Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 9, Number 2, pp. 221–226 (2014)
http://campus.mst.edu/adsa

Multiplicity in Nonlinear Boundary Value Problems for Ordinary Differential Equations

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Abstract

We provide conditions for the existence of multiple solutions of two-point boundary value problems for a system of two nonlinear second-order ordinary differential equations.

AMS Subject Classifications: 34B15.
Keywords: Boundary value problem, multiple solutions, vector fields.

1 Introduction

We present existence and multiplicity results for systems of the type

$$\begin{cases} x'' &= \varphi(t, x, y), \\ y'' &= \psi(t, x, y). \end{cases} \tag{1.1}$$

Our method is the shooting one. For the scalar problem

$$x'' = f(t, x), \quad x(0) = 0, \; x(1) = 0, \tag{1.2}$$

the shooting approach says: if $x(t; \gamma)$ is a solution of the Cauchy problem

$$x'' = f(t, x), \quad x(0) = 0, \; x'(0) = \gamma, \tag{1.3}$$

then the problem (1.2) is solvable if

1) $x(t; \gamma)$ extends to $t = 1$;

Received March 20, 2014; Accepted May 6, 2014
Communicated by Sandra Pinelas

2) $x(t;\gamma)$ continuously depends on γ;

3) there exist γ_1 and γ_2 such that

$$x(1;\gamma_1)x(1;\gamma_2) < 0. \tag{1.4}$$

We provide similar results for the problem (1.1),

$$x(0) = 0,\ y(0) = 0,\ x(1) = 0,\ y(1) = 0. \tag{1.5}$$

Our approach uses two-dimensional vector fields [1, 2].

A solution of the boundary value problem (1.1), (1.5) is a vector function $(x(t), y(t))$ with $C^2[0,1]$ components.

2 Vector Field

We suppose that the right sides of (1.1) are continuous functions and there are also continuous partial derivatives of ϕ and ψ with respect to x and y. This is enough for continuous dependence of solutions on the initial data. We assume also that all solutions of (1.1) extend to the interval $[0,1]$.

Given $(\alpha,\beta) \in \mathbb{R}^2$, we denote by $(x(t;\alpha,\beta), y(t;\alpha,\beta))$ a solution of (1.1) such that

$$x(0) = y(0) = 0, \quad x'(0) = \alpha,\ y'(0) = \beta. \tag{2.1}$$

Define the mapping

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \Phi(\alpha,\beta) = ((x(1;\alpha,\beta), y(1;\alpha,\beta)). \tag{2.2}$$

It is well defined due to the above assumptions on continuous dependence and extendability.

The critical points of the vector field Φ are (α,β) such that $\Phi(\alpha,\beta) = (0,\ 0)$. Any critical point generates a solution to the problem (1.1), (1.5). In order to look for solutions of (1.1), (1.5), we investigate $\Phi(\alpha,\beta)$ and show that under certain conditions, there exist (α,β) such that $\Phi(\alpha,\beta) = (0,\ 0)$.

It is convenient for our purposes to consider the initial conditions in the form

$$\begin{aligned} &x(0) = 0,\ x'(0) = R\cos\Theta,\ y(0) = 0,\ y'(0) = R\sin\Theta, \\ &0 \le \Theta < 2\pi. \end{aligned} \tag{2.3}$$

The initial values are located on circles C_R of radius R, where R varies from zero to infinity.

3 Tools

In what follows we make use of Brower degree

$$\deg(\Phi, C_R, (0,0)),$$

which is identical with the **rotation** number $\gamma(\Phi; C_R)$ of the vector field Φ on circles C_R (the terminology is as in [2, Ch. 1]).

Since our considerations are based on the theory developed in [2], we prefer to formulate the results in terms of rotation numbers.

Theorem 3.1 (See [2, Theorem 3.1])**.** *Suppose a continuous vector field Φ does not have critical points (zero vectors) in a closed domain $\bar{\Omega}$. Then the rotation number $\gamma(\Phi; \Gamma)$ on the boundary Γ of Ω is zero.*

Corollary 3.2. *In the conditions of Theorem 3.1, if $\gamma(\Phi; \Gamma) \neq 0$, then there is a critical point in Ω.*

Proposition 3.3. *Suppose Ω is an annular region with inner and outer boundaries Γ_1 and Γ_2 respectively. Then the rotation number of the vector field Φ over the boundary Γ of Ω is*

$$\gamma(\Phi; \Gamma) = \gamma(\Phi; \Gamma_2) - \gamma(\Phi; \Gamma_1).$$

Corollary 3.4. *If an annular region Ω does not contain critical points of the vector field Φ, then*

$$\gamma(\Phi; \Gamma_2) = \gamma(\Phi; \Gamma_1).$$

The **conclusion** is the following. If

$$\gamma(\Phi; C_{R_1}) \neq \gamma(\Phi; C_{R_2}),$$

where $R_1 > R_2$, then in the respective annular region $D(R_1, R_2)$, there is a critical point of the vector field Φ or, in other words, $D(R_1, R_2)$ contains an initial condition (α, β) such that a solution of the Cauchy problem

$$\begin{cases} x'' &= \varphi(t, x, y), \\ y'' &= \psi(t, x, y), \end{cases}$$

$$x(0) = 0, \quad y(0) = 0, \quad x'(0) = \alpha, \quad y'(0) = \beta,$$

satisfies the boundary condition

$$x(1) = 0, \quad y(1) = 0.$$

4 Results

Theorem 4.1. *Let Φ be the vector field defined in* (2.2). *Suppose $\gamma(\Phi; C_R) \neq 0$ for some R. Then there exists a solution $(x(t), y(t))$ of the problem* (1.1), (1.5) *such that*

$$x'^2(0) + y'^2(0) < R^2.$$

Proof. The proof follows from the results of the previous section. □

Theorem 4.2. *Let Φ be the vector field defined in* (2.2). *Suppose $D(R_1, R_2)$ is an annular region with a boundary $C_{R_1} \cup C_{R_2}$ and*

$$\gamma(\Phi; C_{R_1}) \neq \gamma(\Phi; C_{R_2}).$$

Then there exists a solution $(x(t), y(t))$ of the problem (1.1), (1.5) *such that*

$${R_1}^2 < x'^2(0) + y'^2(0) < {R_2}^2.$$

Proof. The proof follows from the results of the previous section. □

Theorem 4.3. *Let Φ be the vector field defined in* (2.2). *Suppose there are multiple disjoint annular regions D_i with the property described in Theorem 4.2. Then there are multiple solutions (x_i, y_i) of the problem* (1.1), (1.5) *with the initial conditions $(x_i'(0), y_i'(0)) \in D_i$.*

Proof. The proof follows from Theorem 4.2. □

5 Example

Consider the problem

$$\begin{cases} x'' &= y - x^3, \\ y'' &= -x^3 \end{cases} \tag{5.1}$$

$$x(0) = 0,\ y(0) = 0,\ x(1) = 0,\ y(1) = 0. \tag{5.2}$$

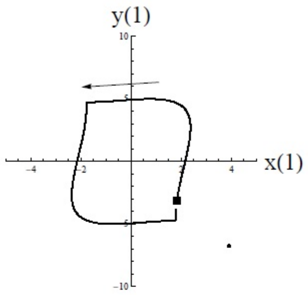
Take the initial conditions

$$x(0) = 0,\ y(0) = 0,\ x'(0) = \alpha,\ y'(0) = \beta,$$

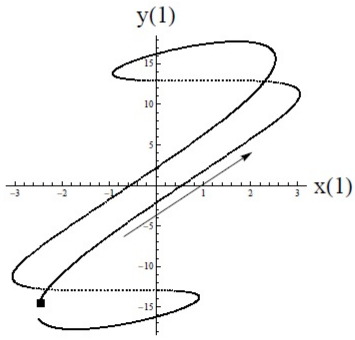
where $\alpha = R\cos\Theta$, $\beta = R\sin\Theta$, $0 \leq \Theta < 2\pi$. Consider the vector field

$$(\alpha, \beta) \to (x(1; \alpha, \beta), y(1; \alpha, \beta)).$$

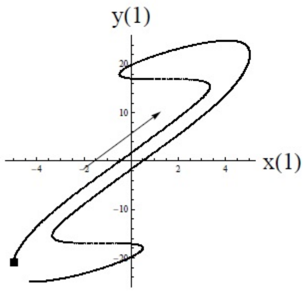
Critical points generate solutions of the BVP (5.1), (5.2).



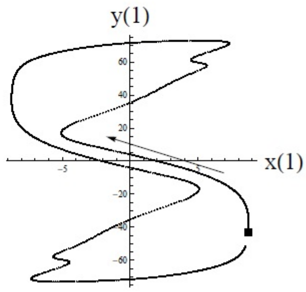
$R = 5.0,\ \gamma = 1$



$R = 13.0,\ \gamma = 1$



$R = 17.0,\ \gamma = -1$



$R = 60.0,\ \gamma = 1.$

The figures show vectors $(x(1;\alpha,\beta), y(1;\alpha,\beta))$, where $\alpha = R\cos\Theta$, $\beta = R\sin\Theta$, $\Theta \in [0, 2\pi - \varepsilon]$, where ε is a small positive number. The images of C_R under the transformation $C_R \to \Phi(C_R)$ intentionally are not closed in order to detect the orientation. Small black squares indicate starting point ($\Theta = 0$). Arrows indicate the orientation as Θ increases from zero to 2π.

It appears that the problem (5.1), (5.2) has at least two nontrivial solutions with $(x'(0), y'(0))$ belonging to the annular regions with the boundaries respectively $C_{13} \cup C_{17}$ and $C_{17} \cup C_{60}$.

6 Final Remarks

If system (1.1) is linear, then the images of circles C_R under the transformation $C_R \to \Phi(C_R)$ are ellipses with positive or negative orientations as shown in [5]. Thus the results are possible for asymptotically linear systems of the form (1.1), where a comparison is made of the linear system of variations with respect to the trivial solution (then $\phi(t, 0, 0)$ and $\phi(t, 0, 0)$ should be identical zero) and the linear system at infinity.

The above technique can be applied to detecting solutions for the fourth order differential systems [3] and equations [4] as well.

Similarly problems with other two-point boundary conditions for the four dimensional systems can be considered. The vector field Φ can be defined appropriately.

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