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Blow-up and Decay Bounds in Hamilton–Jacobi-type Problems

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Abstract

In this paper, we investigate a class of nonlinear parabolic problems, known as viscous Hamilton–Jacobi-type problems. We establish conditions on data sufficient to insure that blow-up occurs in finite time. Moreover, conditions on data and geometry of the spatial domain are derived, ensuring the solution to exist for all time with exponential decay.

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1 Introduction

Qualitative properties of solutions to nonlinear parabolic problems, as blow-up, decay bounds, extinction in finite time, have been largely investigated, due to their interesting applications in physics, chemistry and biology (see [1, 3, 10–12]). In this paper we consider a class of nonlinear parabolic equations which source term depend on the gradient of the solution, known as "viscous Hamilton–Jacobi" equation. These equations are also related to physical theory of growth and roughening of surfaces, known as Kadar–Parisi–Zhang equations (see [4]).

More precisely, we consider the following boundary value problem:

$$\begin{cases} u_t = \Delta u + k(t) f(|\nabla u|^2) & (x,t) \in \Omega \times (0,t^*), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$ and t^* is the possible blow-up time. $u_0(x)$ is a nonnegative C^1 function in Ω which vanishes on $\partial\Omega$, and the time dependent coefficient $k(t)$ is a positive, bounded C^1- function. We consider classical solutions of problem (1), which are nonnegative by maximum principle. It is well known that, as time evolves, the asymptotic behavior of $u(x,t)$ may change in consequence of the choice of k, f and u_0.

The goal of this paper is to show how the choices of the data give rise to solutions with very different behavior: blow-up phenomena or global existence. Compared with Hamilton–Jacobi equation where the source term is $|\nabla u|^p, p > 1$ (see [12]), we have a more general function $k(t)f(|\nabla u|^2)$ and the time dependent coefficient $k(t)$ plays an important role in order to have bounded or unbounded solutions. The paper is organized in this way: in Section 2, we prove that if the initial data u_0 is "large enough" and k and f satisfy appropriate restrictions, the solution blows up in finite time t^* and an upper bound of t^* is derived. On the contrary, in Section 3, conditions on k, f, u_0, and Ω are established, sufficient to obtain exponential decay in time of the solution u.

2 Blow-up in Finite Time

The aim of this section is to introduce conditions on data sufficient to ensure that the solution u blows up in finite time t^* and to derive an upper bound of t^*. Let us introduce the auxiliary function

$$U(t) = k(t) \int_\Omega u^\sigma(x,t)\, dx,\ \sigma \geq 3, \tag{2.1}$$

which, from assumptions on $k(t)$ and $u_0(x)$, has positive initial value

$$U_0 = U(0) = k(0) \int_\Omega u_0(x)^\sigma dx.$$

We prove the following.

Theorem 2.1. *Let $u(x,t)$ be the solution of problem* (1.1) *with Ω a bounded domain in $\mathbb{R}^N$ and $U(t)$ defined in* (2.1). *Assume that*

$$f(s^2) \geq s^{2\xi}, \tag{2.2}$$

with $\xi := \dfrac{\sigma-1}{\sigma-2}$ *and*

$$\frac{k'(t)}{k(t)} \geq \beta > 0. \tag{2.3}$$

If the initial data $u_0(x)$ is "large enough" in the sense of

$$U_0 > \frac{\sigma^2(\sigma-1)^{\sigma-2}|\Omega|}{4\lambda_1}, \tag{2.4}$$

with $|\Omega|$ the N-volume of Ω, and λ_1 the first eigenvalue of the following problem

$$\begin{cases} \Delta\varphi - \lambda\varphi = 0,\ \varphi > 0, \quad x \in \Omega, \\ \varphi(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

then

$$\lim_{t \to t^*} \max_{\Omega} u(x,t) = +\infty,$$

with

$$t^* \leq T^* := -\frac{1}{a(\xi-1)} ln\Big[1 - \frac{a}{b} U_0^{1-\xi}\Big], \tag{2.5}$$

a, b positive constants depending on data.

Proof. Firstly we prove that $U(t)$ is strictly increasing. By using (1.1) and the divergence theorem, we have

$$U'(t) = \frac{k'}{k} U(t) - k\sigma(\sigma-1) \int_\Omega u^{\sigma-2} |\nabla u|^2 dx + k^2 \sigma \int_\Omega u^{\sigma-1} f(|\nabla u|^2).$$

By using (2.3) and (2.2), we obtain (by using for brevity $\xi = \dfrac{\sigma-1}{\sigma-2}$)

$$U'(t) \geq \beta U(t) - k\sigma(\sigma-1) \int_\Omega u^{\sigma-2} |\nabla u|^2 dx + k^2 \sigma \int_\Omega u^{\sigma-1} |\nabla u|^{2\xi} dx \tag{2.6}$$

$$= \beta U(t) - k\sigma(\sigma-1) J + k^2 \sigma I.$$

We now apply Hölder's inequality to obtain

$$J = \int_\Omega u^{\sigma-2} |\nabla u|^2 dx \leq \Big[\int_\Omega u^{\sigma-1} |\nabla u|^{2\xi} dx\Big]^{\frac{1}{\xi}} |\Omega|^{1-\frac{1}{\xi}},$$

from which we obtain

$$I = \int_\Omega u^{\sigma-1} |\nabla u|^{2\xi} dx \geq J^\xi |\Omega|^{1-\xi}. \tag{2.7}$$

We now replace (2.7) in (2.6) and obtain

$$U' \geq \beta U - k\sigma(\sigma-1) J + k^2 \sigma |\Omega|^{1-\xi} J^\xi \tag{2.8}$$

$$= \beta U + \sigma k J \big\{ -(\sigma-1) + k |\Omega|^{1-\xi} J^{\xi-1} \big\}.$$

Then by using the Rayleigh inequality, we have

$$J = \frac{4}{\sigma^2} \int_\Omega |\nabla u^{\frac{\sigma}{2}}|^2 dx \geq \frac{4}{\sigma^2} \lambda_1 \int_\Omega u^\sigma dx. \tag{2.9}$$

Replacing (2.9) in (2.8), we obtain

$$U' \geq \beta U + \sigma k J \left\{ -(\sigma - 1) + \Big[\frac{4}{\sigma^2|\Omega|}\lambda_1\Big]^{\xi-1} k\Big(\int_\Omega u^\sigma dx\Big)^{\xi-1} \right\},$$

and since $k(t) \geq 1$ as a consequence of (2.3), we obtain

$$U' \geq \beta U + \sigma k J \left\{ -(\sigma - 1) + \Big[\frac{4}{\sigma^2|\Omega|}\lambda_1\Big]^{\xi-1} U^{\xi-1} \right\}, \tag{2.10}$$

where the last term is positive if

$$U(t) > \frac{\sigma^2(\sigma-1)^{\sigma-2}|\Omega|}{4\lambda_1},$$

and this is the case since we assume (2.4). Now by using again (2.9) in (2.10), we obtain the differential inequality

$$U' \geq \beta U + \frac{4}{\sigma}\lambda_1 U \left\{ -(\sigma - 1) + \Big[\frac{4}{\sigma^2|\Omega|}\lambda_1\Big]^{\xi-1} U^{\xi-1} \right\} \tag{2.11}$$

$$= \left\{ \beta - \frac{4\lambda_1(\sigma-1)}{\sigma} \right\} U(t) + \Big[\frac{4}{\sigma^2|\Omega|}\lambda_1\Big]^{\xi-1} U^\xi := aU(t) + bU^\xi(t),$$

with $a = \beta - \frac{4\lambda_1(\sigma-1)}{\sigma}$, $b = \Big[\frac{4}{\sigma^2|\Omega|}\lambda_1\Big]^{\xi-1}$. Now integrating the differential inequality

$$U' \geq aU(t) + bU^\xi(t),$$

from 0 to t, by using the substitution $U^{1-\xi}(t) = \eta(t)$, we obtain

$$\eta(t) \leq \left\{ \eta(0) - \frac{b}{a}\big[1 - e^{-a(\xi-1)t}\big] \right\} e^{a(\xi-1)t}.$$

Let T^* be defined by

$$\eta(0) = \frac{b}{a}\big[1 - e^{-a(\xi-1)T^*}\big]. \tag{2.12}$$

Then we have the blow-up of $U(t)$ at time $t^* < T^*$. From (2.12), we easily obtain (2.5). We remark that T^* is an upper bound of t^*. □

3 Decay in Time

From the previous section results, we know that the solution of problem (1.1) may blow up in finite time: the goal of this section is to prove that this is not the case if we introduce suitable restrictions on data u_0, on f and Ω (see [7] for the source term as $k(t)u^p, p > 1$ and for a more general parabolic operator). Throughout this section, the coefficient $k(t)$ is assumed to be a positive bounded function. First we prove that $|\nabla u|$ is bounded for all time under the conditions contained in the following theorem.

Theorem 3.1. *Let* $u(x,t)$ *be a solution of* (1.1) *in a bounded strictly convex domain* $\Omega \subset \mathbb{R}^N$, $N \geq 2$, *with* $C^{2+\epsilon}$ *boundary* $\partial\Omega$. *Assume that* $f(s) \geq 0$ *satisfies the condition*

$$\frac{\partial}{\partial s}\frac{f(s^2)}{s} \geq 0, \quad s > 0, \tag{3.1}$$

and assume that the initial data $u_0(x) \geq 0$ *is small enough in the sense of*

$$\frac{f(G^2)}{G} \leq (N-1)K_{\min}, \tag{3.2}$$

with

$$G := \max_{\Omega} |\nabla u_0|, \tag{3.3}$$

$$K_{\min} := \min_{\partial\Omega} K(x) > 0, \tag{3.4}$$

where $K(x)$ *is the average curvature of* $\partial\Omega$. *Moreover if*

$$\lim_{t\downarrow 0} |\nabla u(x,t)| = |\nabla u_0(x,t)|, \quad (x,t) \in \partial\Omega, \tag{3.5}$$

we then conclude that

$$|\nabla u(x,t)| \leq G, \quad x \in \Omega, \; t > 0. \tag{3.6}$$

Proof. We prove that $|\nabla u|^2$ satisfies a parabolic inequality. In fact, we compute

$$\Delta|\nabla u|^2 - |\nabla u|^2_t + 2k(t)f'(|\nabla u|^2)(\nabla u, \nabla|\nabla u|^2) = 2u_{ik}u_{ik} \geq 0. \tag{3.7}$$

We remark that $k(t)f'(|\nabla u|^2)(\nabla u)$ is bounded by assumptions (3.1) and (3.3) in a time interval $[0, t_1]$. From the parabolic maximum principle (see [2, 8]), we have that the maximum of $|\nabla u|^2$ can be attained or at a boundary point or initially at $t = 0$. If we suppose that the maximum is attained at a boundary point, by (3.2), (3.3) we reach a contradiction (see [5,6,9]). Then the maximum is attained at $t = 0$, i.e., (3.6) holds. $\square$

Now we want to obtain an explicit exponential decay bound for $u(x,t)$ by considering the set of source functions satisfying the condition

$$\frac{\partial}{\partial s}\Big(\frac{f(s)}{s}\Big) \geq 0, \quad s > 0. \tag{3.8}$$

Note that this condition (3.8) implies (3.1).

Theorem 3.2. *Let* $u(x,t)$ *be a solution of problem* (1.1). *Assume that conditions* (3.2)–(3.5) *and* (3.8) *hold. Moreover on* $\partial\Omega \in C^{2+\epsilon}$ *assume that* $u_0 = 0$ *and* $|\nabla u_0|$ *is bounded. Then*

$$u(x,t) \leq \varphi_1(x)e^{-\lambda_1 t}\Big(\frac{1}{k_m H}\max_{\Omega}\frac{e^{k_m H u_0} - 1}{\varphi_1}\Big), \tag{3.9}$$

with $H := \dfrac{f(G^2)}{G^2}$, *and* $k_m = \max_{t>0} k(t)$.

Proof. First we remark that condition (3.8) implies (3.1), then the gradient is bounded. Now the equation in (1.1) can be rewritten as

$$u_t = \Delta u + k(t)\frac{f(|\nabla u|^2)}{|\nabla u|^2}\,|\nabla u|^2.$$

By Theorem 3.1, we have

$$0 = \Delta u + k(t)\frac{f(|\nabla u|^2)}{|\nabla u|^2}\,|\nabla u|^2 - u_t \leq \Delta u + k_m H\,|\nabla u|^2 - u_t. \tag{3.10}$$

Let us introduce the function $\tilde{u}(x,t) = e^{k_m H u} - 1$. By (3.10), $\tilde{u}$ satisfies the inequality

$$\Delta\tilde{u} - \tilde{u}_t = k_m H\, e^{k_m H u}\{\Delta u + k_m H\,|\nabla u|^2 - u_t\} \geq 0$$

and the boundary-initial conditions

$$\tilde{u}(x,t) = 0,\ \ (x,t) \in \partial\Omega \times (t > 0); \quad \tilde{u}(x,0) = e^{k_m H u_0} - 1 \geq 0,\ \ x \in \Omega.$$

It turns out that $\tilde{u}$ is a subsolution of the heat equation $w_t = \Delta w$, which is well known to satisfy

$$w \leq \Big(\max_\Omega \frac{w(x,0)}{\varphi_1}\Big)\varphi_1 e^{-\lambda_1 t}.$$

Then

$$e^{k_m H u} - 1 \leq \Big(\max_\Omega \frac{e^{k_m H u_0} - 1}{\varphi_1}\Big)\varphi_1 e^{-\lambda_1 t},$$

i.e.,

$$k_m H\,u \leq \log(1 + \Big(\max_\Omega \frac{e^{k_m H u_0} - 1}{\varphi_1}\Big)\varphi_1 e^{-\lambda_1 t}) \leq \Big(\max_\Omega \frac{e^{k_m H u_0} - 1}{\varphi_1}\Big)\varphi_1 e^{-\lambda_1 t}.$$

From the previous inequality, we get (3.9). This completes the proof. □

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