Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 9, Number 2, pp. 191–197 (2014)
http://campus.mst.edu/adsa

On the Estimation of Solutions of Nonlinear Difference Equations

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Abstract

In this paper we consider the nonlinear difference equation of the form:

$$\Delta^m y(n) + \sum_{k=0}^{m-1} p_k(n)\Delta^k y(n) = f(n, y(n), \Delta y(n), \ldots, \Delta^{m-1} y(n)),$$

where $m \geq 2$, $n \in \mathbb{N}_{n_0}$, $p_k : \mathbb{N}_{n_0} \to \mathbb{R}$, $k = 0, 1, \ldots, m-1$ and $f : \mathbb{N}_{n_0} \times \mathbb{R}^m \to \mathbb{R}$, and associated with it the linear difference equation

$$\Delta^m x(n) + \sum_{k=0}^{m-1} p_k(n)\Delta^k x(n) = 0.$$

We prove in some comparison theorems how to estimate solutions of the nonlinear difference equation by solutions of the associated linear difference equation. This paper is the continuation of our research on qualitative properties of the above equation.

AMS Subject Classifications: 39A10.
Keywords: Difference equation, Casorati matrix.

1 Introduction

In this paper we are concerned with the estimation of solutions of the nonlinear difference equation of the form:

$$\Delta^m y(n) + \sum_{k=0}^{m-1} p_k(n)\Delta^k y(n) = f(n, y(n), \Delta y(n), \ldots, \Delta^{m-1} y(n)), \tag{1.1}$$

Received March 5, 2014; Accepted April 4, 2014
Communicated by Sandra Pinelas

where $m \geq 2$, $n \in \mathbb{N}_{n_0}$, $p_k : \mathbb{N}_{n_0} \to \mathbb{R}$, $k = 0, 1, \dots, m-1$ and $f : \mathbb{N}_{n_0} \times \mathbb{R}^m \to \mathbb{R}$, and associated with it the linear difference equation

$$\Delta^m x(n) + \sum_{k=0}^{m-1} p_k(n) \Delta^k x(n) = 0. \tag{1.2}$$

As usual for any function $u : \mathbb{N}_{n_0} \to \mathbb{R}$ we define the forward difference operators as follows

$$\Delta^0 u(n) = u(n), \quad \Delta^m u(n) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} u(n+i)$$

for $m \geq 1$, and for all $n \in \mathbb{N}$ we have

$$\sum_{n=r}^{s} u(n) = 0, \quad \prod_{n=r}^{s} u(n) = 1 \quad \text{for} \quad s < r.$$

By a solution of (1.1) we mean a nontrivial sequence y satisfying (1.1) on $\mathbb{N}_{n_0}$. Throughout, we assume that there exists a function $\omega : \mathbb{N}_{n_0} \times \mathbb{R}_+^m \to \mathbb{R}_+$ nondecreasing with respect to last m arguments such that

$$|f(n, a_1, a_2, \dots, a_m)| \leq \omega(n, |a_1|, |a_2|, \dots, |a_m|) \tag{1.3}$$

for $n \in \mathbb{N}_{n_0}$ and $a_k \in \mathbb{R}$ $(k = 1, 2, \dots, m)$.

Let x_k, $k = 0, 1, \dots, m-1$ be linearly independent solutions of (1.2). We denote

$$x(n) = \sum_{i=0}^{m-1} c_i x_i(n), \quad c_i \in R, \quad \sum_{i=0}^{m-1} |c_i| = c \neq 0 \tag{1.4}$$

and

$$\alpha_k(n) = \max\{|\Delta^k x_i(n)| : i = 0, 1, \dots, m-1\},$$

where $k = 0, 1, \dots, m-1$, $n \in \mathbb{N}_{n_0}$.

Moreover, let

$$W(n) = \begin{vmatrix} x_0(n) & x_1(n) & \dots & x_{m-1}(n) \\ \Delta x_0(n) & \Delta x_1(n) & \dots & \Delta x_{m-1}(n) \\ \vdots & \vdots & \dots & \vdots \\ \Delta^{m-1} x_0(n) & \Delta^{m-1} x_1(n) & \dots & \Delta^{m-1} x_{m-1}(n) \end{vmatrix} \tag{1.5}$$

be the Casoratian of the solutions x_i, $i = 0, 1, \dots, m-1$. Without the loss of the generality we may assume that $W(n) > 0$ for $n \in \mathbb{N}_{n_0}$.

We also denote

$$V_k(n,s) = \begin{vmatrix} x_0(s) & x_1(s) & \dots & x_{m-1}(s) \\ \Delta x_0(s) & \Delta x_1(s) & \dots & \Delta x_{m-1}(s) \\ \vdots & \vdots & \dots & \vdots \\ \Delta^{m-2}x_0(s) & \Delta^{m-2}x_1(s) & \dots & \Delta^{m-2}x_{m-1}(s) \\ \Delta^k x_0(n) & \Delta^k x_1(n) & \dots & \Delta^k x_{m-1}(n) \end{vmatrix} \tag{1.6}$$

where $n_0 \le s < n$.

The purpose of this paper is to give an estimation of solutions of (1.1), which would be dependent on solutions of (1.2). We will need the following lemma.

Lemma 1.1. *Consider the following difference inequality*

$$u(n) \le u(n_0) + \sum_{k=n_0}^{n-1} H(k, |u(k)|),$$

and the difference equation

$$w(n) = w(n_0) + \sum_{k=n_0}^{n-1} H(k, |w(k)|) \quad \textit{for} \quad n \in \mathbb{N}_{n_0},$$

where $H : \mathbb{N}_{n_0} \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with respect to the second argument. If $u(n_0) \le w(n_0)$, then $u(n) \le w(n)$ for $n \in \mathbb{N}_{n_0}$.

Proof. The proof follows immediately from [4, Theorem 1.6.2]. □

2 Main Results

Theorem 2.1. *Let ρ be a solution of the first order initial value problem*

$$\begin{aligned} \Delta\rho(n) &= m!\prod_{r=0}^{m-2}\frac{\alpha_r(n+1)}{W(n+1)}\omega(n, |\rho(n)|\alpha_0(n), |\rho(n)|\alpha_1(n), \dots, |\rho(n)|\alpha_{m-1}(n)), \\ \rho(n_0) &= c. \end{aligned} \tag{2.1}$$

Then every solution y of (1.1) *satisfies the following inequality*

$$|\Delta^k y(n)| \le \alpha_k(n)|\rho(n)|, \quad \textit{for} \quad n \ge n_0. \tag{2.2}$$

Proof. First, we notice that any solution y of (1.1) satisfies the equation

$$\Delta^k y(n) = \Delta^k x(n) + \sum_{l=n_0}^{n-1} \frac{V_k(n,l+1)}{W(l+1)} f(l, y(l), \Delta y(l), \dots, \Delta^{m-1} y(l)), \quad n \geq n_0. \tag{2.3}$$

In view of (1.3) and (1.4), we have

$$\begin{aligned}|\Delta^k y(n)| &\leq |\Delta^k x(n)| + \sum_{l=n_0}^{n-1} \frac{|V_k(n,l+1)|}{W(l+1)} |f(l, y(l), \Delta y(l), \dots, \Delta^{m-1} y(l))| \\ &\leq \sum_{j=0}^{m-1} |c_j \Delta^k x_j(n)| + \sum_{l=n_0}^{n-1} \frac{|V_k(n,l+1)|}{W(l+1)} \omega(l, |y(l)|, |\Delta y(l)|, \dots, |\Delta^{m-1} y(l)|) \\ &\leq \alpha_k(n) c + \sum_{l=n_0}^{n-1} \frac{|V_k(n,l+1)|}{W(l+1)} \omega(l, |y(l)|, |\Delta y(l)|, \dots, |\Delta^{m-1} y(l)|). \qquad (2.4)\end{aligned}$$

Notice that

$$|V_k(n,l+1)| \leq \sum_{j=0}^{m-1} |(-1)^{m+j+1}||\Delta^k x_j(n)||M_{m,j+1}| \leq \alpha_k(n) \sum_{j=0}^{m-1} |M_{m,j+1}|,$$

where $M_{m,j+1}$ is the minor obtained from V by the emission of the mth row and the $(j+1)$st column. Then

$$\begin{aligned}|V_k(n,l+1)| &\leq \alpha_k(n) \sum_{j=0}^{m-1} (m-1)! \alpha_0(l+1) \alpha_1(l+1) \cdots \alpha_{m-2}(l+1) \\ &= \alpha_k(n) m! \prod_{r=0}^{m-2} \alpha_r(l+1). \qquad (2.5)\end{aligned}$$

Comparing (2.4) and (2.5) we obtain

$$\begin{aligned}|\Delta^k y(n)| &\leq \alpha_k(n) \Big[c + m! \times \\ &\times \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, |y(l)|, |\Delta y(l)|, \dots, |\Delta^{m-1} y(l)|) \Big]. \qquad (2.6)\end{aligned}$$

Denote for $n \geq n_0$

$$u(n) = c + m! \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, |y(l)|, |\Delta y(l)|, \ldots, |\Delta^{m-1} y(l)|). \quad (2.7)$$

Then

$$|\Delta^k y(n)| \leq \alpha_k(n) u(n), \quad n \geq n_0, \quad k = 0, 1, \ldots, m-1. \quad (2.8)$$

Using now the above inequality in (2.7), we derive

$$\begin{aligned} u(n) &\leq c + m! \times \\ &\times \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, \alpha_0(l)u(l), \alpha_1(l)u(l), \ldots, \alpha_{m-1}(l)u(l)). \quad (2.9) \end{aligned}$$

Let ρ be a solution of (2.1) with the initial condition $\rho(n_0) = c$. Then from (2.8) and (2.9), by Lemma 1.1, we have

$$u(n) \leq \rho(n), \quad n \geq n_0, \quad (2.10)$$

and, by (2.8), we get

$$|\Delta^k y(n)| \leq \alpha_k(n) |\rho(n)|, \quad n \geq n_0, \quad k = 0, 1, \ldots, m-1,$$

which completes the proof. □

Corollary 2.2. *If all solutions of equations* (1.2) *and* (2.1) *are bounded, then every solution of* (1.1) *is also bounded.*

Corollary 2.3. *If every solution of* (2.1) *is bounded and every solution of* (1.2) *belongs to l^p, then every solution of* (1.1) *belongs to l^p.*

Theorem 2.4. *Suppose that there exists a continuous nondecreasing function $B : [\varepsilon, \infty) \to \mathbb{R}_+$ such that*

$$\omega(n, \lambda a_1, \lambda a_2, \ldots, \lambda a_m) \leq B(\lambda) \omega(n, a_1, a_2, \ldots, a_m), \quad (2.11)$$

for $\lambda \geq \varepsilon > 0$, $a_i \in \mathbb{R}_+$ $(i = 1, \ldots, m)$ and $\int_\varepsilon^\infty \frac{ds}{B(s)} = \infty$. Then any solution y of (1.1) *has the property*

$$\begin{aligned} |\Delta^k y(n)| &\leq \alpha_k(n) G^{-1} \times \\ &\times \left[G(c) + m! \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, \alpha_0(l), \ldots, \alpha_{m-1}(l)) \right] \quad (2.12) \end{aligned}$$

for $n \in \mathbb{N}_{n_0}$, where G^{-1} denotes the inverse of the function G defined by

$$G(w) = \int_\varepsilon^w \frac{ds}{B(s)}, \quad w \geq \varepsilon > 0.$$

Proof. Using the same arguments like in the proof of Theorem 2.1 we obtain that

$$u(n) = c + m! \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, |y(l)|, |\Delta y(l)|, \ldots, |\Delta^{m-1} y(l)|).$$

Then $|\Delta^k y(n)| \leq \alpha_k(n) u(n)$. From (2.8) and (2.11) we obtain

$$\begin{aligned} u(n) &\leq c + m! \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, \alpha_0(l)u(l), \alpha_1(l)u(l), \ldots, \alpha_{m-1}(l)u(l)) \\ &\leq c + m! \sum_{l=n_0}^{n-1} B(u(l)) \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, \alpha_0(l), \alpha_1(l), \ldots, \alpha_{m-1}(l)). \end{aligned}$$

Denoting the right-hand side of the above inequality by $z(n)$ we get $u(n) \leq z(n)$, $z(n_0) = c$ and

$$\begin{aligned} \Delta z(n) &= m! B(u(n)) \prod_{r=0}^{m-2} \frac{\alpha_r(n+1)}{W(n+1)} \omega(n, \alpha_0(n), \alpha_1(n), \ldots, \alpha_{m-1}(n)) \\ &\leq B(z(n)) m! \prod_{r=0}^{m-2} \frac{\alpha_r(n+1)}{W(n+1)} \omega(n, \alpha_0(n), \alpha_1(n), \ldots, \alpha_{m-1}(n)). \end{aligned}$$

Dividing both sides of this inequality by $B(z(n))$ and next summing from n_0 to $n-1$ we obtain

$$\sum_{l=n_0}^{n-1} \frac{\Delta z(l)}{B(z(l))} \leq m! \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, \alpha_0(l), \alpha_1(l), \ldots, \alpha_{m-1}(l)). \tag{2.13}$$

In view of the nondecreasing character of z and B we obtain

$$\int_{z(n_0)}^{z(n)} \frac{ds}{B(s)} = \sum_{l=n_0}^{n-1} \int_{z(l)}^{z(l+1)} \frac{ds}{B(s)} \leq \sum_{l=n_0}^{n-1} \frac{1}{B(z(l))} \int_{z(l)}^{z(l+1)} ds = \sum_{l=n_0}^{n-1} \frac{\Delta z(l)}{B(z(l))}. \tag{2.14}$$

From (2.13) and (2.14), we deduce

$$G(z(n)) - G(z(n_0)) \leq m! \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, \alpha_0(l), \alpha_1(l), \ldots, \alpha_{m-1}(l))$$

and

$$z(n) \le G^{-1}\Big(G(c) + m! \sum_{l=n_0}^{n-1} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, \alpha_0(l), \alpha_1(l), \ldots, \alpha_{m-1}(l))\Big). \quad (2.15)$$

Therefore $|\Delta^k y(n)| \le \alpha_k(n)u(n) \le \alpha_k(n)z(n)$, which combining with (2.15) completes the proof of the theorem. □

Corollary 2.5. *If every solution x of* (1.2) *is bounded and*

$$\sum_{l=n_0}^{\infty} \prod_{r=0}^{m-2} \frac{\alpha_r(l+1)}{W(l+1)} \omega(l, \alpha_0(l), \alpha_1(l), \ldots, \alpha_{m-1}(l)) < \infty,$$

then every solution y of (1.1) *is bounded.*

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