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On the Solutions of a System of Nonlinear Difference Equations

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Abstract

We investigate the solutions to a certain system of nonlinear difference equations.

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1 Introduction

Consider the system of difference equations

$$\begin{cases} x_{n+1} = \dfrac{f(y_n)}{x_{n-1}}, \\ y_{n+1} = \dfrac{f(x_n)}{y_{n-1}} \quad \text{for } n \in \mathbb{N}_0, \end{cases} \tag{1.1}$$

where $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, and $y_0 = \mu$ are positive numbers.

Some papers on periodic and positive solutions to some difference equations are as follows. Ma and Lu studied in [3] the periodic solutions of second order nonlinear

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difference equations with singular ϕ-Laplacian Operator. Cengiz Cinar in [1] studied positive solutions of the system of difference equations

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}.$$

Grove and Ladas in [2] studied the existence and behavior of solutions to the rational system of difference equations

$$\begin{cases} x_{n+1} = \dfrac{a}{x_n} + \dfrac{b}{y_n}, \\ y_{n+1} = \dfrac{c}{x_n} + \dfrac{d}{y_n} \quad \text{for } n \in \mathbb{N}_0, \end{cases}$$

where $a, b, c, d \in \mathbb{R}$.

2 Assumptions

The function f will have one of the following forms:

$$f(z) = 1 \tag{2.1}$$

$$f(z) = \begin{cases} A, & \text{if } z > 0 \\ B, & \text{if } z < 0 \end{cases} \tag{2.2}$$

$$f(z) = \begin{cases} Az, & \text{if } z < 0 \\ Bz, & \text{if } z > 0 \end{cases} \tag{2.3}$$

where A, $B \in \mathbb{R}$ such that $A^2 + B^2 \neq 0$.

3 Main Results

Theorem 3.1. *Let* (2.1) *hold and suppose that* x_{-1}, y_{-1}, x_0, *and* y_0 *are positive real numbers. Also, let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, *and* $y_0 = \mu$. *Then all solutions of* (1.1) *are of the following:*

$$x_{4n+1} = \frac{1}{\alpha}, \quad y_{4n+1} = \frac{1}{\beta}$$

$$x_{4n+2} = \frac{1}{\lambda}, \quad y_{4n+2} = \frac{1}{\mu}$$

$$x_{4n+3} = \alpha, \quad y_{4n+3} = \beta$$

$$x_{4n+4} = \lambda, \quad y_{4n+4} = \mu.$$

Proof. The result holds for $n = 0$. Now suppose the result is true for some $k > 0$. Then we have the following:

$$x_{4k+1} = \frac{1}{\alpha}, \quad y_{4k+1} = \frac{1}{\beta}$$
$$x_{4k+2} = \frac{1}{\lambda}, \quad y_{4k+2} = \frac{1}{\mu}$$
$$x_{4k+3} = \alpha, \quad y_{4k+3} = \beta$$
$$x_{4k+4} = \lambda, \quad y_{4k+4} = \mu.$$

Also, for $k+1$, we have the following:

$$x_{4k+5} = \frac{f(y_{4k+4})}{x_{4k+3}} = \frac{f(\mu)}{\alpha} = \frac{1}{\alpha}, \quad y_{4k+5} = \frac{f(x_{4k+4})}{y_{4k+3}} = \frac{f(\lambda)}{\beta} = \frac{1}{\beta}$$

$$x_{4k+6} = \frac{f(y_{4k+5})}{x_{4k+4}} = \frac{f\left(\frac{1}{\beta}\right)}{\lambda} = \frac{1}{\lambda}, \quad y_{4k+6} = \frac{f(x_{4k+5})}{y_{4k+4}} = \frac{f\left(\frac{1}{\alpha}\right)}{\mu} = \frac{1}{\mu}$$

$$x_{4k+7} = \frac{f(y_{4k+6})}{x_{4k+5}} = \frac{f\left(\frac{1}{\mu}\right)}{\lambda} = \frac{1}{\frac{1}{\alpha}} = \alpha, \quad y_{4k+7} = \frac{f(x_{4k+6})}{y_{4k+5}} = \frac{f\left(\frac{1}{\lambda}\right)}{\frac{1}{\beta}} = \beta$$

$$x_{4k+8} = \frac{f(y_{4k+7})}{x_{4k+6}} = \frac{f(\beta)}{\frac{1}{\beta}} = \frac{1}{\frac{1}{\lambda}} = \lambda, \quad y_{4k+8} = \frac{f(x_{4k+7})}{y_{4k+6}} = \frac{f(\alpha)}{\frac{1}{\mu}} = \frac{1}{\frac{1}{\mu}} = \mu.$$

Therefore the result is true for every $k \in \mathbb{N}_0$. This concludes the proof. □

Theorem 3.2. *Suppose that* (2.1) *hold and let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1)*. Also, assume that* x_{-1}*,* y_{-1}*,* x_0*, and* y_0 *are positive real numbers. Then all solutions of* (1.1) *are periodic with period four.*

Proof. By (2.1), we have the following equal:

$$x_{n+1} = \frac{f(y_n)}{x_{n-1}} = \frac{1}{x_{n-1}}$$

$$y_{n+1} = \frac{f(x_n)}{y_{n-1}} = \frac{1}{y_{n-1}}$$

$$x_{n+2} = \frac{f(y_{n+1})}{x_n} = \frac{1}{x_n}$$

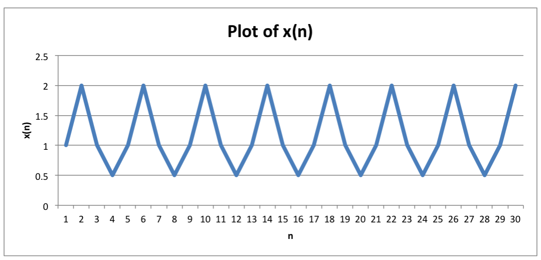
$$y_{n+2} = \frac{f(x_{n+1})}{y_n} = \frac{1}{y_n}$$

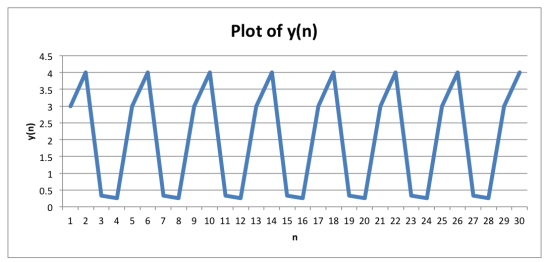
$$x_{n+3} = \frac{f(y_{n+2})}{x_{n+1}} = \frac{1}{x_{n+1}} = \frac{x_{n-1}}{f(y_n)} = x_{n-1}$$

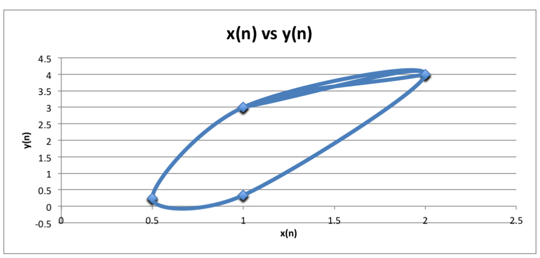
$$y_{n+3} = \frac{f(x_{n+2})}{y_{n+1}} = \frac{1}{y_{n+1}} = \frac{y_{n-1}}{f(x_n)} = y_{n-1}$$
$$x_{n+4} = \frac{f(y_{n+3})}{x_{n+2}} = \frac{1}{x_{n+2}} = \frac{x_n}{f(y_{n+1})} = x_n$$
$$y_{n+4} = \frac{f(x_{n+3})}{y_{n+2}} = \frac{1}{y_{n+2}} = \frac{y_n}{f(x_{n+1})} = y_n.$$

This concludes the proof. □

To see the periodic behavior of $\{x_n, y_n\}$, observe the following three diagrams with $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$:







Theorem 3.3. *Let* (2.2) *hold with* $A, B < 0$ *and suppose that* x_{-1}, y_{-1}, x_0, *and* y_0 *are positive real numbers. Also, let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $x_0 = \lambda$, *and* $y_0 = \mu$. *Then all solutions of* (1.1) *are the following:*

$$x_{4n+1} = \frac{A}{\alpha}\left(\frac{A}{B}\right)^n, \quad y_{4n+1} = \frac{A}{\beta}\left(\frac{A}{B}\right)^n$$
$$x_{4n+2} = \frac{B}{\lambda}\left(\frac{B}{A}\right)^n, \quad y_{4n+2} = \frac{B}{\mu}\left(\frac{B}{A}\right)^n$$
$$x_{4n+3} = \alpha\left(\frac{B}{A}\right)^n, \quad y_{4n+3} = \beta\left(\frac{B}{A}\right)^n$$
$$x_{4n+4} = \lambda\left(\frac{A}{B}\right)^n, \quad y_{4n+4} = \mu\left(\frac{A}{B}\right)^n.$$

Proof. The result follows by the principle of mathematical induction. □

Corollary 3.4. *If* $A \neq B$ *and* $A, B < 0$*, then the solutions of* (1.1) *are oscillatory and nonperiodic.*

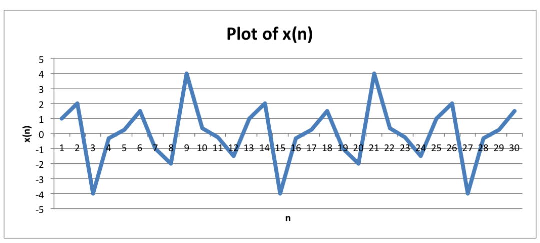
Theorem 3.5. *Suppose* (2.3) *hold and let* $\{x_n, y_n\}$ *be a solution of the system of equations* (1.1)*. Also, assume that* x_{-1}, y_{-1}, x_0*, and* y_0 *are positive real numbers with* $A > 0$ *and* $B < 0$*. Then all solutions of* (1.1) *are periodic with period twelve.*

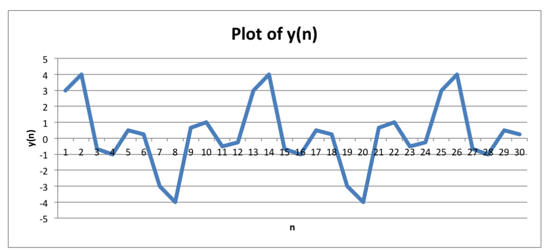
Proof. Let $(\cdot, \cdot)$ be the pair of solutions of (1.1), then the following set

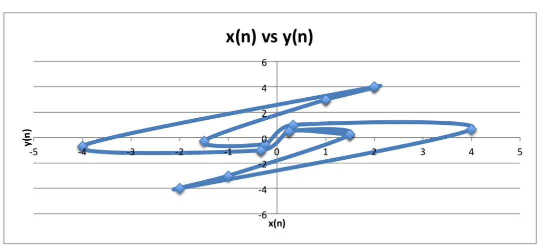
$$\Big\{ (\alpha, \beta), (\lambda, \mu) \left(\frac{B\mu}{\alpha}, \frac{B\lambda}{\beta}\right), \left(\frac{AB}{\beta}, \frac{AB}{\alpha}\right), \left(\frac{A^2}{\mu}, \frac{A^2}{\lambda}\right), \left(\frac{\beta A}{\lambda}, \frac{\alpha A}{\mu}\right),$$
$$\left(\frac{\alpha B}{A}, \frac{\beta B}{A}\right), \left(\frac{\lambda B}{A}, \frac{\mu B}{A}\right), \left(\frac{A\mu}{\alpha}, \frac{A\lambda}{\beta}\right), \left(\frac{A^2}{\beta}, \frac{A^2}{\alpha}\right), \left(\frac{AB}{\mu}, \frac{AB}{\lambda}\right), \left(\frac{B\beta}{\lambda}, \frac{B\alpha}{\mu}\right),$$
$$(\alpha, \beta), (\lambda, \mu), \left(\frac{B\mu}{\alpha}, \frac{B\lambda}{\beta}\right), \ldots \Big\}$$

is periodic with period twelve. This concludes the proof. □

To see the periodic and oscillatory behavior of $\{x_n, y_n\}$, observe the following three diagrams with $A = 1$, $B = -1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$:







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