Asymptotic Stability of a Discrete Version of the Heavy Ball with Friction Dynamical System

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Abstract

In this paper boundedness and asymptotic behavior of a discrete version of nonlinear heavy ball with friction dynamical system is studied. Our results extend the previous results of the first author [7] to the nonhomogeneous case and for more general assumptions on the parameters.

AMS Subject Classifications: 47H05, 39A12.
Keywords: Difference equation, discrete version, maximal monotone operator, boundedness, asymptotic behavior, stability.

1 Introduction

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( |\cdot| \). We denote weak convergence in \( H \) by \( \rightharpoonup \) and strong convergence by \( \rightarrow \). Let \( A \) be a nonempty subset of \( H \times H \) to which we shall refer as a (nonlinear) possibly multivalued operator in \( H \). \( A \) is called monotone (resp. strongly monotone) iff \( \langle y_2 - y_1, x_2 - x_1 \rangle \geq 0 \) (resp. \( \langle y_2 - y_1, x_2 - x_1 \rangle \geq \alpha |x_2 - x_1|^2 \) for some \( \alpha > 0 \)\) for all \( [x_i, y_i] \in A, i = 1, 2 \). \( A \) is called maximal monotone if \( A \) is monotone and \( R(I + A) = H \), where \( I \) is the identity operator of \( H \). Given any function \( \varphi : H \rightarrow ]-\infty, +\infty[ \) (not necessarily convex) with domain \( D(\varphi) \), its subdifferential is the multivalued operator \( \partial \varphi \), defined as

\[
\partial \varphi(x) := \{ w \in H \mid \varphi(x) - \varphi(y) \leq \langle w, x - y \rangle, \ \forall y \in H \}.
\]

The function \( \varphi \) is called proper iff \( \varphi \neq +\infty \). It is a well-known result that if \( \varphi \) is a proper, convex, and lower semicontinuous function, then \( \partial \varphi \) is a maximal monotone
operator. We refer the reader to the book by Morosanu [9] in order to understand monotone operators and subdifferential of convex functions in Hilbert spaces.

Let $A$ be a maximal monotone operator on a real Hilbert space $H$ and $\gamma$ a positive real constant. The following second order dissipative system of maximal monotone type

\[
\begin{aligned}
    u''(t) + \gamma u'(t) + Au(t) &\ni 0, \\
    u(0) = u_0, \quad u'(0) = u_1,
\end{aligned}
\]  

(1.1)

is called heavy ball with friction dynamical system, because when $A = \nabla \varphi$ (the gradient of $\varphi$), this system is a model for damping oscillation of a heavy ball on the graph of $\varphi$. The asymptotic behavior of (1.1) and its discrete version at infinity is a subject of many recent investigations. Attouch and Alvarez [3], Alvarez [2] and Attouch, Goudou and Redont [4] studied the dynamical system (1.1), when $A = \nabla \varphi$, where $\varphi$ is a continuously differentiable and convex function on $H$. When $A = \nabla \varphi$, equation (1.1) provides a dynamical approach to optimization problems, because the solution of (1.1) converges weakly to a minimum point of $\varphi$. To the best of our knowledge, the problem of convergence of solutions to (1.1) for general maximal monotone operator $A$ is still open. For numerical and practical purposes, as well as to get an algorithm for approximation of a zero of a maximal monotone operator or a minimum point of a convex function (when $A = \partial \varphi$), it is useful to consider the discrete version of (1.1). In [2] Alvarez using the following approximations for $u'$ and $u''$:

\[
\begin{aligned}
    u'(t) &= \frac{u(t+h) - u(t)}{h} + O(h), \\
    u''(t) &= \frac{u(t+h) - 2u(t) + u(t-h)}{h} + O(h^2),
\end{aligned}
\]  

(1.2) (1.3)

obtained the following discretization of (1.1).

\[
u_n - u_{n+1} + \alpha_n(u_n - u_{n-1}) \in \lambda_{n+1}Au_{n+1},
\]  

(1.4)

where $0 \leq \alpha_n \leq 1$ and $\lambda_n$ is a positive sequence. When $A = \nabla \varphi$, Alvarez proved convergence of the sequence $u_n$ to a minimum point of $\varphi$. Jules and Maingé [6] considered the iterative method (1.4) for a co-coercive operator $A$ and obtained the weak convergence of the sequence $u_n$ to an element of $A^{-1}(0)$. They also showed that (1.4) has a better rate of convergence than the standard proximal point algorithm (when $\alpha_n \equiv 0$). Alvarez and Attouch [1] obtained the weak convergence of $u_n$ given by (1.4) for general maximal monotone operators with appropriate assumptions on $\lambda_n$ and $\alpha_n$.

In [7] the first author replaced the approximation (1.2) by the following approximation for $u'(t)$

\[
u'(t) = \frac{u(t+h) - u(t-h)}{2h} + O(h^2),
\]  

(1.5)

which is better than (1.2) and obtained the following discretization of (1.1).
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Given a bounded sequence \( \{u_n\} \) based on the resolvent operator as follows

\[
\text{Definition 1.1.} \quad u_{n+1} = \lambda_{n+1} (1 - \alpha_n) u_n + \alpha_n u_{n-1}, \quad u_0, u_1 \in H,
\]

where \( \alpha_n \) (resp. \( \lambda_n \)) is a nonnegative (resp. positive) sequence and \( A \) is a maximal monotone operator. The difference inclusion (1.6) can be rewritten in explicit form based on the resolvent operator as follows

\[
u_{n+1} = J_{\lambda_{n+1}} ((1 - \alpha_n) u_n + \alpha_n u_{n-1}).
\]

It seems that the difference inclusion (1.6) is more stable than (1.4), because if \( \{u_n\} \) is the solution of (1.6) (equivalently (1.7)) with initial data \( u_0, u_1 \), and \( u'_n \) is the solution of (1.6) with initial data \( u'_0, u'_1 \), then by (1.7) since \( J_{\lambda} \) is nonexpansive (see [9, Theorem 1.3 on page 21]) we have:

\[
\begin{align*}
|u_{n+1} - u'_{n+1}| &\leq (1 - \alpha_n) |u_n - u'_n| + \alpha_n |u_{n-1} - u'_{n-1}| \\
&\leq \text{Max} \{|u_n - u'_n|, |u_{n-1} - u'_{n-1}|\} \leq \ldots \leq \text{Max} \{|u_1 - u'_1|, |u_0 - u'_0|\}.
\end{align*}
\]

In this paper, we consider the following nonhomogeneous case of (1.6)

\[
\begin{cases}
(1 - \alpha_n) u_n + \alpha_n u_{n-1} - u_{n+1} + e_n \in \lambda_{n+1} Au_{n+1} \\
u_0, u_1 \in H,
\end{cases}
\]

where \( e_n \) is the error sequence in \( H \). We extend and improve the results of [7] with more general assumptions on the parameters \( \alpha_n, \lambda_n \) and the error \( e_n \).

In Section 2, we prove the boundedness of the sequence \( \{u_n\} \) for coercive maximal monotone operators. We also show the relation between the boundedness of \( \{u_n\} \) and the assumption \( A^{-1}(0) \neq \emptyset \). Section 3 is devoted to the weak convergence of the bounded sequence \( \{u_n\} \) given by (1.8) and its weighted average. In Section 4, we consider the weak convergence and the rate of convergence in the sub-differential case. Under suitable assumptions on the parameters and the operator \( A \), the strong convergence of the algorithm is established in Section 5. We denote the weighted average of the sequence \( u_n \) by \( w_n := \left( \sum_{k=1}^{n} \lambda_k \right)^{-1} \left( \sum_{k=1}^{n} \lambda_k u_k \right) \), and the element \( \frac{(1 - \alpha_n) u_n + \alpha_n u_{n-1} - u_{n+1} + e_n}{\lambda_{n+1}} \) by \( Au_{n+1} \) for simplicity.

**Definition 1.1.** Given a bounded sequence \( \{u_n\} \) in \( H \), the asymptotic center \( c \) of \( \{u_n\} \) is defined as follows (see [5]): for every \( q \in H \), let \( \varphi(q) = \lim_{n \to +\infty} \sup |u_n - q|^2 \). Then \( \varphi \) is a continuous and strictly convex function on \( H \), satisfying \( \varphi(q) \to +\infty \) as \( |q| \to +\infty \). Thus \( \varphi \) achieves its minimum on \( H \) at a unique point \( c \), called the asymptotic center of the sequence \( \{u_n\} \).

Throughout the paper, we assume \( 0 \leq \alpha_n \leq 1 \), for all \( n > 0 \).
2 Boundedness

In this section, we study the boundedness of the sequence \( \{u_n\} \) generated by (1.8). We present the relation between the boundedness of \( \{u_n\} \) and the assumption \( A^{-1}(0) \neq \emptyset \).

**Theorem 2.1.** Let \( A \) be a coercive maximal monotone operator. If the sequence \( \left\{ \frac{|e_n|}{\lambda_{n+1}} \right\} \) is bounded, then the sequence \( \{u_n\} \) is bounded.

**Proof.** Let \( M > 0 \) be such that for each \( n \geq 1 \), \( \frac{|e_n|}{\lambda_{n+1}} < M \). By coerciveness of \( A \), there exist \( K > 0 \) and \( y_0 \in H \) such that for all \( |x, y| \in A \), with \( |x - y_0| > K \), \( \frac{y, x - y_0}{|x - y_0|} > M \). If there exists \( n \) such that \( |u_{n+1} - y_0| > K \), using (1.8), we get

\[
\lambda_{n+1}M|u_{n+1} - y_0| \leq \lambda_{n+1}Au_{n+1}, u_{n+1} - y_0
\]

\[
= \frac{(1 - \alpha_n)|u_n + \alpha_n u_{n-1} + e_n - u_{n+1}, u_{n+1} - y_0|}{\lambda_{n+1}}
\]

\[
< (1 - \alpha_n)(u_n - y_0) + \alpha_n(u_{n-1} - y_0) + e_n, u_{n+1} - y_0 \geq -|u_{n+1} - y_0|^2
\]

\[
\leq |(1 - \alpha_n)|u_n - y_0| + \alpha_n|u_{n-1} - y_0| + e_n||u_{n+1} - y_0| - |u_{n+1} - y_0|^2
\]

Thus

\[
|u_{n+1} - y_0| \leq (1 - \alpha_n)|u_n - y_0| + \alpha_n|u_{n-1} - y_0| + \lambda_{n+1}\frac{|e_n|}{\lambda_{n+1}} - M
\]

\[
\leq (1 - \alpha_n)|u_n - y_0| + \alpha_n|u_{n-1} - y_0| \leq \max\{|u_n - y_0|, |u_{n-1} - y_0|\}
\]

\[
\leq \ldots \leq \max\{|u_1 - y_0|, |u_0 - y_0|, K\}
\]

Hence, for all \( n \geq 1 \), \( |u_{n+1} - y_0| \leq \max\{|u_1 - y_0|, |u_0 - y_0|, K\} \).

**Lemma 2.2.** Suppose that \( a_n \) and \( b_n \) are nonnegative real sequences and that \( \sum_{n=1}^{\infty} b_n < +\infty \). If \( \alpha_{n+1} \leq \alpha_n + b_n \), for all \( n \geq 1 \), then \( \lim_n a_n \) exists.

**Lemma 2.3.** Suppose \( \{\alpha_n\} \) is a sequence in \([0, 1]\) and \( \psi_n \) and \( \delta_n \) are positive real sequences. Suppose that \( \psi_{n+1} \leq (1 - \alpha_n)\psi_n + \alpha_n\psi_{n-1} + \delta_n \) and \( \sum_{n=1}^{\infty} \delta_n < +\infty \). If one of the following conditions holds:

(i) \( \lim_n \alpha_n = \alpha < 1 \), (ii) \( \sum_{n=1}^{\infty} [\alpha_n - \alpha_{n-1}]_+ < +\infty \), where \( [\alpha]_+ := \max\{\alpha, 0\} \), then \( \lim_n \psi_n \) exists.

**Proof.** (i) Set \( \phi_n := \max\{\psi_n, \psi_{n-1}\} \). By assumption, we have \( \psi_{n+1} \leq \phi_n + \delta_n \). On the other hand \( \psi_n \leq \phi_n + \delta_n \), so, \( \phi_{n+1} \leq \phi_n + \delta_n \). Using Lemma 2.2, \( \lim \phi_n \) exists. Taking limsup from both sides of \( \psi_{n+1} \leq \phi_n + \delta_n \), we get \( \limsup_n \psi_n \leq \lim_n \phi_n \). On the other hand
Remark

Suppose that Theorem 2.4.

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\( \psi_{n+1} \leq (1 - \alpha_n) \psi_n + \alpha_n \phi_n + \delta_n \) and also \( \psi_n \leq (1 - \alpha_n) \psi_n + \alpha_n \phi_n + \delta_n \). So \( \phi_{n+1} \leq (1 - \alpha_n) \psi_n + \alpha_n \phi_n + \delta_n \). Taking \( \lim \inf \) as \( n \to \infty \), we get \( \lim \phi_n \leq \lim \inf \psi_n \).

Hence, \( \lim \psi_n \) exists and \( \lim \psi_n = \lim \phi_n \). (ii) By the proof of part (i), \( \psi_n \) is bounded. From the assumption on \( \psi_n \), we have

\[
\psi_{n+1} \leq \psi_n + \alpha_{n-1} \psi_{n-1} - \alpha_n \psi_n + [\alpha_n - \alpha_{n-1}] \psi_{n-1} + \delta_n.
\]

Now the lemma is proved by Lemma 2.2.

\[
\square
\]

**Theorem 2.4.** Suppose that \( A^{-1}(0) \neq \emptyset \) and \( (E_1) \sum_{n=1}^{+\infty} |e_n| < +\infty \). Then:

1. The sequence \( \{u_n\} \) is bounded.

2. If \( (\alpha_1) \sum_{n=1}^{+\infty} [\alpha_n - \alpha_{n-1}]_+ < +\infty \), where \( [\alpha]_+ = \max\{0, \alpha\} \) or \( (\alpha_2) \lim_{n} \alpha_n = \alpha < 1 \), then \( \lim_{n} |u_n - p| \) exists, for each \( p \in A^{-1}(0) \).

**Proof.** Set \( p \in A^{-1}(0) \). First, we prove (1). By the nonexpansivity of the resolvent operator, we have

\[
|u_{n+1} - p| \leq |(1 - \alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p) + e_n| \\
\leq (1 - \alpha_n)|u_n - p| + \alpha_n|u_{n-1} - p| + |e_n| \leq \max\{|u_n - p|, |u_{n-1} - p|\} + |e_n|
\]

\[
\leq \ldots \leq \max\{|u_1 - p|, |u_0 - p|\} + \sum_{n=1}^{+\infty} |e_n| < \infty.
\]

Let us prove (2). By the monotonicity of \( A \), we have

\[
0 \leq \lambda_{n+1} Au_{n+1}, u_{n+1} - p \Rightarrow (1 - \alpha_n)u_n + \alpha_nu_{n-1} + e_n - u_{n+1}, u_{n+1} - p > 0 \\
= (1 - \alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p) + e_n, u_{n+1} - p > -|u_{n+1} - p|^2.
\]

So

\[
|u_{n+1} - p| \leq |(1 - \alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p) + e_n| \\
\leq (1 - \alpha_n)|u_n - p| + \alpha_n|u_{n-1} - p| + |e_n|.
\]

Now the theorem is proved by Lemma 2.3.

\[
\square
\]

**Remark 2.5.** Conditions \( (\alpha_1) \) and \( (\alpha_2) \) are different as the following examples show.

1. If \( \alpha_n = \begin{cases} 1, & n = k^2 \\ 1 - \frac{1}{n}, & n \neq k^2 \end{cases} \), then \( \sum_{n=1}^{+\infty} [\alpha_n - \alpha_{n-1}]_+ < +\infty \), but \( \lim \alpha_n = 1 \).

2. If \( \alpha_n = \begin{cases} \alpha, & n \text{ is even} \\ \alpha - \frac{1}{n}, & n \text{ is odd} \end{cases} \), where \( 0 \leq \alpha < 1 \), then \( \lim_{n} \alpha_n = \alpha < 1 \) and \( \sum_{n=1}^{+\infty} [\alpha_n - \alpha_{n-1}]_+ = +\infty \). Therefore, the assumptions \( (\alpha_1) \) and \( (\alpha_2) \) in Theorem 2.4 are different. Also the condition \( \sum_{n=1}^{+\infty} [\alpha_n - \alpha_{n-1}]_+ < +\infty \) is better than the condition
\[
\sum_{n=1}^{+\infty} |\alpha_n - \alpha_{n-1}| < +\infty \quad \text{assumed in [7]. Take } \alpha_n = \begin{cases} 
\alpha_{n-1} + \frac{1}{y}, & n = k^2 \\
\alpha_{n-1} - \frac{1}{n}, & n \neq k^2
\end{cases}
\]
for \( n \geq 2 \) and \( \alpha_1 = 1. \) Then \( \sum_{n=1}^{+\infty} |\alpha_n - \alpha_{n-1}| = +\infty \) but \( \sum_{n=1}^{+\infty} [\alpha_n - \alpha_{n-1}]_+ < +\infty. \)

**Lemma 2.6** (See [8]). Suppose that \( \{\alpha_n\} \) is a nonnegative sequence and \( \{\lambda_n\} \) is a positive sequence such that \( \sum_{n=1}^{+\infty} \lambda_n = +\infty. \) If \( \frac{\alpha_n}{\lambda_n} \to 0 \) as \( n \to +\infty, \) then \( \sum_{k=1}^{n} \alpha_k \to 0 \) as \( n \to +\infty. \)

**Theorem 2.7.** Suppose that \( \{u_n\} \) is a bounded sequence given by (1.8) and the following conditions are satisfied,

(i) \( (A_1) \sum_{n=1}^{+\infty} \lambda_n = +\infty, \)

(ii) \( (\alpha_1) \) or \( (\alpha_3) \frac{[\alpha_n - \alpha_{n-1}]_+}{\lambda_{n+1}} \to 0, \)

(iii) \( (E_1) \) or \( (E_2) \frac{|e_n|}{\lambda_{n+1}} \to 0. \)

Then \( A^{-1}(0) \neq \emptyset \) and \( \omega_w(w_n) \subset A^{-1}(0), \) where \( \omega_w(w_n) \) is the set of weak cluster points of \( w_n. \)

**Proof.** Suppose \( [x, y] \in A. \) Since \( \{u_n\} \) is bounded, there is a subsequence \( \{w_n\} \) of \( \{w_n\} \) such that \( w_{n_j} \to p \in H. \) On the other hand, by the monotonicity of \( A, \) we get

\[
\lambda_{i+1} < x - u_{i+1}, y > = \lambda_{i+1} \left( < x - u_{i+1}, y - A u_{i+1} > + < x - u_{i+1}, A u_{i+1} > \right) \\
\geq < x - u_{i+1}, \lambda_{i+1} A u_{i+1} >
\]

\[
= < x - u_{i+1}, (1 - \alpha_i)u_i + \alpha_i u_{i-1} + e_i - u_{i+1} > \\
= < x - u_{i+1}, (1 - \alpha_i)(u_i - x) + \alpha_i(u_{i-1} - x) + e_i + x - u_{i+1} > \\
= \left[ |u_{i+1} - x|^2 - < u_{i+1} - x, (1 - \alpha_i)(u_i - x) + \alpha_i(u_{i-1} - x) > - < u_{i+1} - x, e_i > \right] \\
\geq \frac{1}{2} |u_{i+1} - x|^2 - \frac{1}{2} |(1 - \alpha_i)(u_i - x) + \alpha_i(u_{i-1} - x)|^2 - |u_{i+1} - x||e_i| \\
\geq \frac{1}{2} |u_{i+1} - x|^2 - \frac{1}{2} |(1 - \alpha_i)|u_i - x|^2 - \frac{1}{2} \alpha_i |u_{i-1} - x|^2 - |u_{i+1} - x||e_i| \\
= \frac{1}{2} |u_{i+1} - x|^2 - |u_i - x|^2 + \frac{1}{2} \alpha_i |u_i - x|^2 - |u_{i-1} - x|^2 - |e_i||u_{i+1} - x| \\
\geq \frac{1}{2} |u_{i+1} - x|^2 - |u_i - x|^2 + \frac{1}{2} \alpha_i |u_i - x|^2 - |u_{i-1} - x|^2 - |u_{i-1} - x|^2[\alpha_i - \alpha_{i-1}]_+ - |e_i||u_{i+1} - x|
\]
Suppose $\lambda_{n+1} = \frac{1}{\lambda_{n+1}}$, we get
\[
\langle x - w_{n_j}, y \rangle \geq \langle x - (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} \lambda_{i+1} u_{i+1}, y \rangle
\]
\[
\geq (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \left[ -\frac{1}{2} |u_0 - x|^2 - \frac{1}{2} \alpha_{-1} |u_{-1} - x|^2 \right]
\]
\[
- \frac{1}{2} \sum_{i=0}^{n_j-1} |u_{i-1} - x|^2 [\alpha_i - \alpha_{i-1}]_+ - \sum_{i=0}^{n_j-1} e_i |u_{i+1} - x|}
\]
\[
\geq (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \left[ -\frac{1}{2} |u_0 - x|^2 - \frac{1}{2} M^2 \sum_{i=0}^{n_j-1} [\alpha_i - \alpha_{i-1}]_+ - M \sum_{i=0}^{n_j-1} e_i \right],
\]
where $M = \sup_n |u_n - x|$ (in the last inequality, we take $\alpha_{-1} = 0$ and $u_{-1} = 0$). Letting $j \to \infty$, by Lemma 2.6, we get $< x - p, y > \geq 0$. Thus, by the maximality of $A$, we have $p \in A^{-1}(0)$, as desired.

\[\square\]

### 3 Weak Convergence

In this section, we prove the weak convergence of the sequence $\{u_n\}$ and its weighted average to a zero of $A$, which extend the results of [7], under suitable assumptions on the parameters $\alpha_n$ and $\lambda_n$.

**Theorem 3.1.** Suppose $\{u_n\}$ is a bounded sequence generated by (1.8) and conditions $(\Lambda_1)$, $(\alpha_1)$ and $(E_1)$ are satisfied. Then $w_n \to p \in A^{-1}(0)$ as $n \to \infty$, which is also an asymptotic center of $\{u_n\}$.

**Proof.** By Theorem 2.7, $A^{-1}(0) \neq \emptyset$ and $\omega(w_n) \subset A^{-1}(0)$. Thus by part (2) of Theorem 2.4, $\lim_n |u_n - p|$, exists for each $p \in \omega(w_n)$. We show that $\omega(w_n)$ is singleton. Suppose $p, q \in \omega(w_n)$ and $p \neq q$, then by Theorem 2.4, $\lim_n (|u_n - p|^2 - |u_n - q|^2)$ exists and hence $\lim_n < u_n, p - q >$ exists. Thus $\lim_n < w_n, p - q >$ exists. This implies that $< q, p - q > = < p, p - q >$ and hence $p = q$. So, $w_n \to p \in A^{-1}(0)$ as $n \to +\infty$. Now, we show that $p$ is the asymptotic center of $\{u_n\}$. Suppose that $q \in H$ and $q \neq p$, then
\[
|u_n - p|^2 = |u_n - q|^2 + 2 < u_n, q - p > + |p|^2 - |q|^2.
\]
Multiplying both sides of the above equality by \( \lambda_n \), summing up from \( n = 1 \) to \( n = m \), dividing by \( \sum_{n=1}^{m} \lambda_n \) and taking \( \lim \sup \) as \( m \to +\infty \), we get

\[
\lim_{n \to +\infty} |u_n - p|^2 = \lim_{m \to +\infty} \sup \left( \sum_{n=1}^{m} \lambda_n \right)^{-1} \left( \sum_{n=1}^{m} \lambda_n |u_n - q|^2 \right) - |q - p|^2 < \lim_{n \to +\infty} |u_n - q|^2.
\]

This shows that \( p \) is the asymptotic center of the sequence \( \{u_n\} \) as desired.

\( \Box \)

**Theorem 3.2.** Let \( \{u_n\} \) be a bounded sequence given by (1.8).

1. If the following conditions hold:

\[
\begin{align*}
(\Lambda_2) & \sum_{n=1}^{+\infty} \lambda_n^2 = +\infty, \\
(\alpha_1) & \text{ or } (\alpha_4) \frac{\alpha_n - \alpha_{n-1}}{\lambda_{n+1}^2} \to 0, \\
(E_1) & \text{ or } (E_3) \frac{e_n}{\lambda_{n+1}} \to 0, \\
(\alpha_5) & \sum_{n=1}^{+\infty} \frac{\alpha_n^2}{\lambda_{n+1}^2} < +\infty \quad \text{and} \quad (E_4) \sum_{n=1}^{+\infty} \frac{|e_n|^2}{\lambda_{n+1}^2} < +\infty,
\end{align*}
\]

then \( A^{-1}(0) \neq \emptyset \) and \( \omega(u_n) \subset A^{-1}(0) \).

2. If conditions \( (\Lambda_2), (\alpha_1), (\alpha_5), (E_1) \) and \( (E_4) \) are satisfied, then \( u_n \to p \in A^{-1}(0) \).

**Proof.** By Theorem 2.7, \( A^{-1}(0) \neq \emptyset \). In order to prove (1), assume \( p \in A^{-1}(0) \). By the monotonicity of \( A \), \( \lambda_{n+1} A u_{n+1} \geq u_{n+1} \geq 0 \). So by (1.8),

\[
|p - u_{n+1}|^2 + \lambda_{n+1}^2 |Au_{n+1}|^2 \leq |(1 - \alpha_n)(u_n - p) + \alpha_n(u_n - u_{n-1}) + e_n|^2 \\
\leq |(1 - \alpha_n)(u_n - p) + \alpha_n(u_n - u_{n-1}) + e_n|^2 + |e_n|^2 + 2|e_n||(1 - \alpha_n)(u_n - p) + \alpha_n(u_n - u_{n-1})| \leq |u_n - p|^2 + \alpha_n(u_n - u_{n-1})^2 + |e_n|^2 + 2|e_n||(1 - \alpha_n)(u_n - p) + \alpha_n(u_n - u_{n-1})|.
\]

Therefore

\[
\begin{align*}
\lambda_{n+1}^2 |Au_{n+1}|^2 & \leq |u_n - p|^2 - |u_{n+1} - p|^2 + \alpha_n|u_n - u_{n-1}|^2 + |e_n|^2, \\
-\alpha_n|u_n - p|^2 & + M^2[\alpha_n - \alpha_{n-1}] + |e_n|^2 + 2M|e_n|,
\end{align*}
\]

where \( M = \sup_n |u_n - p| \). On the other hand, by the monotonicity of \( A \), we have

\[
0 \leq \langle Au_{n+1} - Au_n, u_{n+1} - u_n \rangle = \langle Au_{n+1} - Au_n, \alpha_n(u_{n-1} - u_n) + e_n - \lambda_{n+1}Au_{n+1} \rangle,
\]

thus
\[ |Au_{n+1}|^2 \leq \langle Au_n, Au_{n+1} \rangle + \langle Au_{n+1} - Au_n, \alpha_n \frac{u_{n-1} - u_n}{\lambda_{n+1}} + \frac{e_n}{\lambda_{n+1}} \rangle \]
\[ \leq \frac{1}{2} |Au_n|^2 + \frac{1}{2} |Au_{n+1}|^2 - \frac{1}{2} |Au_n - Au_{n+1}|^2 + \frac{1}{2} |Au_n - Au_{n+1}|^2 \]
\[ + \frac{1}{2} \alpha_n \left( u_{n-1} - u_n \right) + \frac{e_n}{\lambda_{n+1}} \]
\[ \leq \frac{1}{2} |Au_n|^2 + \frac{1}{2} |Au_{n+1}|^2 + \frac{\alpha_n}{\lambda_{n+1}} |u_{n-1} - u_n|^2 + \frac{|e_n|^2}{\lambda_{n+1}}. \]

Therefore

\[ |Au_{n+1}|^2 \leq |Au_n|^2 + \frac{1}{2} \alpha_n |u_{n-1} - u_n|^2 + \frac{2|e_n|^2}{\lambda_{n+1}}. \]

By \((\alpha_5)\) and \((E_4)\), \( \lim_{n \to +\infty} |Au_n| \) exists. Sum up both sides of (3.2) from \( n = 1 \) to \( n = k \), next divide by \( \sum_{n=1}^{k} \lambda_{n+1}^2 \) and then letting \( k \to \infty \), by Lemma 2.6, we obtain

\[ \lim_{k} |Au_k| = 0. \]

Now, if \( u_{n_j} \to q \), then by the demiclosedness of \( A \), we have \( q \in A^{-1}(0) \). Hence \( \omega_w(u_n) \subset A^{-1}(0) \).

(2) The proof is similar to that of Theorem 3.1 and part (2) of Theorem 2.4. \( \square \)

## 4 Subdifferential Case

In this section, under different conditions, we present the weak convergence of \( \{u_n\} \), when \( A = \partial \varphi \), where \( \varphi : H \to [-\infty, +\infty] \) is a proper, convex and lower semi-continuous function. Also the rate of convergence of the sequence \( \{\varphi(u_n)\} \) to the minimum value of \( \varphi \) is discussed.

**Theorem 4.1.** Let \( \{u_n\} \) be a bounded sequence given by (1.8) with \( A = \partial \varphi \), where \( \varphi : H \to [-\infty, +\infty] \) is a proper, convex and lower semi-continuous function. If conditions

\( (\Lambda_1), (\alpha_1), (E_1), \) and \( (E_5) \sum_{n=1}^{+\infty} \frac{|e_n|^2}{\lambda_{n+1}} < +\infty \) are satisfied, then \( u_n \rightharpoonup p \in (\partial \varphi)^{-1}(0) \).

**Proof.** By Theorem 2.7, \( A^{-1}(0) \neq \emptyset \). Assume \( p \in A^{-1}(0) \). By the subdifferential inequality and (1.8), we get

\[
\lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) \leq \langle (1 - \alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1} + e_n, u_{n+1} - p \rangle
\]
\[
= \langle (1 - \alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p), u_{n+1} - p \rangle + \langle e_n, u_{n+1} - p \rangle - |u_{n+1} - p|^2
\]
\[
\leq \frac{1}{2} \langle (1 - \alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p), u_{n+1} - p \rangle^2 - \frac{1}{2} |u_{n+1} - p|^2 + |\alpha_n||u_{n+1} - p|
\]
\[
\leq \frac{1}{2} \langle (1 - \alpha_n)(u_n - p), u_{n+1} - p \rangle^2 + \frac{1}{2} \alpha_n |u_{n-1} - p|^2 - \frac{1}{2} |u_{n+1} - p|^2 + |\alpha_n||u_{n+1} - p|
\]
\[
= \frac{1}{2} |u_n - p|^2 - |u_{n+1} - p|^2 + \frac{1}{2} \alpha_n |u_{n-1} - p|^2 - |u_n - p|^2 + |\alpha_n||u_{n+1} - p|
\]
\[
\begin{align*}
&\leq \frac{1}{2}(|u_n - p|^2 - |u_{n+1} - p|^2) + \frac{1}{2}\alpha_{n-1}|u_{n-1} - p|^2 - \frac{1}{2}\alpha_n|u_n - p|^2 \\
&+ \frac{1}{2}|u_{n-1} - p|^2[\alpha_n - \alpha_{n-1}] + |e_n||u_{n+1} - p|.
\end{align*}
\]

Hence
\[
\sum_{n=1}^{+\infty} \lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) < \infty. \quad (4.1)
\]

By (4.1), \(\liminf_n \varphi(u_n) = \varphi(p)\). On the other hand, by the convexity of \(\varphi\) and the subdifferential inequality, we have
\[
\varphi(u_{n+1}) - (1 - \alpha_n)\varphi(u_n) - \alpha_n\varphi(u_{n-1}) \leq \varphi(u_{n+1}) - \varphi((1 - \alpha_n)u_n + \alpha_nu_{n-1})
\]
\[
\leq \partial\varphi(u_{n+1})_n - (1 - \alpha_n)u_n - \alpha_nu_{n-1} > 0
\]
\[
= \frac{1}{\lambda_{n+1}}(1 - \alpha_n)u_n + \alpha_nu_{n-1} - u_{n+1} + e_n, u_{n+1} - (1 - \alpha_n)u_n - \alpha_nu_{n-1} > 0
\]
\[
= \frac{1}{\lambda_{n+1}}e_n, u_{n+1} - (1 - \alpha_n)u_n - \alpha_nu_{n-1} > \frac{1}{\lambda_{n+1}}|u_{n+1} - (1 - \alpha_n)u_n - \alpha_nu_{n-1}|^2
\]
\[
\leq \frac{|e_n|^2}{2\lambda_{n+1}}.
\]
So
\[
\varphi(u_{n+1}) \leq (1 - \alpha_n)\varphi(u_n) + \alpha_n\varphi(u_{n-1}) + \frac{|e_n|^2}{2\lambda_{n+1}}. \quad (4.2)
\]

Hence, by (4.2), we get
\[
\varphi(u_{n+1}) \leq \varphi(u_n) + \alpha_n\varphi(u_{n-1}) - \alpha_n\varphi(u_n) + [\alpha_n - \alpha_{n-1}]\varphi(u_{n-1}) + \frac{|e_n|^2}{2\lambda_{n+1}}.
\]
Thus, by Lemma 2.3, \(\lim_n \varphi(u_n)\) exists and hence \(\lim_n \varphi(u_n) = \varphi(p)\). Now, if \(u_n \to q\), then \(\varphi(q) \leq \liminf_k \varphi(u_k) = \varphi(p)\) implies \(q \in (\partial\varphi)^{-1}(0)\). So, \(\omega_w(u_n) \subset (\partial\varphi)^{-1}(0)\).

In order to prove \(u_n \to p\), we show \(\omega_w(u_n)\) is singleton. By part (2) of Theorem 2.4, \(\lim_{n} |u_n - p|, \text{ exists for each } p \in \omega_w(u_n)\). Let \(p, q \in \omega_w(u_n)\) and \(p \neq q\), then
\[
\lim_{n \to +\infty} <u_n, p - q> \text{ exists. So } p = q \text{ and hence } \omega_w(u_n) \text{ is singleton.}
\]

**Lemma 4.2** (See [8]). Suppose that \(\{a_n\}\) and \(\{b_n\}\) are two positive real sequences such that \(\{a_n\}\) is non-increasing and convergent to zero and \(\sum_{n=1}^{+\infty} a_nb_n < +\infty\). Then
\[
(\sum_{k=1}^{n} b_k)a_n \to 0 \text{ as } n \to +\infty.
\]

**Theorem 4.3.** Suppose that \(\{u_n\}\) is a bounded sequence given by (1.8) with \(e_n \equiv 0\) and \(A = \partial \varphi\), where \(\varphi : H \to [-\infty, +\infty]\) is a proper, convex and lower semi-continuous
function. If \( \sum_{n=1}^{+\infty} \min\{\lambda_n, \lambda_{n+1}\} = +\infty \) and \((\alpha_1)\) are satisfied, then
\[
\varphi(u_n) - \varphi(p) = o\left(\left(\sum_{i=1}^{n+1} \min\{\lambda_i, \lambda_{i+1}\}\right)^{-1}\right),
\]
where \( p \) is a minimum point of \( \varphi \).

**Proof.** Set \( y_n = \max\{\varphi(u_n) - \varphi(p), \varphi(u_{n-1}) - \varphi(p)\} \). By Theorem 4.1, \( y_n \to 0 \). By (4.2), we get \( \varphi(u_{n+1}) - \varphi(p) \leq y_n \), therefore \( y_{n+1} \leq y_n \). On the other hand, by (4.1), we get
\[
\sum_{n=1}^{+\infty} \min\{\lambda_n, \lambda_{n+1}\} y_{n+1} \leq \sum_{n=1}^{+\infty} \lambda_n \left(\varphi(u_n) - \varphi(p)\right) + \sum_{n=1}^{+\infty} \lambda_{n+1} \left(\varphi(u_{n+1}) - \varphi(p)\right) < \infty.
\]
So, by Lemma 4.2, \( y_{n+1} = o\left(\sum_{i=1}^{n+1} \min\{\lambda_i, \lambda_{i+1}\}\right)^{-1}\). Since \( \varphi(u_n) - \varphi(p) \leq y_{n+1} \). Hence, \( \varphi(u_n) - \varphi(p) = o\left(\sum_{i=1}^{n+1} \min\{\lambda_i, \lambda_{i+1}\}\right)^{-1}\).

## 5 Strong Convergence

In this section, the strong convergence of \( \{u_n\} \) is obtained by additional conditions on the maximal monotone operator \( A \).

**Theorem 5.1.** Assume that \((I + A)^{-1}\) is a compact operator and conditions \((\Lambda_2), (\alpha_1)\) and \((E_1)\) are satisfied. Then \( u_n \to p \in A^{-1}(0) \) if and only if \( \{u_n\} \) is bounded.

**Proof.** By (3.2) and the assumptions, we get \( \lim \inf |Au_n| = 0 \). Therefore, there exists a subsequence \( \{Au_{n_j}\} \) of \( \{Au_n\} \) such that \( |Au_{n_j}| \to 0 \) and \( \{u_{n_j} - Au_{n_j}\} \) is bounded. Since \((I + A)^{-1}\) is compact, \( \{u_{n_j}\} \) has a strongly convergent subsequence (we denote this again by \( \{u_{n_j}\} \)) to \( p \in H \). By the monotonicity of \( A \), we have \( < Au_n - Au_{n_j}, u_n - u_{n_j} > \geq 0 \), so \( < Au_n, u_n - p > \geq 0 \) as \( j \to \infty \). The maximality of \( A \) implies \( p \in A^{-1}(0) \). On the other hand, by part (2) of Theorem 2.4, \( \lim_n |u_n - p| \) exists. Hence \( u_n \to p \in A^{-1}(0) \).

**Lemma 5.2** (See [8]). Assume \( \{y_n\} \) is a positive real sequence satisfying the following inequality:
\[
b_n y_n \leq y_{n-1} - y_n + a_n,
\]
where \( \{b_n\} \) and \( \{a_n\} \) are positive sequences.

(i) If \( \left\{ \frac{b_n}{a_n} \right\} \) is bounded, then the sequence \( \{y_n\} \) is bounded.
(ii) If \( \lim_{n} \frac{a_n}{b_n} = 0 \), then \( \lim_{n} y_n \) exists.

(iii) If \( \lim_{n} \frac{a_n}{b_n} = 0 \) and \( \sum_{n=1}^{+\infty} b_n = +\infty \), then \( \lim_{n} y_n = 0 \).

**Theorem 5.3.** Let \( \{u_n\} \) be bounded and \( A \) be a maximal monotone and strongly monotone operator. If the conditions

\[
[(\Lambda_1), (E_1) \text{ and } (\alpha_1)] \text{ or } [(\Lambda_1), (E_2) \text{ and } (\alpha_6) \frac{\alpha_n}{\lambda_{n+1}} \to 0]
\]

are satisfied, then \( u_n \to p \), where \( p \) is the unique element of \( A^{-1}(0) \).

**Proof.** By Theorem 2.7, \( A^{-1}(0) \neq \emptyset \). Assume that \( p \) is the single element of \( A^{-1}(0) \). By the strong monotonicity of \( A \) and (1.8), we get

\[
2\alpha\lambda_{n+1}|u_{n+1} - p|^2 \leq 2 < u_n - u_{n+1} - \alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p >.
\]

It follows that

\[
2\alpha\lambda_{n+1}|u_{n+1} - p|^2 \leq |u_n - p|^2 - |u_{n+1} - p|^2 + \alpha_n(|u_{n-1} - p|^2 - |u_n - p|^2) + 2|e_n||u_{n+1} - p|.
\]  

(5.1)

If the conditions \( [(\Lambda_1), (E_2), (\alpha_6) \frac{\alpha_n}{\lambda_{n+1}} \to 0] \) are satisfied, then by Lemma 5.2, the theorem follows. If the conditions \( [(\Lambda_1), (E_1) \text{ and } (\alpha_1)] \) are satisfied, then from (5.1), we have

\[
2\alpha\lambda_{n+1}|u_{n+1} - p|^2 \leq |u_n - p|^2 - |u_{n+1} - p|^2 + \alpha_{n-1}|u_{n-1} - p|^2 - \alpha_n|u_n - p|^2 + |u_{n-1} - p|^2 + |e_n||u_{n+1} - p|.
\]

So, \( \sum_{n=1}^{+\infty} \lambda_{n+1}|u_{n+1} - p|^2 < \infty \). Thus \( \lim_{n} |u_{n+1} - p|^2 = 0 \). Since, by part (2) of Theorem 2.4, \( \lim_{n} |u_n - p| \) exists, it is \( \lim_{n} |u_n - p| = 0 \).

**Acknowledgement**

The authors are grateful to the referee for his/her careful reading and valuable comments and suggestions.

**References**


