $I_1 := \Big(f_1(s, X_1(s)) - f_1(s, X_2(s))\Big)^2$ and $I_2 := \Big(f_2(s, X_1(s)) - f_2(s, X_2(s))\Big)^2$. Now

$$\begin{aligned} I_1 &= \Big(f_1(s, X_1(s)) - f_1(s, X_2(s))\Big)^2 \\ &= \left[\frac{a_1(s)x_1(t)}{b_1(s) + c_1(s)x_1(s) + p(s)y_1(s)} - \frac{a_1(s)x_2(s)}{b_1(s) + c_1(s)x_2(s) + p(s)y_2(s)}\right]^2 \\ &= \left[\frac{a_1(s)x_1(t)}{b_1(s) + c_1(s)x_1(s) + p(s)y_1(s)} - \frac{a_1(s)x_2(t)}{b_1(s) + c_1(s)x_1(s) + p(s)y_1(s)}\right. \\ &+ \left.\frac{a_1(s)x_2(t)}{b_1(s) + c_1(s)x_1(s) + p(s)y_1(s)} - \frac{a_1(s)x_2(s)}{b_1(s) + c_1(s)x_2(s) + p(s)y_2(s)}\right]^2 \\ &\leq 2\Big(\frac{a_1(s)}{b_1(s) + c_1(s)x_1(s) + p(s)y_1(s)}\Big)^2\Big(x_1(s) - x_2(s)\Big)^2 + 2\Big(a_1(s)x_2(s)\Big)^2 \\ &\times \Big(\frac{1}{b_1(s) + c_1(s)x_1(s) + p(s)y_1(s)} - \frac{1}{b_1(s) + c_1(s)x_2(s) + p(s)y_2(s)}\Big)^2 \\ &\leq 2\Big(\frac{a_1(s)}{b_1(s)}\Big)^2\Big(x_1(s) - x_2(s)\Big)^2 + 2\Big(a_1(s)x_2(s)\Big)^2 \\ &\times \left(\frac{c_1(s)\Big(x_2(s) - x_1(s)\Big) + p_1(s)\Big(y_2(s) - y_1(s)\Big)}{\Big(b_1(s) + c_1(s)x_1(s) + p(s)y_1(s)\Big)\Big(b_1(s) + c_1(s)x_2(s) + p(s)y_2(s)\Big)}\right)^2 \\ &\leq 2\Big(\frac{a_1(s)}{b_1(s)}\Big)^2\Big(x_1(s) - x_2(s)\Big)^2 \\ &+ 2\Big(a_1(s)x_2(s)\Big)^2\left[\frac{c_1(s)\Big(x_2(s) - x_1(s)\Big) + p_1(s)\Big(y_2(s) - y_1(s)\Big)}{b_1^2(s) + b_1(s)c_1(s)x_2(s)}\right]^2. \end{aligned}$$

If $x_2 = 0$, then we obtain

$$\begin{aligned} I_1 &\leq 2\Big(\frac{a_1(s)}{b_1(s)}\Big)^2\Big(x_1(s) - x_2(s)\Big)^2 \\ &\leq 2\Big(\frac{a_1(s)}{b_1(s)}\Big)^2\Bigg(\Big(x_1(s) - x_2(s)\Big)^2 + \Big(y_1(s) - y_2(s)\Big)^2\Bigg) \\ &\leq 2\Big(\frac{A_1}{m_1}\Big)^2 d_2^2(X_1(s), X_2(s)), \end{aligned}$$

where $A_1 = \sup\limits_{s\in\mathbb{Z}} a_1(s)$. If $x_2 \neq 0$, then we obtain

$$\begin{aligned} I_1 &\leq 2\Big(\frac{a_1(s)}{b_1(s)}\Big)^2\Big(x_1(s) - x_2(s)\Big)^2 \\ &+ 2(a_1(s)x_2(s))^2\Big(\frac{c_1(s)\Big(x_2(s) - x_1(s)\Big) + p(s)\Big(y_2(s) - y_1(s)\Big)}{b_1(s)c_1(s)x_2(s)}\Big)^2 \\ &\leq 6\Big(\frac{a_1(s)}{b_1(s)}\Big)^2\Big(x_1(s) - x_2(s)\Big)^2 + 4\Big(\frac{p(s)}{b_1(s)c_1(s)}\Big)^2\Big(y_1(s) - y_2(s)\Big)^2 \\ &\leq 6\Big(\frac{A_1}{m_1}\Big)^2\Big(x_1(s) - x_2(s)\Big)^2 + 4\Big(\frac{P}{m_1\alpha_1}\Big)^2\Big(y_1(s) - y_2(s)\Big)^2 \\ &\leq \widetilde{M}\Bigg(\Big(x_1(s) - x_2(s)\Big)^2 + \Big(y_1(s) - y_2(s)\Big)^2\Bigg) \\ &\leq \widetilde{M} d_2^2(X_1(s), X_2(s)), \end{aligned}$$

where $A_1 = \sup\limits_{s\in\mathbb{Z}} a_1(s)$, $P = \sup\limits_{s\in\mathbb{Z}} p(s)$ and $\widetilde{M} = \max\Bigg(6\Big(\frac{A_1}{m_1}\Big)^2, 4\Big(\frac{P}{m_1\alpha_1}\Big)^2\Bigg)$. In view of the above, letting $L_1 = \widetilde{M}$, one obtains

$$I_1 \leq L_1 d_2^2(X_1(s), X_2(s)).$$

Similarly, one can obtain that

$$I_2 \leq L_2 d_2^2(X_1(s), X_2(s)),$$

where $L_2 = \max\Bigg(6\Big(\frac{A_2}{m_2}\Big)^2, 4\Big(\frac{Q}{m_2\alpha_2}\Big)^2\Bigg)$ with $A_2 = \sup\limits_{s\in\mathbb{Z}} a_2(s)$ and $Q = \sup\limits_{s\in\mathbb{Z}} q(s)$. Consequently,

$$\begin{aligned} d_2^2(F(s, X_1(s)), F(s, X_2(s))) &= I_1 + I_2 \\ &\leq L d_2^2(X_1(s), X_2(s)), \end{aligned}$$

where $L = \max\Big(L_1, L_2\Big)$. In view of the above, it follows that F is a Lipschitzian function with Lipschitz constant $\sqrt{L}$. To complete the proof, it remains to show that Ψ has a fixed-point. Indeed, for all $X_1, X_2 \in AA(\mathbb{Z})$ and $t \in \mathbb{Z}$,

$$\begin{aligned} d_2\Big(\Psi X_1(t), \Psi X_2(t)\Big) &\leq \sum_{r=-\infty}^{t-1} d_2\Big(X(t,r+1)F(r,X_1(r)), X(t,r+1)F(r,X_2(r))\Big) \\ &\leq \left(\sum_{r=-\infty}^{t-1} \sqrt{L}\theta(t,r+1)\right) d_\infty\Big(X_1, X_2\Big) \\ &\leq \theta_0\sqrt{L} d_\infty\Big(X_1, X_2\Big) \end{aligned}$$

and hence

$$d_\infty\Big(\Psi X_1, \Psi X_2\Big) \leq \theta_0\sqrt{L} d_\infty\Big(X_1, X_2\Big).$$

Consequently, if $\sqrt{L} < \theta_0^{-1}$, then Ψ is a strict contraction. Therefore, using the Banach fixed-point principle it follows that it has a fixed point which constitutes the only almost automorphic solutions to (1.2). This completes the proof. □

4 Leslie–Gower Competition Model with Survival Rate

We now go back to the study of the Leslie–Gower competition model with survival rate in an almost automorphic environment. For that, we consider 1.2 in which we let,

$$a(t) = \gamma_{11}(t),\ \ b(t) = \gamma_{12}(t),\ \ c(t) = \gamma_{21}(t),\ \ d(t) = \gamma_{22}(t),$$

$$a_1(t) = (1-\gamma_{11}(t))\mu_1 K_1(t),\ \ a_2(t) = (1-\gamma_{22}(t))\mu_2 K_2(t),\ \ b_1(t) = (1-\gamma_{11}(t))K_1(t),$$

$$b_2(t) = (1-\gamma_{22}(t))K_2(t),\ \ c_1(t) = \mu_1 - 1 + \gamma_{11}(t),\ \ c_2(t) = \mu_2 - 1 + \gamma_{22}(t),$$

and

$$A(t) = \begin{pmatrix} \gamma_{11}(t) & \gamma_{12}(t) \\ \gamma_{21}(t) & \gamma_{22}(t) \end{pmatrix}$$

for all $t \in \mathbb{Z}$.

Corollary 4.1. *Under previous assumptions on $A(\cdot)$, suppose that γ_{ij}, q, p and K_i for $i, j = 1, 2$, are almost automorphic such that there exist constants $K^i > 0$ satisfying $\inf_{t\in\mathbb{Z}} K_i(t) \geq K^i$, $\sup_{t\in\mathbb{Z}} \gamma_{ii}(t) < 1$ for $i = 1, 2$. Then* (1.1) *has a unique almost automorphic solution whenever $\sqrt{L} < \theta_0^{-1}$ where*

$$L = 2\max\left[3\Big(\frac{A_1}{m_1}\Big)^2, 3\Big(\frac{A_2}{m_2}\Big)^2, 2\Big(\frac{P}{m_1\alpha_1}\Big)^2, 2\Big(\frac{Q}{m_2\alpha_2}\Big)^2\right]$$

with $A_i = \sup_{t\in\mathbb{Z}} a_i(t)$, $m_i = K^i(1 - \sup_{t\in\mathbb{Z}} \gamma_{ii}(t))$, $\alpha_i = \mu_i - 1$, $P = \sup_{t\in\mathbb{Z}} p(t)$, $Q = \sup_{t\in\mathbb{Z}} q(t)$ *for* $i = 1, 2$.

5 Example

In this section we give an example to illustrate our abstract results.

Consider (1.1) in which, $\gamma_{11}(t) = 0.25 - 0.05\sin(t)$, $\gamma_{22}(t) = 0.21 - 0.05\sin(t)$, $\gamma_{21}(t) = 0.2 - 0.05\sin(t)$, $\gamma_{12}(t) = 0.3 - 0.05\sin(t)$, $\mu_1 = \mu_2 = 2$,

$$K_1(t) = 1 - 0.5\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}, \quad K_2(t) = 2 - \sin\frac{1}{2+\cos t+\cos\sqrt{3}t},$$

$$p(t) = \left|\sin\frac{R}{2+\cos t+\cos\sqrt{2}t}\right| \text{ and } q(t) = \left|\sin\frac{R}{2+\cos t+\cos\sqrt{5}t}\right| \text{ with } R > 0.$$

Let

$$A(t) = \begin{pmatrix} \gamma_{11}(t) & \gamma_{12}(t) \\ \gamma_{21}(t) & \gamma_{22}(t) \end{pmatrix} = \begin{pmatrix} 0.25 - 0.05\sin(t) & 0.3 - 0.05\sin(t) \\ 0.2 - 0.05\sin(t) & 0.21 - 0.05\sin(t) \end{pmatrix},$$

for $t \in \mathbb{Z}$. Using the fact that

$$\begin{aligned} ||A||_\infty &:= \sup_{t\in\mathbb{Z}}\left(\sup_{0\neq x\in\mathbb{R}^2}\frac{\|A(t)x\|_2}{\|x\|_2}\right) \\ &< 0.45 \\ &< 1, \end{aligned}$$

we deduce that

$$\begin{aligned} \theta_0 &:= \sup_{t\in\mathbb{Z}}\left(\sum_{s=-\infty}^{t-1}\left\|\prod_{r=s}^{t-1}A(r)\right\|\right) \\ &\leq \frac{1}{1-||A||_\infty} \end{aligned}$$

which clearly yields all assumptions of Corollary 4.1 are all fulfilled with $K^1 = 0.5$, $K^2 = 1$,

$$a_1(t) = R\mu_1(0.75 + 0.05\sin(t))\left(1 - 0.5\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) \leq 2.4R,$$

$$a_2(t) = R\mu_2(0.8 + 0.05\sin(t))\left(2 - \sin\frac{1}{2+\cos t+\cos\sqrt{3}t}\right) \leq 5.1R,$$

$$b_1(t) = (0.75 + 0.05\sin(t))\left(1 - 0.5\sin\frac{1}{2+\cos t+\cos\sqrt{2}t}\right) > 0,$$
$$b_2(t) = (0.8 + 0.05\sin(t))\left(2 - \sin\frac{1}{2+\cos t+\cos\sqrt{3}t}\right) > 0,$$
$$c_1(t) = 1.25 - 0.05\sin(t) > 0, \ \ c_2(t) = 1.2 - 0.05\sin(t) > 0.$$

It can be shown that, if $R < 0.032$, then Corollary 4.1 yields the existence, and the uniqueness of an almost automorphic solution to (1.1), with the above-mentioned coefficients.

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