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Control in Systems of Delay Hyperbolic Equations

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Abstract

A control problem for delay partial differential equations is considered. The Fourier method and special functions, called the delayed sine and delayed cosine functions, are used for solving.

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1 Introduction

Partial functional differential equations arise from various problems in biology, medicine, control theory, climate modelling, and many others, which are characterized by both spatial and temporal variables and exhibit various spatio-temporal patterns [17]. Despite of their wide range of applications, the systematic study of such equations began in the 70s [4], but still there are no general methods for obtaining solutions for partial functional differential equations or solving control problems for them. Some results on controllability properties of systems of linear ordinary differential equations with a constant delay were obtained in [6, 12]. In the mentioned articles, the authors used special delayed matrix functions, and analogues of some of these functions will be used in the present paper. Controllability of linear discrete systems of ODE with constant coefficient and a pure delay was considered in [7], where results were obtained using the delayed matrix exponential. Also it might be useful to provide some references on the solutions for different types of delay equations obtained using the spacial delay matrix functions [2, 5, 11]. Some results on control problems for delay PDE using semigroups theory were obtained in [1, 3]. The control problem for the delay parabolic equation

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was solved in [13], and these results were extended to a more general class of equations in [9].

In this paper, a control problem is solved for a wave equation with a single delay

$$\frac{\partial^2\xi(x,t)}{\partial t^2} = a^2\frac{\partial^2\xi(x,t-\tau)}{\partial x^2} + c\xi\left(x,t-\tau\right) + u\left(x,t\right) \tag{1.1}$$

with initial conditions

$$\begin{aligned} \xi\left(x,t\right) = \varphi\left(x,t\right),\ (x,t) \in [0,l] \times [-\tau,0], \\ \xi'\left(x,t\right) = \varphi'\left(x,t\right),\ (x,t) \in [0,l] \times [-\tau,0] \end{aligned} \tag{1.2}$$

and boundary conditions

$$\begin{aligned} \xi\left(0,t\right) = \mu_1\left(t\right),\ t \in [-\tau,T], \\ \xi\left(l,t\right) = \mu_2\left(t\right),\ t \in [-\tau,T], \end{aligned} \tag{1.3}$$

where $\tau > 0$, $\tau =$const, $a, c \in \mathbb{R}$, $l, T \in \mathbb{R}^+$, functions $\mu_i : [-\tau, T] \to \mathbb{R}$, $i = 1, 2$, $\varphi : [0, l] \times [-\tau, 0] \to \mathbb{R}$ are twice continuously differentiable such that

$$\mu_1\left(t\right) = \varphi\left(0,t\right),\ \mu_2\left(t\right) = \varphi\left(l,t\right),\ t \in [-\tau,0].$$

The functions

$$\xi, u : [0,l] \times [-\tau,T] \to \mathbb{R}$$

are twice continuously differentiable with respect to x and t for $(x,t) \in [0,l] \times [-\tau,T]$.

We want to find a control function $u\left(x,t\right)$ such that the solution $\xi\left(x,t\right)$ of the first boundary value problem (1.1), (1.2), (1.3) at a set moment of time $t = T$ will achieve the set condition

$$\xi\left(x,T\right) = \Omega\left(x\right), \tag{1.4}$$

where $\Omega : [0,l] \to \mathbb{R}$, $\mu_1\left(T\right) = \Omega\left(0\right)$, $\mu_2\left(T\right) = \Omega\left(l\right)$.

The purpose of this paper is to describe a method of constructing the control function and to give a formal solution of the control problem. The results on existence, uniqueness and convergence of solutions of the wave equation with delay obtained in the form of Fourier series were discussed in [10] and references therein. To find a solution of the wave equation with delay, we need to provide some preliminary results from the theory of ordinary delay differential equations. That is done in Section 2. The delayed sine and delayed cosine functions (defined in Section 2 together with the description of their main properties) is used in order to analytically solve auxiliary initial problems arising when the Fourier method is applied for ordinary linear differential equations of the second order with a single delay. The control problem for delay hyperbolic equations is solved in Section 3.

2 Preliminaries: Second-order Ordinary DDE

Before solving the hyperbolic equation with a delay, we must recall some results from the theory of ordinary delay differential equations. A solution of the equation (1.1) satisfying all boundary and initial conditions (1.2), (1.3) will be constructed by the classical method of separation of variables (Fourier method) [16]. Nevertheless, due to delayed arguments, complications arise in solving analytically auxiliary initial Cauchy problems for second-order linear differential equations with a single delay. We overcome this circumstance by using special functions called the delayed sine and delayed cosine functions [8]. Here we give a definition of the delayed sine and delayed cosine functions, their basic properties, and solutions of the initial problems for second-order homogeneous and nonhomogeneous linear differential equations with a single delay. It must be mentioned that the delayed sine and delayed cosine functions can be used for the representation of solutions only for certain class of linear DDE, in other case a method proposed in [14] could be used.

2.1 Definition of Delayed Sine and Cosine Functions

Here we introduce the special functions called the delayed sine and delayed cosine functions.

It is well known that trigonometrical functions $\sin bt$ and $\cos bt$ can be represented in the form of power series

$$\sin bt = b\frac{t}{1!} - b^3\frac{t^3}{3!} + b^5\frac{t^5}{5!} - \ldots + (-1)^k\, b^{k+1}\frac{t^{2k+1}}{(2k+1)!} + \ldots$$

$$\cos bt = 1 - b^2\frac{t^2}{2!} + b^4\frac{t^4}{4!} - \ldots + (-1)^k\, b^{2n}\frac{t^{2k}}{(2k+1)!} + \ldots$$

We introduce "similar" functions represented not by series but by partial sums.

Definition 2.1. Let $b \in \mathbb{R}$. The delayed cosine function $\cos_\tau\{b,t\} : \mathbb{R} \to \mathbb{R}$ is a function defined as

$$\cos_\tau\{b,t\} = \begin{cases} 0, & \text{if} \quad -\infty < t < -\tau, \\ 1, & \text{if} \quad -\tau \le t < 0, \\ 1 - b^2\dfrac{t^2}{2!}, & \text{if} \quad 0 \le t < \tau, \\ \ldots & \\ 1 - b^2\dfrac{t^2}{2!} + b^4\dfrac{(t-\tau)^4}{4!} + \cdots & \\ +(-1)^k b^{2k}\dfrac{[t-(k-1)\tau]^{2k}}{(2k)!}, & \text{if} \quad (k-1)\,\tau \le t < k\tau, \\ \ldots & \end{cases}$$

a $2k$-degree polynomial on intervals $(k-1)\tau < t \leq k\tau$ “merged” in points $t = k\tau, \quad k = 0, 1, 2, \ldots$.

Definition 2.2. Let $b \in \mathbb{R}$. The delayed sine function $\sin_\tau\{b,t\} : \mathbb{R} \to \mathbb{R}$ is a function defined as

$$\sin_\tau\{b,t\} = \begin{cases} 0, & \text{if} \quad -\infty < t < -\tau, \\ b(t+\tau), & \text{if} \quad -\tau \leq t < 0, \\ b(t+\tau) - b^3\dfrac{t^3}{3!}, & \text{if} \quad 0 \leq t < \tau, \\ \cdots & \\ b(t+\tau) - b^3\dfrac{t^3}{3!} + \cdots & \\ +(-1)^k b^{2k+1}\dfrac{[t-(k-1)\tau]^{2k+1}}{(2k+1)!}, & \text{if} \quad (k-1)\tau \leq t < k\tau, \\ \cdots & \end{cases}$$

a $(2k+1)$-degree polynomial on intervals $(k-1)\tau < t \leq k\tau$ “merged” in points $t = k\tau, \quad k = 0, 1, 2, \ldots$.

2.1.1 Properties of Delayed Sine and Cosine Functions

Lemma 2.3. *Within intervals $(k-1)\tau < t < k\tau$, $k = 0, 1, 2, \ldots$, the rule of differentiation for the delayed cosine function can be formulated as*

$$\frac{d}{dt}\cos_\tau\{b,t\} = -b\sin_\tau\{b,t-\tau\}\,, \quad \frac{d^2}{dt^2}\cos_\tau\{b,t\} = -b^2\cos_\tau\{b,t-\tau\}\,.$$

Lemma 2.4. *Within intervals $(k-1)\tau < t \leq k\tau$, $k = 0, 1, 2, \ldots$, the rule of differentiation for the delayed sine function can be formulated as*

$$\frac{d}{dt}\sin_\tau\{b,t\} = b\cos_\tau\{b,t-\tau\}\,, \quad \frac{d^2}{dt^2}\sin_\tau\{b,t\} = -b^2\sin_\tau\{b,t-\tau\}\,.$$

Lemma 2.5. *Within intervals $(k-1)\tau < t < k\tau$, $k = 0, 1, 2, \ldots$, the rules of integration hold:*

$$\int_0^t \cos_\tau\{b,s\}\,ds = \frac{1}{b}\left\{\sin_\tau\{b,t\} - \sin_\tau\{b,0\}\right\},$$

$$\int_0^t \sin_\tau\{b,s\}\,ds = -\frac{1}{b}\left\{\cos_\tau\{b,t+\tau\} - \cos_\tau\{b,\tau\}\right\}.$$

2.2 Second-Order Linear Differential Equations with a Singe Delay

Let us consider a homogeneous linear DDE

$$x''(t) + b^2 x\,(t-\tau) = 0, \tag{2.1}$$

where $b \in \mathbb{R}$, $\tau > 0$ together with the initial Cauchy conditions

$$x(t) = \beta(t),\ x'(t) = \beta'(t),\ t \in [-\tau, 0]. \tag{2.2}$$

From Lemma 2.3, it follows that the delayed cosine function $\cos_\tau\{b,t\}$ is a solution of the initial Cauchy problems (2.1), (2.2) when $\beta(t) \equiv 1$, $t \in [-\tau, 0]$. Moreover, from Lemma 2.4, we can conclude that the delayed sine function $\sin_\tau\{b,t\}$ is a solution of (2.1), (2.2) when $\beta(t) \equiv b(t+\tau)$, $t \in [-\tau, 0]$.

Based on these results, the following theorem is given.

Theorem 2.6 (See [8])**.** *Let $\beta : [-\tau, 0] \to \mathbb{R}$ be a twice continuously differentiable function. Then the unique solution of the initial Cauchy problems* (2.1)*,* (2.2) *can be represented as*

$$x(t) = \beta\,(0) \cos_\tau \{b, t-\tau\} + \frac{1}{b}\beta'(0) \sin_\tau \{b, t-\tau\} - b\int_{-\tau}^{0} \sin_\tau \{b, t-2\tau-s\}\,\beta(s)ds, \tag{2.3}$$

where $t \in [-\tau, \infty]$.

Next we recall some results on a nonhomogeneous linear DDE

$$x''(t) + b^2 x\,(t-\tau) = f(t), \tag{2.4}$$

where $b \in \mathbb{R}$, $\tau > 0$, $f : [0, \infty] \to \mathbb{R}$. For our further calculations, we need to consider the Cauchy problem only with zero initial conditions

$$x(t) = 0,\ x'(t) = 0,\ t \in [-\tau, 0]. \tag{2.5}$$

Theorem 2.7 (See [8])**.** *The unique solution of the problems* (2.4)*,* (2.5) *is given by the formula*

$$x(t) = \frac{1}{b}\int_{0}^{t} \sin_\tau \{b, t-\tau-s\}\, f(s)ds, \tag{2.6}$$

where $t \in [-\tau, \infty]$.

Now, having these results, we can consider the control problem for the delay wave equation.

3 Hyperbolic Delay Equation

To find the control function for which the linear homogeneous delay partial differential equation with a single delay (1.1) with initial (1.2) and boundary (1.3) conditions has a solution, which satisfies the condition (1.4) at $t = T$, first we need to find an analytical solution $\xi(x, t)$ for the boundary value problem (1.1), (1.2), (1.3) which depends on the control function $u(x, t)$.

3.1 Constructing a Solution of (1.1)

We construct a solution in the form of sum

$$\xi(x,t) = \xi_0(x,t) + \xi_1(x,t) + \mu_1(t) + \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right], \tag{3.1}$$

where $(x, t) \in [0, l] \times [-\tau, T]$, $\xi_0(x, t)$ is a solution of a homogeneous equation

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = a^2 \frac{\partial^2 \xi(x,t-\tau)}{\partial x^2} + c\xi(x, t-\tau) \tag{3.2}$$

with zero boundary conditions

$$\xi_0(0,t) = 0,\ \xi_0(l,t) = 0,\ t \in [-\tau, T] \tag{3.3}$$

and nonzero initial conditions

$$\xi_0(x,t) = \Phi(x,t),\ \xi'_{0t}(x,t) = \Phi'(x,t),\ (x,t) \in [0,l] \times [-\tau, T], \tag{3.4}$$

where

$$\Phi(x,t) = \varphi(x,t) - \mu_1(t) - \frac{x}{l}\left[\mu_2(t) - \mu_1(t)\right]; \tag{3.5}$$

$\xi_1(x, t)$ is a solution of nonhomogeneous equation

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = a^2 \frac{\partial^2 \xi(x,t-\tau)}{\partial x^2} + c\xi(x, t-\tau) + F(x,t), \tag{3.6}$$

where

$$\begin{aligned} F(x,t) =& c\left\{\mu_1(t-\tau) + \frac{x}{l}\left[\mu_2(t-\tau) - \mu_1(t-\tau)\right]\right\} \\ &- \mu_1''(t) - \frac{x}{l}\left[\mu_2''(t) - \mu_1''(t)\right] + u(x,t) \end{aligned} \tag{3.7}$$

with zero boundary conditions

$$\xi_1(0,t) = 0,\ \xi_1(l,t) = 0,\ t \in [-\tau, T] \tag{3.8}$$

and zero initial conditions

$$\xi_1(x,t) = 0,\ \xi_1'(x,t) = 0,\ (x,t) \in [0,l] \times [-\tau, 0]. \tag{3.9}$$

3.1.1 Solving a Homogeneous DPDE (3.2)

For finding a solution $\xi = \xi_0(x,t)$ of (3.2), (3.3), (3.4), we will use the method of separation of variables. The solution $\xi_0(x,t)$ is seen as the product of two unknown functions $X(x)$ and $T(t)$, that is,

$$\xi_0(x,t) = X(x)\,T(t)\,. \tag{3.10}$$

Substituting (3.10) into the equation (3.2), we get

$$X(x)\,T''(t) = a^2 X''(x)\,T(t-\tau) + cX(x)\,T(t-\tau)\,.$$

Separating variables, we obtain

$$\frac{T''(t) - cT(t-\tau)}{a^2 T(t-\tau)} = \frac{X''(x)}{X(x)} = -\omega^2,$$

where ω is a constant. We consider two differential equations

$$T''(t) + \left(a^2\omega^2 - c\right) T(t-\tau) = 0,\ X''(x) + \omega^2 X(x) = 0. \tag{3.11}$$

Nonzero solutions of the second equation of (3.11) that satisfy zero boundary conditions

$$X(0) = 0,\ X(l) = 0,$$

exist for

$$\omega^2 = \omega_n^2 = \left(\frac{\pi n}{l}\right)^2,\ n = 1,2,\ldots, \tag{3.12}$$

and are defined as

$$X(x) = X_n(x) = A_n \sin\frac{\pi n}{l}x,\ n = 1,2,\ldots, \tag{3.13}$$

where A_n are arbitrary constants. Now we consider the first equation of (3.11) with $\omega = \omega_n$

$$T_n''(t) + k_n^2 T_n(t-\tau) = 0, \tag{3.14}$$

$$k_n = \sqrt{(\omega a)^2 - c} = \sqrt{\left(\frac{\pi n}{l}a\right)^2 - c},\ n = 1,2,\ldots. \tag{3.15}$$

Each of the equations of (3.14) represents a linear second-order delay differential equation with constant coefficients. We will specify initial conditions for each of them. To obtain such initial conditions, we expand the corresponding initial condition $\Phi(x,t)$ and its derivative (see (3.5)) into Fourier series

$$\begin{aligned}
\Phi(x,t) &= \sum_{n=1}^{\infty} \Phi_n(t) \sin\frac{\pi n}{l}x,\ (x,t) \in [0,l]\times[-\tau,0],\\
\Phi_t'(x,t) &= \sum_{n=1}^{\infty} \Phi_n'(t) \sin\frac{\pi n}{l}x,\ (x,t) \in [0,l]\times[-\tau,0],
\end{aligned} \tag{3.16}$$

$$\begin{aligned}\Phi_n(t) &= \frac{2}{l}\int_0^l \varphi(s,t)\sin\frac{\pi n}{l}sds - \frac{2}{\pi n}\left[\mu_1(t) + (-1)^n\mu_2(t)\right], t \in [-\tau, 0],\\ \Phi_n'(t) &= \frac{2}{l}\int_0^l \varphi'(s,t)\sin\frac{\pi n}{l}sds - \frac{2}{\pi n}\left[\mu_1'(t) + (-1)^n\mu_2'(t)\right], n = 1, 2, \ldots.\end{aligned} \tag{3.17}$$

We will find an analytical solution of the Cauchy initial problem for each of the equations (3.14) with conditions (3.16). That is, we will find an analytical solution of the Cauchy initial problem

$$\begin{aligned}&T_n''(t) + k_n^2 T_n(t-\tau) = 0,\\ &T_n(t) = \Phi_n(t),\ T_n'(t) = \Phi_n'(t),\ t \in [-\tau, 0],\end{aligned} \tag{3.18}$$

for every $n = 1, 2, \ldots.$ Using the results from Section 2, we will solve the problem (3.18). According to the formula (2.3), we get

$$\begin{aligned}T_n(t) =& \Phi_n(0)\cos_\tau\{k_n, t-\tau\} + \frac{1}{k_n}\Phi_n'(0)\sin_\tau\{k_n, t-\tau\}\\ &- k_n\int_{-\tau}^0 \sin_\tau\{k_n, t-2\tau-s\}\Phi_n(s)ds.\end{aligned} \tag{3.19}$$

Thus, the solution $\xi_0(x,t)$ of the homogeneous equation (3.2) that satisfies zero boundary conditions (3.3) and nonzero initial conditions (3.4) (to satisfy (3.4) we chose $A_n = 1, n = 1, 2, \ldots$) is

$$\begin{aligned}\xi_0(x,t) =& \sum_{n=1}^{\infty}\left\{\Phi_n(0)\cos_\tau\{k_n, t-\tau\} + \frac{1}{k_n}\Phi_n'(0)\sin_\tau\{k_n, t-\tau\}\right.\\ &\left.-k_n\int_{-\tau}^0 \sin_\tau\{k_n, t-2\tau-s\}\Phi_n(s)ds\right\}\sin\frac{\pi n}{l}x,\end{aligned} \tag{3.20}$$

where k_n defined in (3.15), Φ_n in (3.17), and $(x,t) \in [0,l] \times [-\tau, T]$.

3.1.2 Solving a Nonhomogeneous DPDE (3.6)

Further, we will consider the nonhomogeneous equation (3.6) with zero boundary conditions (3.8) and zero initial conditions (3.9). We will try to find the solution in the form of an expansion

$$\xi_1(x,t) = \sum_{n=1}^{\infty} T_n^0(t)\sin\frac{\pi n}{l}x, \tag{3.21}$$

where $(x,t) \in [0,l] \times [-\tau, T]$, and $T_n^0 : [-\tau, T] \to \mathbb{R}$ are unknown functions. Substituting (3.21) into (3.6) and equating the coefficients of the same functional terms, we will obtain a system of equations

$$\left(T_n^0\right)''(t) + \left[\left(\frac{\pi n}{l}a\right)^2 - c\right]T_n^0(t-\tau) = f_n(t) + u_n(t), \tag{3.22}$$

where $f_n : [-\tau, T] \to \mathbb{R}$ are Fourier coefficients of the function $F(x,t)$ (see (3.7)), that is,

$$\begin{aligned} f_n(t) =& \frac{2}{l}\int_0^l F(s,t)\sin\frac{\pi n}{l} s ds \\ =& \frac{2}{l}\int_0^l \Big\{ c\Big[\mu_1(t-\tau) + \frac{s}{l}[\mu_2(t-\tau) - \mu_1(t-\tau)]\Big] \\ & -\mu''(t) - \frac{s}{l}[\mu_2''(t) - \mu_1''(t)] + u(x,t)\Big\} \sin\frac{\pi n}{l} s ds \\ =& \frac{2}{\pi n}\left(c\{(-1)^n \mu_2(t-\tau) + \mu_1(t-\tau)\} - \{(-1)^n \mu_2''(t) + \mu_1''(t)\}\right), \end{aligned} \tag{3.23}$$

$u_n(t)$ is Fourier coefficients of the control function

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{\pi n}{l}x, \tag{3.24}$$

In accordance with (3.9), we assume zero initial conditions

$$T_n^0(t) = 0,\ t \in [-\tau, 0],\ n = 1, 2, \dots \tag{3.25}$$

for every equation (3.22). Then, by formula (2.6), a solution of each of problems (3.22), (3.25) can be written as

$$T_n^0(t) = \frac{1}{k_n}\int_0^t \sin_\tau\{k_n, t-\tau-s\}(f_n(s) + u_n(s))\, ds,\ t \in [-\tau, T], n = 1, 2, \dots, \tag{3.26}$$

where k_n are defined in (3.15).

Hence, the solution of the nonhomogeneous equation (3.6) with zero boundary conditions (3.8) and zero initial conditions (3.9) is

$$\xi_1(x,t) = \sum_{n=1}^{\infty}\left\{\frac{1}{k_n}\int_0^t \sin_\tau\{k_n, t-\tau-s\}(f_n(s) + u_n(s))\, ds\right\}\sin\frac{\pi n}{l}x, \tag{3.27}$$

where f_n is given by the formula (3.23), u_n in (3.24).

3.1.3 Formal Solution of the Boundary Value Problem

Using all above results, since the solutions $\xi_0(x,t)$ and $\xi_1(x,t)$ are formally twice differentiable with respect to x and t, functions $\mu_1(t)$, $\mu_2(t)$ are twice differentiable, we conclude that a formal solution of the boundary value problem (1.1), (1.2), (1.3) can be represented as a sum of solutions of separate problems

$$\begin{aligned}\xi(x,t) = \sum_{n=1}^{\infty} \Bigg\{ & \Phi_n(-\tau)\cos_\tau\{k_n,t\} + \frac{1}{k_n}\Phi_n'(-\tau)\sin_\tau\{k_n,t\} \\ & + \frac{1}{k_n}\int_{-\tau}^{0}\sin_\tau\{k_n,t-\tau-s\}\Phi_n''(s)\,ds + \frac{1}{k_n}\int_0^t \sin_\tau\{k_n,t-\tau-s\} f_n(s)\,ds \\ & + \frac{2}{\pi n}\left[(-1)^n\mu_2(t)+\mu_1(t)\right] + \frac{1}{k_n}\int_0^t \sin_\tau\{k_n,t-\tau-s\}u_n(s)\,ds \Bigg\}\sin\frac{\pi n}{l}x,\end{aligned} \tag{3.28}$$

where $(x,t)\in[0,l]\times[-\tau,T]$, coefficients Φ_n are defined in (3.5), f_n in (3.23), u_n in (3.24), and the numbers k_n in (3.15).

Results on existence, uniqueness and convergence of solutions of the wave equation with delay can be found in [10].

3.2 Control of Hyperbolic Delay Equations

We construct a control function $u(x,t)$ for the problem (1.1), (1.2), (1.3) in the form (3.24), for which a solution (3.28) in the moment of time $t=T$ will satisfy the condition (1.4).

We denote the combined initial and boundary conditions (they are all known functions) by

$$\begin{aligned}\Theta_n(t) = & \Phi_n(-\tau)\cos_\tau\{k_n,t\} + \frac{1}{k_n}\Phi_n'(-\tau)\sin_\tau\{k_n,t\} \\ & + \frac{1}{k_n}\int_{-\tau}^{0}\sin_\tau\{k_n,t-\tau-s\}\Phi_n''(s)\,d\xi \\ & + \int_0^t \sin_\tau\{k_n,t-\tau-s\}f_n(s)\,ds + \frac{2}{\pi n}\left[(-1)^n\mu_1(t)-\mu_2(t)\right].\end{aligned} \tag{3.29}$$

Then the solution $\xi(x,t)$ (3.28) can be rewritten as

$$\xi(x,t) = \sum_{n=1}^{\infty}\left[\Theta_n(t) + \int_0^t \sin_\tau\{k_n,t-\tau-s\}u_n(s)\,ds\right]\sin\frac{\pi n}{l}x.$$

We expand the final function $\Omega(x)$ into the Fourier series

$$\Omega(x) = \sum_{n=1}^{\infty}\Omega_n\sin\frac{\pi n}{l}x,\ \Omega_n = \frac{2}{l}\int_0^l \Omega(s)\sin\frac{\pi n}{l}s ds. \tag{3.30}$$

Each of the harmonic components $u_n(s)$, $n=1,2,\ldots$ can be obtained from the integral equations

$$\Theta_n(t) + \int_0^t \sin_\tau\{k_n,t-\tau-s\}u_n(s)\,ds = \Omega_n, n=1,2,\ldots. \tag{3.31}$$

For $t = T > 0$, equations (3.31) have no unique solutions, and we consider two possible approaches to obtain $u_n(t)$.

I. Let $\Omega(x) = \varphi(x, 0)$, which means that the function $\xi(x, t)$ already behaves in desirable way and we need to "maintain" this behaviour on $[0, T]$, so we need to solve (3.31) for $t \in [-\tau, T]$. It is possible to use the Laplace transform to find the solutions of integral equations (3.31) [15]. We denote the Laplace transform operator as $\mathfrak{L}\{f(x), p\} = \tilde{f}(p)$, and the inverse operator as $\mathfrak{L}^{-1}$, then applying the Laplace transform to the equation (3.31), we have

$$\mathfrak{L}\left\{\int_0^t \sin_\tau\{k_n, t-\tau-s\}\, u_n(s)\, ds, p\right\} = \frac{\Omega_n}{p} - \mathfrak{L}\{\Theta_n(t), p\},$$

using the convolution property, we get

$$\widetilde{\sin}_\tau\{k_n, p-\tau\}\, \tilde{u}_n(p) = \frac{\Omega_n}{p} - \tilde{\Theta}_n(p).$$

Therefore,

$$\tilde{u}_n(p) = \frac{\Omega_n}{p\,\widetilde{\sin}_\tau\{k_n, p-\tau\}} - \frac{\tilde{\Theta}_n(p)}{\widetilde{\sin}_\tau\{k_n, p-\tau\}}.$$

Applying the inverse transform, we obtain

$$u_n(t) = \mathfrak{L}^{-1}\left\{\frac{\Omega_n}{p\,\widetilde{\sin}_\tau\{k_n, p-\tau\}}, t\right\} - \mathfrak{L}^{-1}\left\{\frac{\tilde{\Theta}_n(p)}{\widetilde{\sin}_\tau\{k_n, p-\tau\}}, t\right\}, \tag{3.32}$$

where $t \in [-\tau, T]$ $n = 1, 2, \ldots$.

II. Let $\Omega(x) \neq \varphi(x, 0)$. We choose the functions $u_n = u_n(T)$, as coefficients, which depend on $t = T$, like on parameter. Then,

$$\Theta_n(T) + \int_0^T \sin_\tau\{k_n, T-\tau-s\}\, u_n(T)\, ds = \Omega_n, n = 1, 2, \ldots,$$

$u_n(T)$ can be removed from the integral

$$u_n(T) \int_0^T \sin_\tau\{k_n, T-\tau-s\}\, ds = \Omega_n - \Theta_n(T).$$

From Lemma 2.5, we know that

$$\int_0^T \sin_\tau\{k_n, T-\tau-s\}\, ds = -\frac{1}{k_n}\left[\cos_\tau\{k_n, T\} - 1\right],$$

so, coefficients $u_n(T)$ are

$$u_n(T) = -\frac{k_n\left[\Omega_n - \Theta_n(T)\right]}{\cos_\tau\{k_n, T\} - 1},\ n = 1, 2, \ldots. \tag{3.33}$$

The control function $u(x,t)$ for the equation (1.1) with initial (1.2) and boundary (1.3) conditions can be found in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{\pi n}{l} x,$$

where u_n defined in (3.33) or (3.32) depend on the values of initial, boundary and finite functions.

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