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Asymptotic Approximations of the Stable and Unstable Manifolds of Fixed Points of a Two-dimensional Cubic Map

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Abstract

We find the asymptotic approximations of the stable and unstable manifolds of the saddle equilibrium solutions of the following difference equation $x_{n+1} = ax_n^3 + bx_{n-1}^3 + cx_n + dx_{n-1}, n = 0, 1, \dots$ where the parameters a, b, c and d are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary numbers. These manifolds determine completely the global dynamics of this equation.

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1 Introduction

In this paper we consider the difference equation

$$x_{n+1} = ax_n^3 + bx_{n-1}^3 + cx_n + dx_{n-1}, \tag{1.1}$$

where the parameters a, b, c and d are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary numbers. Equation (1.1) is a special case of a general second order difference equation with cubic terms considered in [2], where the global dynamics was established in the case of all non-negative parameters and the initial conditions, in the hyperbolic case. In [2] we found precisely the basins of attraction of all attractors, which are either equilibrium points, period-two solutions or the point at infinity. The boundaries of these basins of attraction are the global stable manifolds of neighboring saddle equilibrium points (resp. nonhyperbolic equilibrium points of stable type) or the saddle period-two points (resp. nonhyperbolic period-two points of stable type). The unstable manifolds of neighboring saddle equilibrium points (resp. nonhyperbolic equilibrium points of stable type) play the role of carrying simplex, that is of the manifold which eventually carries the solutions toward its attractor. See [1] for similar results on second order difference equation with quadratic terms.

In this paper we demonstrate the computational procedure for finding the local stable and unstable manifold for equation (1.1). The method can be extended in a straightforward manner to the general second order difference equation with cubic terms considered in [2], but it will be computationally extensive and it will contain 10 parameters.

The paper is organized as follows. The rest of this section contains the result on global behavior of solutions of equation (1.1) from [2]. Section 2 contains some preliminary results about cooperative maps needed to establish the smoothness of stable and unstable manifolds and so justify the use of a method of undetermined coefficients. Section 3 contains a computational procedure and asymptotic expansions of two invariant manifolds, obtained by using *Mathematica*. Finally Section 4 contains some numerical examples and the comparison of the asymptotic expansions of global stable manifolds with the basins of attraction obtained by *Dynamica* 3 [7]. Appendix gives the values of some coefficients in asymptotic expansions of two invariant manifolds, obtained by using *Mathematica*.

Set

$$u_n = x_{n-1} \text{ and } v_n = x_n \text{ for } n = 0, 1, \dots \tag{1.2}$$

and write Eq.(1.1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= av_n^3 + bu_n^3 + cv_n + du_n. \end{aligned} \tag{1.3}$$

Let T be the corresponding map defined by:

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ av^3 + bu^3 + cv + du \end{pmatrix}. \tag{1.4}$$

The following result was established in [2]:

Theorem 1.1. *If*

$$c+d<1 \text{ and } \left(\left(c>\frac{(d-1)(2a-b)}{2b-a} \text{ and } 2a<b\right) \text{ or } 2a\geq b\right) \tag{1.5}$$

then Eq.(1.1) *has three distinct equilibrium points* $\bar{x}_- = -\frac{\sqrt{1-c-d}}{\sqrt{a+b}}$, $\bar{x}_0 = 0$ *and* $\bar{x}_+ = \frac{\sqrt{1-c-d}}{\sqrt{a+b}}$, *and the following holds:*

i) $\bar{x}_-$ *and* $\bar{x}_+$ *are the saddle points;*

ii) $\bar{x}_0$ *is locally asymptotically stable.*

Further, there exist four continuous curves $\mathcal{W}^s(\bar{x}_-), \mathcal{W}^s(\bar{x}_+)$ *(stable manifolds of* $\bar{x}_-$ *and* $\bar{x}_+$*),* $\mathcal{W}^u(\bar{x}_-), \mathcal{W}^u(\bar{x}_+)$*, (unstable manifolds of* $\bar{x}_-$ *and* $\bar{x}_-$*) where* $\mathcal{W}^s(\bar{x}_-)$ *and* $\mathcal{W}^s(\bar{x}_+)$ *are passing through the points* $E_-(\bar{x}_-, \bar{x}_-)$ *and* $E_+(\bar{x}_+, \bar{x}_+)$ *respectively, and are graphs of decreasing functions. The curves* $\mathcal{W}^u(\bar{x}_-)$ *are the graphs of increasing functions, and it has endpoints* $E_-(\bar{x}_-, \bar{x}_-)$ *and* $E_0(0,0)$*. The curve* $\mathcal{W}^u(\bar{x}_+)$ *is the graphs of increasing function and it has the endpoints* $E_0(0,0)$ *and* $E_+(\bar{x}_+, \bar{x}_+)$*. Every solution* $\{x_n\}$ *which starts below* $\mathcal{W}^s(\bar{x}_+)$ *and above* $\mathcal{W}^s(\bar{x}_-)$ *in North-east ordering converges to* $E_0(0,0)$ *and every solution* $\{x_n\}$ *which starts above* $\mathcal{W}^s(\bar{x}_+)$ *or below* $\mathcal{W}^s(\bar{x}_-)$ *in North-east ordering satisfies* $\lim x_n = \infty$*. The set of initial conditions* $\mathbb{R}^2$ *is the union of four disjoint basins of attraction, namely*

$$\mathbb{R}^2 = \mathcal{B}(E_-) \cup \mathcal{B}(E_+) \cup \mathcal{B}(E) \cup \mathcal{B}(E_\infty),$$

where E_-, E, E_+*and* E_∞ *denote the points* (x_-, x_-), $(0,0)$, (x_+, x_+) *and* (∞, ∞) *respectively, and*

$$\begin{aligned}
\mathcal{B}(E_-) =& \mathcal{W}^s(E_-),\\
\mathcal{B}(E_+) =& \mathcal{W}^s(E_+),\\
\mathcal{B}(E_0) =& \big\{(x,y)|(x_{E_-}, y_{E_-}) \preceq_{ne} (x,y) \preceq_{ne} (x_{E_+}, y_{E_+}) \quad \text{for some}\\
& (x_{E_+}, y_{E_+}) \in \mathcal{W}^s(E_+) \text{ and } (x_{E_-}, y_{E_-}) \in \mathcal{W}^s(E_-)\big\},\\
\mathcal{B}(E_\infty) =& \big\{(x,y)|(x_{E_+}, y_{E_+}) \preceq_{ne} (x,y) \quad \text{for some} \quad (x_{E_+}, y_{E_+}) \in \mathcal{W}^s(E_+)\big\}\\
& \cup \big\{(x,y)|(x,y) \preceq_{ne} (x_{E_-}, y_{E_-}) \quad \text{for some} \quad (x_{E_-}, y_{E_-}) \in \mathcal{W}^s(E_-)\big\}.
\end{aligned}$$

As one may see from Theorem 1.1 the boundaries of the basins of attraction of all attractors of Eq.(1.1) are the stable manifolds of equilibrium points. In addition, by using the results from [10] one can see that the solutions which are asymptotic to the locally asymptotically stable equilibrium solutions are approaching the unstable manifolds of the neighboring saddle equilibrium points. The monotonicity and smoothness of stable and unstable manifolds for the map T given with (1.4) is guaranteed by Theorems 2.1, 2.3, 2.4 of [10]. See [5, 8, 10, 13, 14] for related results about the stable manifolds for competitive maps. Our main goal here is to get the local asymptotic estimates for these manifolds for both equilibrium solutions. We will bring the considered map to the normal form around the equilibrium solutions and then use the method of undetermined coefficients to find the local approximations of the considered manifolds. Since the map T is cooperative, it is guaranteed that both stable and unstable manifolds are as smooth as the functions of the considered map and that are monotonic such that the stable manifold is decreasing and unstable manifold is increasing, see [3, 10]. See [5, 11, 15] for similar local approximations of stable and unstable manifolds. See [4, 6, 7, 12, 15] for basic results on stable and unstable manifolds for general maps.

2 Preliminaries

In this section we present some basic results for the cooperative maps which describe the existence and the properties of their invariant manifolds.

A first order system of difference equations

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, 2, \ldots, \quad (x_0, y_0) \in \mathcal{S}\,, \tag{2.1}$$

where $\mathcal{S} \subset \mathbb{R}^2$, $(f, g) : \mathcal{S} \to \mathcal{S}$, f, g are continuous functions is *cooperative* if $f(x, y)$ and $g(x, y)$ are non-decreasing in x and y. *Strongly cooperative* systems of difference equations or strongly cooperative maps are those for which the functions f and g are coordinate-wise strictly monotone.

If $\mathbf{v} = (u, v) \in \mathbb{R}^2$, we denote with $\mathcal{Q}_\ell(\mathbf{v})$, $\ell \in \{1, 2, 3, 4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, i.e., $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 \ : x \geq u,\ y \geq v\}$, $\mathcal{Q}_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 \ : x \leq u,\ y \geq v\}$, and so on. Define the *South-East* partial order $\preceq_{se}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{se} (s, t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x, y) \preceq_{ne} (s, t)$ if and only if $x \leq s$ and $y \leq t$. For $\mathcal{A} \subset \mathbb{R}^2$ and $\mathrm{x} \in \mathbb{R}^2$, define the *distance from* x *to* $\mathcal{A}$ as $\operatorname{dist}(\mathrm{x}, \mathcal{A}) := \inf\{\|\mathrm{x} - \mathrm{y}\| \ : \mathrm{y} \in \mathcal{A}\}$. By $\operatorname{int}\mathcal{A}$ we denote the interior of a set $\mathcal{A}$.

It is easy to show that a map F is cooperative if it is non-decreasing with respect to the North-East partial order, that is if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \tag{2.2}$$

The following five results were proved by Kulenović and Merino [10] for competitive systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or non-hyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. We give the analogue versions for cooperative maps.

A region $\mathcal{R} \subset \mathbb{R}^2$ is *rectangular* if it is the cartesian product of two intervals in $\mathbb{R}$.

Theorem 2.1. *Let T be a cooperative map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \operatorname{int}\left(\mathcal{Q}_2(\overline{\mathrm{x}}) \cup \mathcal{Q}_4(\overline{\mathrm{x}})\right)$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NE or SW vertex of $\mathcal{R}$), and T is strongly cooperative on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian matrix of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^{λ} associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^{λ} at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly decreasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

Corollary 2.2. *If T has no fixed point nor periodic points of minimal period two in Δ, then the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$.*

As is well known for maps that are strongly cooperative near the fixed point, hypothesis (b). of Theorem 2.1 reduces just to $|\lambda| < 1$, see [10]. Also, one can show that in such a case no associated eigenvector is aligned with a coordinate axis.

Theorem 2.3. *Under the hypotheses of Theorem 2.1, suppose there exists a neighborhood $\mathcal{U}$ of $\overline{\mathrm{x}}$ in $\mathbb{R}^2$ such that T is of class C^k on $\mathcal{U} \cup \Delta$ for some $k \geq 1$, and that the Jacobian of T at each $\mathrm{x} \in \Delta$ is invertible. Then the curve $\mathcal{C}$ in the conclusion of Theorem 2.1 is of class C^k.*

The following result gives a description of the global stable and unstable manifolds of a saddle point of a cooperative map. The result is the modification of Theorem 5 from [8]. See also [9].

Theorem 2.4. *In addition to the hypotheses of Theorem 2.1, suppose that $\mu > 1$ and that the eigenspace E^{μ} associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 2.1 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the global stable manifold $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the global unstable manifold $\mathcal{W}^u(\overline{\mathrm{x}})$ is a curve in $\mathcal{R}$ that is tangential to E^{μ} at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly increasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Theorem 2.5. *Assume the hypotheses of Theorem 2.1, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2.1. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\begin{aligned} \mathcal{W}_- &:= \{\mathrm{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathrm{y} \in \mathcal{C} \textit{ with } \mathrm{x} \preceq_{ne} \mathrm{y}\} \\ \mathcal{W}_+ &:= \{\mathrm{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathrm{y} \in \mathcal{C} \textit{with } \mathrm{y} \preceq_{ne} \mathrm{x}\}, \end{aligned} \tag{2.3}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ *is invariant, and* $\operatorname{dist}(T^n(\mathrm{x}), \mathcal{Q}_1(\overline{\mathrm{x}})) \to 0$ *as* $n \to \infty$ *for every* $\mathrm{x} \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ *is invariant, and* $\operatorname{dist}(T^n(\mathrm{x}), \mathcal{Q}_3(\overline{\mathrm{x}})) \to 0$ *as* $n \to \infty$ *for every* $\mathrm{x} \in \mathcal{W}_+$.

If, in addition, $\overline{\mathrm{x}}$ *is an interior point of* $\mathcal{R}$ *and* T *is* C^2 *and strongly cooperative in a neighborhood of* $\overline{\mathrm{x}}$*, then* T *has no periodic points in the boundary of* $\mathcal{Q}_2(\overline{\mathrm{x}}) \cup \mathcal{Q}_4(\overline{\mathrm{x}})$ *except for* $\overline{\mathrm{x}}$*, and the following statements are true.*

(iii) For every $\mathrm{x} \in \mathcal{W}_-$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_1(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

(iv) For every $\mathrm{x} \in \mathcal{W}_+$ *there exists* $n_0 \in \mathbb{N}$ *such that* $T^n(\mathrm{x}) \in \operatorname{int} \mathcal{Q}_3(\overline{\mathrm{x}})$ *for* $n \geq n_0$.

Remark 2.6. The map T defined with (1.4) is strongly cooperative in $\mathbb{R}^2$. Theorems 2.1, 2.3 and 2.4 show that the stable and unstable manifolds of cooperative maps, which satisfies certain conditions, are simple monotonic curves which are as smooth as the functions of the map. Thus the assumed forms of these manifolds are justified. As is well-known the stable and unstable manifolds of general maps can have complicated structure consisting of many branches or being strange attractors, see [4, 6, 15] for some examples of polynomial maps such as Henon with unstable manifold being a strange attractor. Finally, see [14] for examples of competitive and so cooperative maps in the plane with chaotic attractors.

3 Invariant Manifolds and Normal Forms

Let

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + \begin{pmatrix} g_1(\xi_n, \eta_n) \\ g_2(\xi_n, \eta_n) \end{pmatrix}, \tag{3.1}$$

where $g_1(0,0) = 0, \quad g_2(0,0) = 0, \quad Dg_1(0,0) = 0$ and $Dg_2(0,0) = 0$. Suppose that $|\mu_1| < 1$ and $|\mu_2| > 1$. Then, there are two unique invariant manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ tangents to $(1,0)$ and $(0,1)$ at (0,0), which are graphs of the maps $\varphi : E_1 \to E_2$ and $\psi : E_1 \to E_2$, such that $\varphi(0) = \psi(0) = 0$ and $\varphi'(0) = \psi'(0) = 0$. See [5, 6, 11, 15]. Letting $\eta_n = \varphi(\xi_n)$ yields

$$\eta_{n+1} = \varphi(\xi_{n+1}) = \varphi(\mu_1 \xi_n + g_1(\xi_n, \varphi(\xi_n))). \tag{3.2}$$

On the other hand by (3.1)

$$\eta_{n+1} = \mu_2 \varphi(\xi_n) + g_2(\xi_n, \varphi(\xi_n)). \tag{3.3}$$

Equating equations (3.2) and (3.3) yields

$$\varphi(\mu_1 \xi_n + g_1(\xi_n, \varphi(\xi_n))) = \mu_2 \varphi(\xi_n) + g_2(\xi_n, \varphi(\xi_n)). \tag{3.4}$$

Similarly, letting $\xi_n = \psi(\eta_n)$ yields

$$\xi_{n+1} = \psi(\eta_{n+1}) = \psi(\mu_2\eta_n + g_2(\psi(\eta_n), \eta_n)). \tag{3.5}$$

By using (3.1) we obtain

$$\xi_{n+1} = \mu_1\psi(\eta_n) + g_1(\psi(\eta_n), \eta_n). \tag{3.6}$$

Equating equations (3.5) and (3.6) yields

$$\psi(\mu_2\eta_n + g_2(\psi(\eta_n), \eta_n)) = \mu_1\psi(\eta_n) + g_1(\psi(\eta_n), \eta_n). \tag{3.7}$$

Thus the functional equations (3.4) and (3.7), define the local stable manifold $\mathcal{W}^s = \{(\xi, \eta) \in \mathbb{R}^2 : \eta = \varphi(\xi)\}$, and the local unstable manifold $\mathcal{W}^u = \{(\xi, \eta) \in \mathbb{R}^2 : \xi = \psi(\eta)\}$. Without loss generality, we can assume that solutions of the functional equations (3.4) and (3.7) take the forms $\varphi(\xi) = \alpha_1\xi^2 + \beta_1\xi^3 + O(|\xi|^4)$ and $\psi(\eta) = \alpha_2\eta^2 + \beta_2\eta^3 + O(|\eta|^4)$.

3.1 Normal Form of the Map T at the Saddle Points $\bar{x}_-$ and $\bar{x}_+$

Let $\bar{x}$ denote one of the saddle points $\bar{x}_-$ or $\bar{x}_+$. Put $y_n = x_n - \bar{x}$. Then Eq(1.1) becomes

$$y_{n+1} = a\,(\bar{x} + y_n)^3 + b\,(\bar{x} + y_{n-1})^3 + c\,(\bar{x} + y_n) + d\,(\bar{x} + y_{n-1}) - \bar{x}. \tag{3.8}$$

Set $u_n = y_{n-1}$ and $v_n = y_n$ for $n = 0, 1, \ldots$ and write Eq(3.8) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= a\,(\bar{x} + v_n)^3 + b\,(\bar{x} + u_n)^3 + c\,(\bar{x} + v_n) + d\,(\bar{x} + u_n) - \bar{x}. \end{aligned} \tag{3.9}$$

Let F be the function defined by:

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ a\,(\bar{x} + v)^3 + b\,(\bar{x} + u)^3 + c\,(\bar{x} + v) + d\,(\bar{x} + u) - \bar{x} \end{pmatrix}. \tag{3.10}$$

Then F has the fixed point $(0, 0)$, which corresponds to the fixed point $(\bar{x}, \bar{x})$ of the map T. The Jacobian matrix of F is given by

$$Jac_F(u, v) = \begin{pmatrix} 0 & 1 \\ d + 3b\,(u + \bar{x})^2 & c + 3a\,(v + \bar{x})^2 \end{pmatrix}.$$

At $(0, 0)$, $Jac_F(u, v)$ has the form

$$\begin{aligned} J_0 = Jac_F(0, 0) &= \begin{pmatrix} 0 & 1 \\ d + 3b\bar{x}^2 & c + 3a\bar{x}^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ \dfrac{-3cb - 2db + 3b + ad}{a + b} & \dfrac{-2ca - 3da + 3a + bc}{a + b} \end{pmatrix}. \end{aligned} \tag{3.11}$$

The eigenvalues of (3.11) are $\mu_{1,2}$ where

$$\mu_1 = -\frac{a(2c+3d-3)-bc+A}{2(a+b)} \text{ and } \mu_2 = -\frac{a(2c+3d-3)-bc-A}{2(a+b)},$$

and

$$A = \sqrt{(a(-2c-3d+3)+bc)^2+4(a+b)(ad+b(-3c-2d+3))},$$

and the corresponding eigenvectors are given by

$$v_1 = \left(\frac{-2ac-3ad+3a+bc+A}{b(6c+4d-6)-2ad}, 1\right)^T$$

and

$$v_2 = \left(\frac{-2ac-3ad+3a+bc-A}{b(6c+4d-6)-2ad}, 1\right)^T,$$

respectively.

Then we have that

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ d+3b\bar{x}^2 & c+3a\bar{x}^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u,v) \\ g_1(u,v) \end{pmatrix}, \tag{3.12}$$

and

$$\begin{aligned} f_1(u,v) =& 0 \\ g_1(u,v) =& \bar{x}\left(3av^2+3bu^2-1+c+d\right)+(a+b)\bar{x}^3+av^3+bu^3. \end{aligned}$$

Then, the system (3.9) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ d+3b\bar{x}^2 & c+3a\bar{x}^2 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1(u_n,v_n) \\ g_1(u_n,v_n) \end{pmatrix}. \tag{3.13}$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \cdot \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}$$

where

$$P = \begin{pmatrix} -\frac{2ca+3da-3a-A-bc}{2(3cb+2db-3b-ad)} & \frac{-2ca-3da+3a-A+bc}{2(3cb+2db-3b-ad)} \\ 1 & 1 \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} \frac{3cb+2db-3b-ad}{A} & \frac{2ca+3da-3a+A-bc}{2A} \\ \frac{-3cb-2db+3b+ad}{A} & \frac{-2ca-3da+3a+A+bc}{2A} \end{pmatrix}.$$

Then system (3.13) is equivalent to

$$\begin{pmatrix}\xi_{n+1}\\ \eta_{n+1}\end{pmatrix} = \begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix} + \begin{pmatrix}\tilde{f}_1(\xi_n,\eta_n)\\ \tilde{g}_1(\xi_n,\eta_n)\end{pmatrix}, \tag{3.14}$$

where

$$\begin{pmatrix}\tilde{f}_1(u,v)\\ \tilde{g}_1(u,v)\end{pmatrix} := P^{-1}\cdot H_1\left(P\cdot\begin{pmatrix}u\\ v\end{pmatrix}\right)$$

and

$$H_1\begin{pmatrix}u\\ v\end{pmatrix} := \begin{pmatrix}f_1(u,v)\\ g_1(u,v)\end{pmatrix}.$$

By straightforward calculation we obtain that

$$\tilde{f}_1(u,v) = \frac{a(2c+3d-3)+A-bc}{2A}\Upsilon_1(u,v),$$

$$\tilde{g}_1(u,v) = \frac{a(-2c-3d+3)+A+bc}{2A}\Upsilon_1(u,v),$$

where

$$\begin{aligned}\Upsilon_1(u,v) = \bar{x}&\left(\frac{3b(-a(2c+3d-3)(u+v)+A(u-v)+bc(u+v))^2}{4(ad+b(-3c-2d+3))^2}\right)\\ &+\bar{x}\left(3a(u+v)^2+c+d-1\right)+a(u+v)^3\\ &+\bar{x}^3(a+b)+\frac{b(-a(2c+3d-3)(u+v)+A(u-v)+bc(u+v))^3}{8(b(3c+2d-3)-ad)^3}.\end{aligned}$$

3.2 Stable Manifolds at $\bar{x}_-$ and $\bar{x}_+$

Assume that (1.5) holds and that the local stable manifold corresponding to the saddle point E_+ is the graph of the function φ_1 of the form

$$\varphi_1(\xi) = \alpha_1\xi^2 + \beta_1\xi^3 + O(|\xi|^4), \quad \alpha_1,\beta_1\in\mathbb{R},$$

and that the local stable manifold corresponding to the saddle point E_- is the graph of the function φ_2 of the form

$$\varphi_2(\xi) = \alpha_2\xi^2 + \beta_2\xi^3 + O(|\xi|^4), \quad \alpha_2,\beta_2\in\mathbb{R},$$

Now we compute the constants α_1, α_2, β_1 and β_2. The function φ_1 must satisfy the stable manifold equation

$$\varphi_1\left(\mu_1\xi + \tilde{f}_1(\xi,\varphi_1(\xi))\right) = \mu_2\varphi_1(\xi) + \tilde{g}_1(\xi,\varphi_1(\xi)),$$

for $\bar{x} = \dfrac{\sqrt{1-c-d}}{\sqrt{a+b}}$. This leads to the following polynomial equation

$$p_1\xi^2 + p_2\xi^3 + \cdots + p_{26}\xi^{27} = 0.$$

Substituting $\bar{x}_2$ into p_1 and p_2 and solving system $p_1 = 0$ and $p_2 = 0$, we obtain the values

$$\alpha_1 = \frac{\Upsilon_1}{\Upsilon_2}, \quad \beta_1 = \frac{\Upsilon_3}{\Upsilon_4}$$

where the coefficients Υ_3 and Υ_4, generated by *Mathematica* are in Appendix A and

$$\begin{aligned}\Upsilon_1 = {} & 3(a+b)^{3/2}\sqrt{1-c-d}(a(2c+3d-3)+A-bc) \\ & \left(4a^3d^2 + a^2b\left(4c^2-12c(d+1)+(6-7d)d+9\right) + 2ab\,(A(2c+3d-3)\right. \\ & \left. +b\left(16c^2+3c(7d-11)+2(3-2d)^2\right)\right) + b(A-bc)^2\right),\end{aligned}$$

$$\begin{aligned}\Upsilon_2 = {} & 2A(ad+b(3-3c-2d))^2\left(a^2(2c+3d-3)(2c+3d-1) - 2a(3d(A\right. \\ & \left. +b(c-1)) + 2(c-2)(A+bc)+3b) + A^2 + 2Ab(c+1) + b^2(c-2)c\right).\end{aligned}$$

Since $\eta_n = \alpha_1\xi_n^2 + \beta_1\xi_n^3$, and

$$\begin{pmatrix}\xi_n \\ \eta_n\end{pmatrix} = P^{-1}\cdot\begin{pmatrix}x_{n-1}-\bar{x}_+ \\ x_n-\bar{x}_+\end{pmatrix} \tag{3.15}$$

we can approximate locally the local stable manifold $\mathcal{W}^s_{loc}(\bar{x}_+,\bar{x}_+)$ as the graph of the map $\tilde{\varphi}_1(x)$ such that $S_+(x,\tilde{\varphi}_1(x)) = 0$ where

$$\begin{aligned}S_+(x,y) := {} & \frac{(y-\bar{x}_+)\,(a(3-2c-3d)+A+bc) + 2\,(x-\bar{x}_+)\,(ad+b(3-3c-2d))}{2A} \\ & - \frac{\beta_1\left((A-bc+a(2c+3d-3))\,(y-\bar{x}_+) + (x-\bar{x}_+)\,(b(6c+4d-6)-2ad)\right)^3}{8A^3} \\ & - \frac{\alpha_1\left((A-bc+a(2c+3d-3))\,(y-\bar{x}_+) + (x-\bar{x}_+)\,(b(6c+4d-6)-2ad)\right)^2}{4A^3}\end{aligned} \tag{3.16}$$

and which satisfies

$$\tilde{\varphi}_1(\bar{x}_+) = \bar{x}_+ \text{ and } \tilde{\varphi}_1'(\bar{x}_+) = \frac{b(6c+4d-6)-2ad}{-2ac-3ad+3a+bc+A}.$$

The function φ_2 must satisfy the stable manifold equation

$$\varphi_2\left(\mu_1\xi + \tilde{f}_1\,(\xi,\varphi_2(\xi))\right) = \mu_2\varphi_2(\xi) + \tilde{g}_1\,(\xi,\varphi_2(\xi))\,,$$

for $\bar{x} = -\dfrac{\sqrt{1-c-d}}{\sqrt{a+b}}$. This leads to the following polynomial equation

$$p_1'\xi^2 + p_2'\xi^3 + \cdots + p_{26}'\xi^{27} = 0.$$

Substituting $\bar{x}_2$ into p_1' and p_2' and solving system $p_1' = 0$ and $p_2' = 0$, we obtain the values

$$\alpha_2 = -\alpha_1 = -\frac{\Upsilon_1}{\Upsilon_2}, \quad \beta_2 = \frac{\Upsilon_5}{\Upsilon_4}$$

where the coefficient Υ_5, generated by *Mathematica* is in appendix A. Since $\eta_n = \alpha_2\xi_n^2 + \beta_2\xi_n^3$, (3.15) we can approximate locally the local stable manifold $\mathcal{W}^s_{loc}(\bar{x}_-, \bar{x}_-)$ as the graph of the map $\tilde{\varphi}_2(x)$ such that $S_-(x, \tilde{\varphi}_2(x)) = 0$ where

$$\begin{aligned} S_-(x,y) := & \frac{(y-\bar{x}_-)\left(a(3-2c-3d)+A+bc\right)+2\left(x-\bar{x}_-\right)\left(ad+b(3-3c-2d)\right)}{2A} \\ & - \frac{\beta_2\left(\left(A-bc+a(2c+3d-3)\right)(y-\bar{x}_-)+(x-\bar{x}_-)\left(b(6c+4d-6)-2ad\right)\right)^3}{8A^3} \\ & - \frac{\alpha_2\left(\left(A-bc+a(2c+3d-3)\right)(y-\bar{x}_-)+(x-\bar{x}_-)\left(b(6c+4d-6)-2ad\right)\right)^2}{4A^3} \end{aligned} \tag{3.17}$$

and which satisfies

$$\tilde{\varphi}_2(\bar{x}_-) = \bar{x}_- \text{ and } \tilde{\varphi}_2'(\bar{x}_-) = \frac{b(6c+4d-6)-2ad}{-2ac-3ad+3a+bc+A}.$$

Theorem 3.1. *Consider Eq.*(1.1). *Then the local stable manifolds corresponding to the saddle points $\bar{x}_+$ and $\bar{x}_-$ are given with the asymptotic expansions $S_+(x, \tilde{\varphi}_1(x)) = 0$ and $S_-(x, \tilde{\varphi}_2(x)) = 0$ respectively.*

3.3 Unstable Manifolds at $\bar{x}_-$ and $\bar{x}_+$

Assume (1.5) and that the local unstable manifold, that corresponds to the saddle point $\bar{x}_+ = \dfrac{\sqrt{1-c-d}}{\sqrt{a+b}}$, is the graph of the function ψ_1 that has the form

$$\psi_1(\eta) = \gamma_1\eta^2 + \delta_1\eta^3 + O(|\eta|^4), \quad \gamma_1, \delta_1 \in \mathbb{R}$$

and that the local unstable manifold, that corresponds to the saddle point

$$\bar{x}_- = -\frac{\sqrt{1-c-d}}{\sqrt{a+b}},$$

is the graph of the function ψ_2 that has the form

$$\psi_2(\eta) = \gamma_2\eta^2 + \delta_2\eta^3 + O(|\eta|^4), \quad \gamma_2, \delta_2 \in \mathbb{R}.$$

Now we compute the constants γ_1 and δ_1.

The function ψ_1 must satisfy the unstable manifold equation

$$\psi_1(\mu_2\eta + \tilde{g}_1(\psi_1(\eta), \eta)) = \mu_1\psi_1(\eta) + \tilde{f}_1(\psi_1(\eta), \eta),$$

for $\bar{x} = \bar{x}_+ = \dfrac{\sqrt{1-c-d}}{\sqrt{a+b}}$. This leads to the following polynomial equation

$$q_1\eta^2 + q_2\eta^3 + \cdots + q_{26}\eta^{27} = 0.$$

Substituting $\bar{x}_+$ into q_1 and q_2 and solving system $q_1 = 0$ and $q_2 = 0$, we obtain the values

$$\gamma_1 = \frac{\Gamma_1}{\Gamma_2}, \quad \delta_1 = \frac{\Gamma_3}{\Gamma_4}.$$

where the coefficients Γ_3 and Γ_4, generated by *Mathematica* are in Appendix A and

$$\begin{aligned}\Gamma_1 &= 3(a+b)^{3/2}\sqrt{-c-d+1}(a(2c+3d-3)+A-bc)\left(4a^3d^2+a^2b\left(4c^2-12c(d+1)\right.\right.\\&\left.+(6-7d)d+9\right)+2ab\left(A(2c+3d-3)+b\left(16c^2+3c(7d-11)+2(3-2d)^2\right)\right)\\&\left.+b(A-bc)^2\right)\end{aligned}$$

$$\begin{aligned}\Gamma_2 &= 2A(ad+b(-3c-2d+3))^2\left(a^2(2c+3d-3)(2c+3d-1)-2a(3d(A+b(c-1))\right.\\&\left.+2(c-2)(A+bc)+3b)+A^2+2Ab(c+1)+b^2(c-2)c\right).\end{aligned}$$

Since $\xi_n = \gamma_1\eta_n^2 + \delta_1\eta_n^3$, and (3.15) we can approximate locally the local unstable manifold $\mathcal{W}^u_{loc}(\bar{x}_+, \bar{x}_+)$ as the graph of the map $\tilde{\psi}_1(y)$ such that $U(\tilde{\psi}_1(y), y) = 0$ where

$$\begin{aligned}U_+(x,y) :=& \frac{(y-\bar{x}_+)(a(2c+3d-3)+A-bc)+(x-\bar{x}_+)(2ad+b(6c+4d-6))}{2A}\\&-\frac{\delta_1\left((y-\bar{x}_+)(a(3-2c-3d)+A+bc)+2(x-\bar{x}_+)(ad+b(3-3c-2d))\right)^3}{8A^3}\\&-\frac{\gamma_1\left((y-\bar{x}_+)(a(3-2c-3d)+A+bc)+2(x-\bar{x}_+)(ad+b(3-3c-2d))\right)^2}{4A^2}\end{aligned} \tag{3.18}$$

and which satisfies

$$\tilde{\psi}_1(\bar{x}_+) = \bar{x}_+ \text{ and } \tilde{\psi}_1'(\bar{x}_+) = \frac{b(6c+4d-6)-2ad}{-2ac-3ad+3a+bc-A}.$$

Now we compute the constants γ_2 and δ_2.

The function ψ_2 must satisfy the unstable manifold equation

$$\psi_2(\mu_2\eta + \tilde{g}_1(\psi_2(\eta),\eta)) = \mu_1\psi_2(\eta) + \tilde{f}_1(\psi_2(\eta),\eta),$$

for $\bar{x} = \bar{x}_- = -\dfrac{\sqrt{1-c-d}}{\sqrt{a+b}}$.

This leads to the following polynomial equation

$$q_1'\eta^2 + q_2'\eta^3 + \cdots + q_{26}'\eta^{27} = 0.$$

Substituting $\bar{x}_-$ into q_1 and q_2 and solving system $q_1 = 0$ and $q_2 = 0$, we obtain the values

$$\gamma_2 = -\gamma_1 = \frac{\Gamma_1}{\Gamma_2}, \quad \delta_2 = \frac{\Gamma_5}{\Gamma_4}.$$

where the coefficient Γ_5, generated by *Mathematica* is in appendix A.

Since $\xi_n = \gamma_2\eta_n^2 + \delta_2\eta_n^3$, and (3.15) we can approximate locally the local unstable manifold $\mathcal{W}^u_{loc}(\bar{x}_-, \bar{x}_-)$ as the graph of the function $\tilde{\psi}_2(y)$ such that $U'(\tilde{\psi}_2(y), y) = 0$, where

$$\begin{aligned} U_-(x,y) :=& \frac{(y-\bar{x}_-)(a(2c+3d-3)+A-bc)+(x-\bar{x}_-)(2ad+b(6c+4d-6))}{2A} \\ &- \frac{\delta_2\left((y-\bar{x}_-)(a(3-2c-3d)+A+bc)+2(x-\bar{x}_-)(ad+b(3-3c-2d))\right)^3}{8A^3} \\ &- \frac{\gamma_2\left((y-\bar{x}_-)(a(3-2c-3d)+A+bc)+2(x-\bar{x}_-)(ad+b(3-3c-2d))\right)^2}{4A^2} \end{aligned} \tag{3.19}$$

and which satisfies

$$\tilde{\psi}_2(\bar{x}_-) = \bar{x}_- \text{ and } \tilde{\psi}_2'(\bar{x}_-) = \frac{b(6c+4d-6)-2ad}{-2ac-3ad+3a+bc-A}.$$

Thus we proved the following result.

Theorem 3.2. *Consider Eq.*(1.1)*. Then the local unstable manifolds corresponding to the saddle points $\bar{x}_-$ and $\bar{x}_+$ are given with the asymptotic expansions $U_-(\tilde{\psi}_2(y), y) = 0$ and $U_+(\tilde{\psi}_1(y), y) = 0$ respectively.*

4 Numerical Examples

In this section we provide some numerical examples and we compare visually the asymptotic approximations of stable and unstable manifolds, obtained by using *Mathematica*, with the boundaries of the basins of attraction obtained by using the software package *Dynamica* 3 [7].

For $a = 1.0$, $b = 1.0$, $c = 0.3$ and $d = 0.2$ we have that

$$\begin{aligned} S^1_+(x,y) =& 0.00687369(1.(-3.4(x-0.5)-0.3(y-0.5))-0.4(x-0.5) \\ &+2.62832(y-0.5))^3-0.0430681(1.(-3.4(x-0.5)-0.3(y-0.5)) \\ &-0.4(x-0.5)+2.62832(y-0.5))^2-0.11291(3.8(x-0.5) \\ &+6.52832(y-0.5)), \end{aligned}$$

$$\begin{aligned} S^1_-(x,y) =& 0.00687369(1.(-3.4(x+0.5)-0.3(y+0.5))-0.4(x+0.5) \\ &+2.62832(y+0.5))^3+0.0430681(1.(-3.4(x+0.5)-0.3(y+0.5)) \\ &-0.4(x+0.5)+2.62832(y+0.5))^2-0.11291(3.8(x+0.5) \\ &+6.52832(y+0.5)) \end{aligned}$$

and for $a = 16.0$, $b = 2.0$, $c = 0.5$ and $d = 0.1$

$$\begin{aligned} S_+^2(x,y) = &-0.0150918(8.4(x-0.149071)+61.3307(y-0.149071)) \\ &+0.00186278(2.(-2.6(x-0.149071)-0.5(y-0.149071)) \\ &-3.2(x-0.149071)+5.93065(y-0.149071))^3 \\ &-0.00686115(2.(-2.6(x-0.149071)-0.5(y-0.149071)) \\ &-3.2(x-0.149071)+5.93065(y-0.149071))^2, \end{aligned}$$

$$\begin{aligned} S_-^2(x,y) = &-0.0150918(8.4(0.149071+x)+61.3307(0.149071+y)) \\ &+0.00686115(-3.2(0.149071+x)+5.93065(0.149071+y) \\ &+2.(-2.6(0.149071+x)-0.5(0.149071+y)))^2 \\ &+0.00186278(-3.2(0.149071+x)+5.93065(0.149071+y) \\ &+2.(-2.6(0.149071+x)-0.5(0.149071+y)))^3. \end{aligned}$$

Figures 4.1 and 4.2 show the graph of the functions $S_+^1(x,y) = 0$, $S_-^1(x,y) = 0$, $S_+^2(x,y) = 0$, and $S_-^2(x,y) = 0$ with the basins of attraction created with *Dynamica* 3.

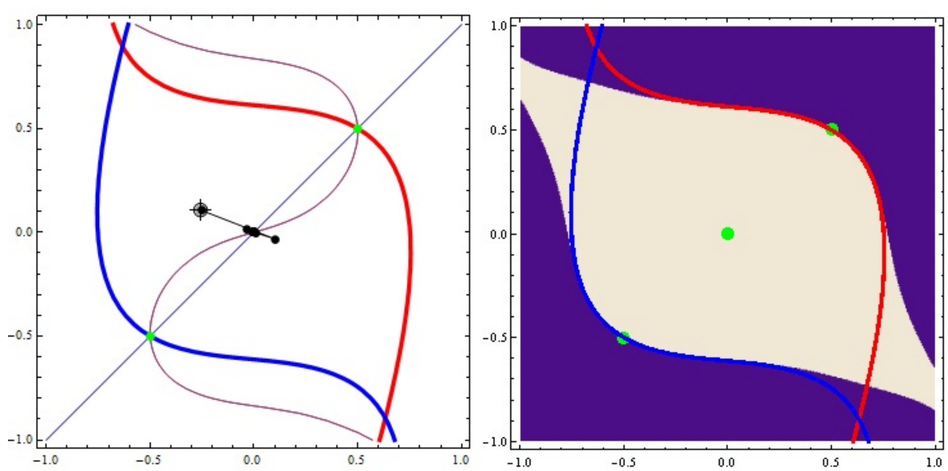

Figure 4.1: The graph of the function $S_+^1(x,y) = 0$ (red curves) and $S_-^1(x,y) = 0$ (blue curves) for $a = 1.0$, $b = 1.0$, $c = 0.3$ and $d = 0.2$ with the basins of attraction generated by *Dynamica* 3.

For $a = 1.0$, $b = 1.0$, $c = 0.3$ and $d = 0.2$ we have that

$$\begin{aligned} U_+^1(x,y) = &-0.11291(1.(-3.4(x-0.5)-0.3(y-0.5))-0.4(x-0.5) \\ &+2.62832(y-0.5))+0.00213032(3.8(x-0.5)+6.52832(y-0.5))^2 \\ &-0.000149822(3.8(x-0.5)+6.52832(y-0.5))^3, \end{aligned}$$

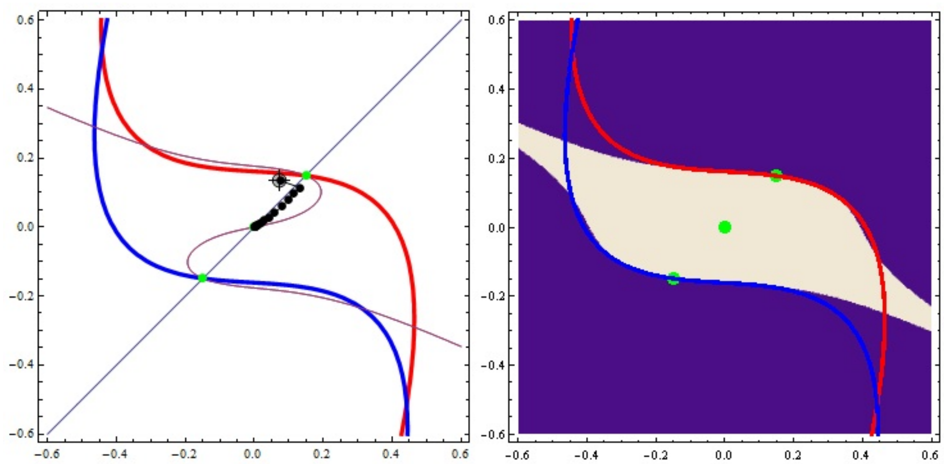

Figure 4.2: The graph of the function $S^2_+(x,y) = 0$ (red curves) and $S^2_-(x,y) = 0$ (blue curves) for $a = 16.0$, $b = 2.0$, $c = 0.5$ and $d = 0.1$ with the basins of attraction generated by*Dynamica* 3.

$$
\begin{aligned}
U^1_-(x,y) = &- 0.11291(1.(-3.4(x+0.5) - 0.3(y+0.5)) - 0.4(x+0.5) \\
&+ 2.62832(y+0.5)) - 0.000149822(3.8(x+0.5) + 6.52832(y+0.5))^3 \\
&- 0.00213032(3.8(x+0.5) + 6.52832(y+0.5))^2,
\end{aligned}
$$

and for $a = 16.0$, $b = 2.0$, $c = 0.5$ and $d = 0.1$

$$
\begin{aligned}
U^2_+(x,y) = &- 0.0150918(2.(-2.6(x-0.149071) - 0.5(y-0.149071)) \\
&- 3.2(x-0.149071) + 5.93065(y-0.149071)) \\
&+ 0.0000416195(8.4(x-0.149071) + 61.3307(y-0.149071))^2 \\
&- 2.0226418 \cdot 10^{-6}(8.4(x-0.149071) + 61.3307(y-0.149071))^3,
\end{aligned}
$$

$$
\begin{aligned}
U^2_-(x,y) = &0.0150918(2.(-2.6(x+0.149071) - 0.5(y+0.149071)) \\
&- 3.2(x+0.149071) + 5.93065(y+0.149071)) \\
&- 2.0226418 \cdot 10^{-6}(8.4(x+0.149071) + 61.3307(y+0.149071))^3 \\
&- 0.0000416195(8.4(x+0.149071) + 61.3307(y+0.149071))^2.
\end{aligned}
$$

Figures 4.3 and 4.4 show the graph of the functions $U^1_+(x,y) = 0$, $U^1_-(x,y) = 0$, $U^2_+(x,y) = 0$, and $U^2_-(x,y) = 0$ with the basins of attraction created with *Dynamica* 3.

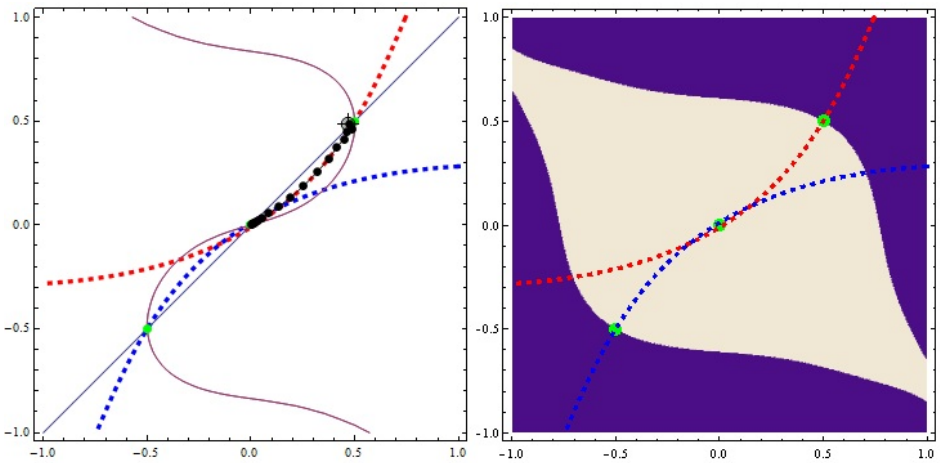

Figure 4.3: The graph of the function $U_{+}^{1}(x, y) = 0$ (red curve) and $U_{-}^{1}(x, y) = 0$ (blue curve) for $a = 1.0$, $b = 1.0$, $c = 0.3$ and $d = 0.2$ with the basins of attraction generated by *Dynamica* 3.

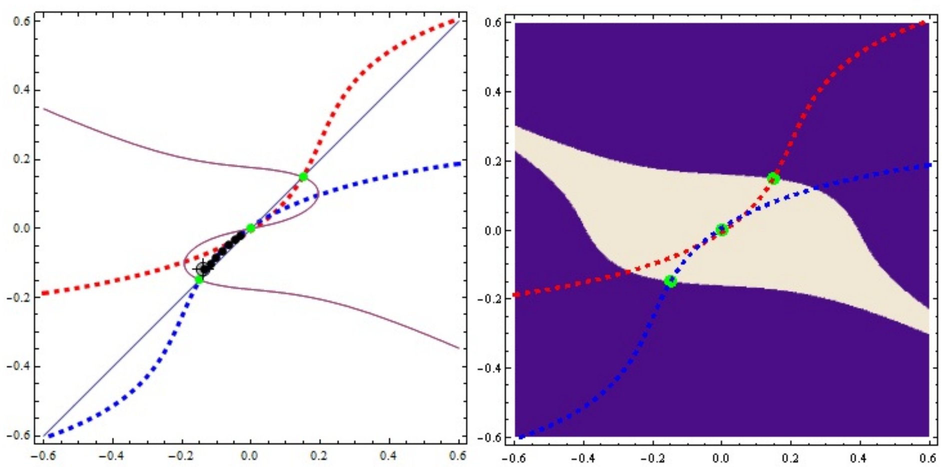

Figure 4.4: The graph of the function $U_{+}^{2}(x, y) = 0$ (red curve) and $U_{-}^{2}(x, y) = 0$ (blue curve) for $a = 16.0$, $b = 2.0$, $c = 0.5$ and $d = 0.1$ with the basins of attraction generated by*Dynamica* 3.

References

[1] J. Bektešević, M. R. S. Kulenović and E. Pilav, Global Dynamics of Quadratic Second Order Difference Equation in the First Quadrant, *Appl. Math. Comp.*, 227(2014), 50–65.

[2] J. Bektešević, M. R. S. Kulenović and E. Pilav, Global Dynamics of Cubic Second Order Difference Equation in the First Quadrant, submitted.

[3] A. Brett and M. R. S. Kulenović, Basins of attraction of equilibrium points of monotone difference equations. *Sarajevo J. Math.* 5(18) (2009), 211–233.

[4] S. Elaydi, *Discrete chaos. With applications in science and engineering*. Second edition. With a foreword by Robert M. May., Chapman & Hall/CRC, Boca Raton, FL, 2008.

[5] M. Guzowska, R. Luis and S. Elaydi, Bifurcation and invariant manifolds of the logistic competition model, *J. Difference Equ. Appl.*, 2011, 17:12, 1851–1872.

[6] J. K. Hale and H. Kocak, *Dynamics and bifurcations*. Texts in Applied Mathematics, 3. Springer-Verlag, New York, 1991.

[7] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.

[8] M. R. S. Kulenović and O. Merino, Competitive-Exclusion versus Competitive-Coexistence for Systems in the Plane, *Discrete Contin. Dyn. Syst. Ser. B* 6(2006), 1141-1156.

[9] M. R. S. Kulenović and O. Merino, Global Bifurcation for Competitive Systems in the Plane, *Discrete Contin. Dyn. Syst. B* 12(2009), 133-149.

[10] M. R. S. Kulenović and O. Merino, Invariant manifolds for competitive discrete systems in the plane. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 20 (2010), no. 8, 2471–2486.

[11] H. A. Lauwerier, Two-dimensional iterative maps. *Chaos, 5895, Nonlinear Sci. Theory Appl.*, Manchester Univ. Press, Manchester, 1986.

[12] C. Robinson, *Stability, Symbolic Dynamics, and Chaos*, CRC Press, Boca Raton, 1995.

[13] H. L. Smith, Periodic competitive differential equations and the discrete dynamics of competitive maps, *J. Differential Equations* 64 (1986), 165-194.

[14] H. L. Smith, Planar Competitive and Cooperative Difference Equations,*J. Differ. Equ. Appl.* 3(1998), 335-357.

[15] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*. Second edition. Texts in Applied Mathematics, 2. Springer-Verlag, New York, 2003.

A Values of Coefficients Υ_3, Υ_4, Υ_5, Γ_3, Γ_4 and Γ_5

$$
\begin{aligned}
\Upsilon_4 =& 2A(ad-3bc-2bd+3b)^3 \big(8a^3c^3+36a^3c^2d-36a^3c^2+54a^3cd^2-108a^3cd+46a^3c+27a^3\\
& d^3-81a^3d^2+69a^3d-15a^3+12a^2Ac^2+36a^2Acd-36a^2Ac+27a^2Ad^2-54a^2Ad+31a^2\\
& A-12a^2bc^3-36a^2bc^2d+36a^2bc^2-27a^2bcd^2+54a^2bcd-39a^2bc-24a^2bd+24a^2b+6a\\
& A^2c+9aA^2d-9aA^2-12aAbc^2-18aAbcd+18aAbc+8aAb+6ab^2c^3+9ab^2c^2d-9ab^2c^2\\
& -12ab^2d+12ab^2+A^3-3A^2bc+3Ab^2c^2+4Ab^2-b^3c^3+4b^3c\big)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_4 =& 2A(ad+b(-3c-2d+3))^3 \big(a^3(-(2c+3d-5))(2c+3d-3)(2c+3d-1)\\
& +a^2 \big(A\big(12c^2+36c(d-1)+27(d-2)d+31\big)+3b\big(c\big(4c^2+12c(d-1)\\
& +9(d-2)d+13)+8(d-1)\big)\big)-a\big(A^2(6c+9d-9)\\
& +2Ab(3c(2c+3d-3)-4)+3b^2\big(c^2(2c+3d-3)-4d+4\big)\big)+A^3+3A^2bc\\
& +Ab^2\big(3c^2+4\big)+b^3c\big(c^2-4\big)\big)
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_3 =& (a+b)^3\big(8d^3(2c+3d-3)a^5+\big(b\big(16c^4+96(d-1)c^3+72\big(d^2-6d+3\big)c^2 -\\
& 8\big(13d^3+36d^2-81d+27\big)c-9\big(7d^4-4d^3-30d^2+36d-9\big)\big)-8Ad^3\big)a^4\\
& -4b\big(A\big(8c^3+36(d-1)c^2+18\big(2d^2-6d+3\big)c+3\big(5d^3-21d^2+27d-9\big)\big)\\
& +\big(8c^4-36(2d+1)c^3-54\big(5d^2-5d-1\big)c^2-3\big(83d^3-207d^2+117d+9\big)\\
& c-18(3-2d)^2(d-1)d\big)\big)a^3-2b\big(\big(204c^4+36(23d-26)c^3+27\big(39d^2-98d+59\big)c^2\\
& +4(3-2d)^2(34d-33)c+12(d-1)(2d-3)^3\big)b^2-6A\big(4c^3-6(d+2)\\
& c^2+\big(-15d^2+18d+9\big)c-2(3-2d)^2d\big)b-3A^2(2c+3d-3)^2\big)a^2+4b\big((-2c-3d+3)\\
& A^3-3bc(2c+3d-3)A^2+b^2\big(48c^3+9(11d-17)c^2+18(3-2d)^2c+2(2d-3)^3\big)A\\
& +b^3c\big(52c^3+3(35d-53)c^2+18(3-2d)^2c+2(2d-3)^3\big)\big)a+b(A+bc)^4\big)\\
& -6(a+b)^{3/2}\sqrt{-c-d+1}(b(-3c-2d+3)+ad)\big(4d^2\big(4c^2+4(3d-4)c\\
& +3\big(3d^2-8d+5\big)\big)a^5+\big(8A(2c+3d-2)d^2+b\big(16c^4-112c^3-8\big(19d^2+3d-36\big)c^2\\
& -4\big(54d^3-95d^2-30d+81\big)c-9\big(7d^4-22d^3+16d^2+14d-15\big)\big)\big)a^4\\
& +2\big(\big(56c^4+4(72d-89)c^3+6\big(80d^2-204d+135\big)c^2\\
& +3\big(102d^3-403d^2+564d-267\big)c+9\big(8d^4-43d^3+91d^2-89d+33\big)\big)b^2\\
& +A\big(-48d^3+117d^2-66d+c^2(4-48d)-12c\big(9d^2-9d+1\big)+9\big)b+2A^2d^2\big)a^3\\
& -2b\big(\big(4c^2+2(12d-7)c+17d^2-33d+12\big)A^2-3b\big(24c^3+(76d-72)c^2\\
& +2\big(32d^2-75d+35\big)c+16d^3-61d^2+70d-21\big)A+b^2\big(60c^4+6(30d-31)c^3\\
& +\big(157d^2-261d+108\big)c^2+\big(48d^3-43d^2-126d+117\big)c+12(3-2d)^2(d-1)\big)\big)\\
& a^2+2b\big(-A^3+b\big(22c^2+(30d-41)c+8d^2-21d+15\big)A^2\\
& +b^2\big(-36c^3+(105-48d)c^2-2\big(8d^2-45d+51\big)c\\
& +4(3-2d)^2\big)A+b^3c\big(14c^3+(18d+1)c^2+\big(8d^2+15d-45\big)c+4(3-2d)^2\big)\big)a\\
& +b(A+bc)^2\big(A^2-2b(c+1)A+b^2c(c+2)\big)\big)\alpha_1
\end{aligned}
$$

$$
\begin{aligned}
\Upsilon_5 =& \left(8d^3(2c+3d-3)a^5 + \left(b\left(16c^4+96(d-1)c^3+72\left(d^2-6d+3\right)c^2\right.\right.\right.\\
&-8\left(13d^3+36d^2-81d+27\right)c-9\left(7d^4-4d^3-30d^2+36d-9\right)\left.\right)-8Ad^3\left.\right)a^4\\
&-4b\left(A\left(8c^3+36(d-1)c^2+18\left(2d^2-6d+3\right)c+3\left(5d^3-21d^2+27d-9\right)\right)\right.\\
&+b\left(8c^4-36(2d+1)c^3-54\left(5d^2-5d-1\right)c^2-3\left(83d^3-207d^2+117d+9\right)c\right.\\
&-18(3-2d)^2(d-1)d\left.\right)\left.\right)a^3-2b\left(\left(204c^4+36(23d-26)c^3+27\left(39d^2-98d+59\right)c^2\right.\right.\\
&+4(3-2d)^2(34d-33)c+12(d-1)(2d-3)^3\left.\right)b^2-6A\left(4c^3-6(d+2)c^2+\left(-15d^2+18d\right.\right.\\
&+9\left.\right)c-2(3-2d)^2d\left.\right)b-3A^2(2c+3d-3)^2\left.\right)a^2\\
&+4b\left((-2c-3d+3)A^3-3bc(2c+3d-3)A^2\right.\\
&+b^2\left(48c^3+9(11d-17)c^2+18(3-2d)^2c+2(2d-3)^3\right)A\\
&+b^3c\left(52c^3+3(35d-53)c^2+18(3-2d)^2c\right.\\
&+2(2d-3)^3\left.\right)\left.\right)a+b(A+bc)^4\left.\right)(a+b)^3+6\sqrt{-c-d+1}(b(-3c-2d+3)+ad)\\
&\left(4d^2\left(4c^2+4(3d-4)c+3\left(3d^2-8d+5\right)\right)a^5+\left(8A(2c+3d-2)d^2+b\left(16c^4-112c^3\right.\right.\right.\\
&-8\left(19d^2+3d-36\right)c^2-4\left(54d^3-95d^2-30d+81\right)c\\
&-9\left(7d^4-22d^3+16d^2+14d-15\right)\left.\right)\left.\right)a^4+2\left(\left(56c^4+4(72d-89)c^3\right.\right.\\
&+6\left(80d^2-204d+135\right)c^2+3\left(102d^3-403d^2+564d-267\right)c\\
&+9\left(8d^4-43d^3+91d^2-89d+33\right)\left.\right)b^2\\
&+A\left(-48d^3+117d^2-66d+c^2(4-48d)-12c\left(9d^2-9d+1\right)+9\right)b+2A^2d^2\left.\right)a^3\\
&-2b\left(\left(4c^2+2(12d-7)c+17d^2-33d+12\right)A^2\right.\\
&-3b\left(24c^3+(76d-72)c^2+2\left(32d^2-75d+35\right)\right.\\
&c+16d^3-61d^2+70d-21\left.\right)A+b^2\left(60c^4+6(30d-31)c^3+\left(157d^2-261d+108\right)c^2\right.\\
&+\left(48d^3-43d^2-126d+117\right)c+12(3-2d)^2(d-1)\left.\right)\left.\right)a^2\\
&+2b\left(-A^3+b\left(22c^2+(30d-41)c+8d^2-21d+15\right)A^2+b^2\left(-36c^3+(105-48d)c^2\right.\right.\\
&-2\left(8d^2-45d+51\right)c+4(3-2d)^2\left.\right)A+b^3c\left(14c^3+(18d+1)c^2+\left(8d^2+15d-45\right)c\right.\\
&+4(3-2d)^2\left.\right)\left.\right)a+b(A+bc)^2\left(A^2-2b(c+1)A+b^2c(c+2)\right)\left.\right)\alpha_2(a+b)^{3/2}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_3 =& 2(a+b)^3\left(8d^3(2c+3d-3)a^5+\left(8Ad^3+b\left(16c^4+96(d-1)c^3+72\left(d^2-6d+3\right)\right.\right.\right.\\
&c^2-8\left(13d^3+36d^2-81d+27\right)c-9\left(7d^4-4d^3-30d^2+36d-9\right)\left.\right)\left.\right)a^4\\
&+4b\left(A\left(8c^3+36(d-1)c^2+18\left(2d^2-6d+3\right)c\right.\right.\\
&+3\left(5d^3-21d^2+27d-9\right)\left.\right)+b\left(-8c^4+36(2d+1)c^3+54\left(5d^2-5d-1\right)c^2\right.\\
&+3\left(83d^3-207d^2+117d+9\right)c+18(3-2d)^2(d-1)d\left.\right)\left.\right)a^3\\
&+2b\left(-\left(204c^4+36(23d-26)\right.\right.\\
&c^3+27\left(39d^2-98d+59\right)c^2+4(3-2d)^2(34d-33)c+12(d-1)(2d-3)^3\left.\right)b^2\\
&-6A\left(4c^3-6(d+2)c^2+\left(-15d^2+18d+9\right)c-2(3-2d)^2d\right)b+3A^2(2c+3d-3)^2\left.\right)a^2\\
&+4b\left((2c+3d-3)A^3-3bc(2c+3d-3)A^2\right.\\
&-b^2\left(48c^3+9(11d-17)c^2+18(3-2d)^2c+2(2d-3)^3\right)A+b^3c\left(52c^3+3(35d-53)c^2\right.\\
&+18(3-2d)^2c+2(2d-3)^3\left.\right)\left.\right)a+b(A-bc)^4\left.\right)-6\gamma_1(a+b)^{3/2}\sqrt{1-c-d}\\
&(b(-3c-2d+3)+ad)\left(4d^2\left(4c^2+4(3d-4)c+3\left(3d^2-8d+5\right)\right)a^5+\left(b\left(16c^4-112c^3\right.\right.\right.\\
&-8\left(19d^2+3d-36\right)c^2-4\left(54d^3-95d^2-30d+81\right)c
\end{aligned}
$$

$$
\begin{aligned}
&-9\left(7d^4-22d^3+16d^2+14d-15\right)\big)-8Ad^2(2c+3d-2)\big)\,a^4\\
&+2\left(\left(56c^4+4(72d-89)c^3+6\left(80d^2-204d+135\right)c^2+3\left(102d^3-403d^2+564d-267\right)c\right.\right.\\
&+9\left(8d^4-43d^3+91d^2-89d+33\right)\big)\,b^2+A\left(48d^3-117d^2+66d+c^2(48d-4)\right.\\
&+12c\left(9d^2-9d+1\right)-9\big)\,b+2A^2d^2\big)\,a^3-2b\left(\left(4c^2+2(12d-7)c+17d^2-33d+12\right)A^2\right.\\
&+3b\left(24c^3+(76d-72)c^2+2\left(32d^2-75d+35\right)c+16d^3-61d^2+70d-21\right)A\\
&+b^2\left(60c^4+6(30d-31)c^3+\left(157d^2-261d+108\right)c^2\right.\\
&+\left(48d^3-43d^2-126d+117\right)c+12(3-2d)^2(d-1)\big)\big)\,a^2\\
&+2b\left(A^3+b\left(22c^2+(30d-41)c+8d^2-21d+15\right)A^2+b^2\left(36c^3+3(16d-35)c^2\right.\right.\\
&+2\left(8d^2-45d+51\right)c-4(3-2d)^2\big)\,A\\
&+b^3c\left(14c^3+(18d+1)c^2+\left(8d^2+15d-45\right)c\right.\\
&+4(3-2d)^2\big)\big)\,a+b(A-bc)^2\left(A^2+2b(c+1)A+b^2c(c+2)\right)\big)\,.
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_5=&\left(8d^3(2c+3d-3)a^5+\left(8Ad^3+b\left(16c^4+96(d-1)c^3+72\left(d^2-6d+3\right)c^2\right.\right.\right.\\
&-8\left(13d^3+36d^2-81d+27\right)c-9\left(7d^4-4d^3-30d^2+36d-9\right)\big)\big)\,a^4\\
&+4b\left(A\left(8c^3+36(d-1)c^2+18\left(2d^2-6d+3\right)c+3\left(5d^3-21d^2+27d-9\right)\right)\right.\\
&+b\left(-8c^4+36(2d+1)c^3+54\left(5d^2-5d-1\right)c^2\right.\\
&+3\left(83d^3-207d^2+117d+9\right)c+18(3-2d)^2(d-1)d\big)\big)\,a^3\\
&+2b\left(-\left(204c^4+36(23d-26)c^3+27\left(39d^2-98d+59\right)c^2\right.\right.\\
&+4(3-2d)^2(34d-33)c+12(d-1)(2d-3)^3\big)\,b^2-6A\left(4c^3-6(d+2)c^2\right.\\
&+\left(-15d^2+18d+9\right)c-2(3-2d)^2d\big)\,b+3A^2(2c+3d-3)^2\big)\,a^2+4b\left((2c+3d-3)A^3\right.\\
&-3bc(2c+3d-3)A^2-b^2\left(48c^3+9(11d-17)c^2+18(3-2d)^2c+2(2d-3)^3\right)A\\
&+b^3c\left(52c^3+3(35d-53)c^2+18(3-2d)^2c+2(2d-3)^3\right)\big)\,a+b(A-bc)^4\big)\,(a+b)^3\\
&+6\gamma_2\sqrt{-c-d+1}(b(-3c-2d+3)+ad)\left(4d^2\left(4c^2+4(3d-4)c+3\left(3d^2-8d+5\right)\right)a^5\right.\\
&+\left(b\left(16c^4-112c^3-8\left(19d^2+3d-36\right)c^2\right.\right.\\
&-4\left(54d^3-95d^2-30d+81\right)c-9\left(7d^4-22d^3+16d^2+14d-15\right)\big)\\
&-8Ad^2(2c+3d-2)\big)\,a^4+2\left(\left(56c^4+4(72d-89)c^3+6\left(80d^2-204d+135\right)c^2\right.\right.\\
&+3\left(102d^3-403d^2+564d-267\right)c+9\left(8d^4-43d^3+91d^2-89d+33\right)\big)\,b^2\\
&+A\left(48d^3-117d^2+66d+c^2(48d-4)+12c\left(9d^2-9d+1\right)-9\right)b\\
&+2A^2d^2\big)\,a^3-2b\left(\left(4c^2+2(12d-7)c+17d^2-33d+12\right)A^2+3b\left(24c^3+(76d-72)c^2\right.\right.\\
&+2\left(32d^2-75d+35\right)c+16d^3-61d^2+70d-21\big)\,A+b^2\left(60c^4+6(30d-31)c^3\right.\\
&+\left(157d^2-261d+108\right)c^2+\left(48d^3-43d^2-126d+117\right)c+12(3-2d)^2(d-1)\big)\big)\,a^2\\
&+2b\left(A^3+b\left(22c^2+(30d-41)c+8d^2-21d+15\right)A^2+b^2\left(36c^3+3(16d-35)c^2\right.\right.\\
&+2\left(8d^2-45d+51\right)c-4(3-2d)^2\big)\,A+b^3c\left(14c^3+(18d+1)c^2+\left(8d^2+15d-45\right)c\right.\\
&+4(3-2d)^2\big)\big)\,a+b(A-bc)^2\left(A^2+2b(c+1)A+b^2c(c+2)\right)\big)\,(a+b)^{3/2}
\end{aligned}
$$