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Noether's Theorem with Momentum and Energy Terms for Cresson's Quantum Variational Problems

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Abstract

We prove a DuBois–Reymond necessary optimality condition and a Noether symmetry theorem to the recent quantum variational calculus of Cresson. The results are valid for problems of the calculus of variations with functionals defined on sets of nondifferentiable functions. As an application, we obtain a constant of motion for a linear Schrödinger equation.

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1 Introduction

Quantum calculus, sometimes called "calculus without limits", is analogous to traditional infinitesimal calculus without the notion of limits [17]. Several dialects of quantum calculus are available in the literature, including Jackson's quantum calculus [17, 22], Hahn's quantum calculus [6, 18, 19], the time-scale q-calculus [5, 21], the

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power quantum calculus [1], and the symmetric quantum calculi [7–9]. Here we consider the recent quantum calculus of Cresson.

Motivated by Nottale's theory of scale relativity without the hypothesis of spacetime differentiability [23, 24], Cresson introduced in 2005 his quantum calculus on a set of Hölder functions [11]. This calculus attracted attention due to its applications in physics and the calculus of variations, and has been further developed by several different authors – see [2, 10, 12] and references therein. Cresson's calculus of 2005 [11] presents, however, some problems, and in 2011 Cresson and Greff improved it [13, 14]. It is this new version of 2011 that we consider here, a brief review of it being given in Section 2. Along the text, by *Cresson's calculus* we mean this quantum version of 2011 [13, 14].

There is a close connection between quantum calculus and the calculus of variations. For the state of the art on the quantum calculus of variations we refer the reader to the recent book [20]. With respect to Cresson's approach, the quantum calculus of variations is still in its infancy: see [3, 4, 13–16]. In [13] a Noether type theorem is proved but only with the momentum term. In [14] nondifferentiable Euler–Lagrange equations are used in the study of PDEs. It is proved that nondifferentiable characteristics for the Navier–Stokes equation correspond to extremals of an explicit nondifferentiable Lagrangian system, and that the solutions of the Schrödinger equation are nondifferentiable extremals of the Newton Lagrangian. Euler–Lagrange equations for variational functionals with Lagrangians containing multiple quantum derivatives, depending on a parameter, or containing higher-order quantum derivatives, are studied in [3]. Variational problems with constraints, with one and more than one independent variable, of first and higher-order type, are investigated in [4]. Recently, Hamilton–Jacobi equations were obtained [15] and problems of the calculus of variations and optimal control with time delay were considered [16]. Here we extend the available nondifferentiable Noether's theorem of [13] by considering invariance transformations that also change the time variable, and thus obtaining not only the generalized momentum term of [13] but also a new energy term. For that we first obtain a new necessary optimality condition of DuBois–Reymond type.

The text is organized as follows. In Section 2 we recall the notions and results of Cresson's quantum calculus needed in the sequel. Our main results are given in Section 3: the nondifferentiable DuBois–Reymond necessary optimality condition (Theorem 3.7) and the nondifferentiable Noether type symmetry theorem (Theorem 3.8). We end with an application of our results to the linear Schrödinger equation (Section 4).

2 Cresson's Quantum Calculus

We briefly review the necessary concepts and results of the quantum calculus [14]. Let $\mathbb{X}^d$ denote the set $\mathbb{R}^d$ or $\mathbb{C}^d$, $d \in \mathbb{N}$, and I be an open set in $\mathbb{R}$ with $[t_1, t_2] \subset I$, $t_1 < t_2$. Denote by $\mathcal{G}\left(I, \mathbb{X}^d\right)$ the set of functions $f : I \to \mathbb{X}^d$ and by $\mathcal{C}^0\left(I, \mathbb{X}^d\right)$ the subset of functions of $\mathcal{G}\left(I, \mathbb{X}^d\right)$ that are continuous.

Definition 2.1 (The ϵ-left and ϵ-right quantum derivatives [14])**.** Let $f \in \mathcal{C}^0\left(I, \mathbb{R}^d\right)$. For all $\epsilon > 0$, the ϵ-left and ϵ-right quantum derivatives of f, denoted respectively by $\Delta_\epsilon^- f$ and $\Delta_\epsilon^+ f$, are defined by

$$\Delta_\epsilon^- f(t) = \frac{f(t) - f(t-\epsilon)}{\epsilon} \quad \text{and} \quad \Delta_\epsilon^+ f(t) = \frac{f(t+\epsilon) - f(t)}{\epsilon}.$$

Remark 2.2. The ϵ-left and ϵ-right quantum derivatives of a continuous function f correspond to the classical derivative of the ϵ-mean function f_ϵ^σ defined by

$$f_\epsilon^\sigma(t) = \frac{\sigma}{\epsilon} \int_t^{t+\sigma\epsilon} f(s) ds, \quad \sigma = \pm.$$

The next operator generalizes the classical derivative.

Definition 2.3 (The ϵ-scale derivative [14])**.** Let $f \in \mathcal{C}^0\left(I, \mathbb{R}^d\right)$. For all $\epsilon > 0$, the ϵ-scale derivative of f, denoted by $\frac{\Box_\epsilon f}{\Box t}$, is defined by

$$\frac{\Box_\epsilon f}{\Box t} = \frac{1}{2}\left[\left(\Delta_\epsilon^+ f + \Delta_\epsilon^- f\right) + i\mu\left(\Delta_\epsilon^+ f - \Delta_\epsilon^- f\right)\right],$$

where i is the imaginary unit and $\mu = \{-1, 1, 0, -i, i\}$.

Remark 2.4. If f is differentiable, then one can take the limit of the scale derivative when ϵ goes to zero. We then obtain the classical derivative $\frac{df}{dt}$ of f.

We also need to extend the scale derivative to complex valued functions.

Definition 2.5 (See [14])**.** Let $f \in \mathcal{C}^0\left(I, \mathbb{C}^d\right)$ be a continuous complex valued function. For all $\epsilon > 0$, the ϵ-scale derivative of f, denoted by $\frac{\Box_\epsilon f}{\Box t}$, is defined by

$$\frac{\Box_\epsilon f}{\Box t} = \frac{\Box_\epsilon \mathrm{Re}(f)}{\Box t} + i \frac{\Box_\epsilon \mathrm{Im}(f)}{\Box t},$$

where Re(f) and Im(f) denote the real and imaginary part of f, respectively.

In Definition 2.3, the ϵ-scale derivative depends on ϵ, which is a free parameter related to the smoothing order of the function. This brings many difficulties in applications to physics, when one is interested in particular equations that do not depend on an extra parameter. To solve these problems, the authors of [14] introduced a procedure to extract information independent of ϵ but related with the mean behavior of the function.

Definition 2.6 (See [14])**.** Let $C^0_{conv}\left(I \times (0,1), \mathbb{R}^d\right) \subseteq C^0\left(I \times (0,1), \mathbb{R}^d\right)$ be such that for any function $f \in C^0_{conv}\left(I \times (0,1), \mathbb{R}^d\right)$ the $\lim_{\epsilon \to 0} f(t, \epsilon)$ exists for any $t \in I$; and

E be a complementary of $C^0_{conv}\left(I \times (0,1), \mathbb{R}^d\right)$ in $C^0\left(I \times (0,1), \mathbb{R}^d\right)$. We define the projection map π by

$$\begin{array}{rccl} \pi: & C^0_{conv}\left(I \times (0,1), \mathbb{R}^d\right) \oplus E & \to & C^0_{conv}(I \times \left(0,1), \mathbb{R}^d\right) \\ & f_{conv} + f_E & \mapsto & f_{conv} \end{array}$$

and the operator $\langle \cdot \rangle$ by

$$\begin{array}{rccl} \langle \cdot \rangle: & C^0\left(I \times (0,1), \mathbb{R}^d\right) & \to & C^0\left(I, \mathbb{R}^d\right) \\ & f & \mapsto & \langle f \rangle : t \mapsto \lim\limits_{\epsilon \to 0} \pi(f)(t, \epsilon). \end{array}$$

The quantum derivative of f without the dependence of ϵ is introduced in [14].

Definition 2.7 (See [14])**.** The quantum derivative of f in the space $\mathcal{C}^0\left(I, \mathbb{R}^d\right)$ is given by the rule

$$\frac{\Box f}{\Box t} = \left\langle \frac{\Box_\epsilon f}{\Box t} \right\rangle. \tag{2.1}$$

The scale derivative (2.1) has some nice properties. Namely, it satisfies a Leibniz and a Barrow rule. First let us recall the definition of an α-Hölderian function.

Definition 2.8 (Hölderian function of exponent α [14])**.** Let $f \in C^0\left(I, \mathbb{R}^d\right)$. We say that f is α-Hölderian, $0 < \alpha < 1$, if for all $\epsilon > 0$ and all $t, t' \in I$ there exists a constant $c > 0$ such that $|t - t'| \leqslant \epsilon$ implies $\|f(t) - f(t')\| \leqslant c\epsilon^\alpha$, where $\|\cdot\|$ is a norm in $\mathbb{R}^d$. The set of Hölderian functions of Hölder exponent α, for some α, is denoted by $H^\alpha(I, \mathbb{R}^d)$.

In what follows, we frequently use $\Box$ to denote the scale derivative operator $\dfrac{\Box}{\Box t}$.

Theorem 2.9 (The quantum Leibniz rule [14])**.** *Let $\alpha + \beta > 1$. For $f \in H^\alpha\left(I, \mathbb{R}^d\right)$ and $g \in H^\beta\left(I, \mathbb{R}^d\right)$, one has*

$$\Box(f \cdot g)(t) = \Box f(t) \cdot g(t) + f(t) \cdot \Box g(t). \tag{2.2}$$

Remark 2.10. For $f \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$ and $g \in \mathcal{C}^1\left(I, \mathbb{R}^d\right)$, one obtains from (2.2) the classical Leibniz rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$. For convenience of notation, we sometimes write (2.2) as $(f \cdot g)^\Box(t) = f^\Box(t) \cdot g(t) + f(t) \cdot g^\Box(t)$.

Theorem 2.11 (The quantum Barrow rule [14])**.** *Let $f \in \mathcal{C}^0([t_1, t_2], \mathbb{R})$ be such that $\Box f / \Box t$ is continuous and*

$$\lim_{\epsilon \to 0} \int_{t_1}^{t_2} \left(\frac{\Box_\epsilon f}{\Box t}\right)_E (t) dt = 0. \tag{2.3}$$

Then,

$$\int_{t_1}^{t_2} \frac{\Box f}{\Box t}(t)\, dt = f(b) - f(a). \tag{2.4}$$

The next theorem gives the analogous of the derivative of a composite function for the quantum derivative.

Theorem 2.12 (See [14])**.** *Let* $f \in C^2\left(\mathbb{R}^d \times I, \mathbb{R}\right)$ *and* $x \in H^\alpha(\mathbb{R}^d, I)$ *with* $\frac{1}{2} \leq \alpha < 1$. *Then,*

$$\frac{\Box f}{\Box t}(x(t),t) = \frac{\partial f}{\partial t}(x(t),t) + \nabla_x f(x(t),t) \cdot \nabla_\Box x(t) + \sum_{k=1}^{d}\sum_{j=1}^{d} \frac{1}{2}\frac{\partial^2 f}{\partial x_k x_j}\,(x(t),t)\, a_{k,j}(x(t)),$$

where

$$\nabla_\Box x(t) = \left(\frac{\Box x_1}{\Box t}(t), \ldots, \frac{\Box x_n}{\Box t}(t)\right)^T$$

and $a_{k,j}(x(t))$ *denotes*

$$\left\langle \pi\left(\frac{\epsilon}{2}\left(\left(\Delta_\epsilon^+ x_k(t)\right)\left(\Delta_\epsilon^+ x_k(t)\right)(1+i\mu) - \left(\Delta_\epsilon^- x_k(t)\right)\left(\Delta_\epsilon^- x_k(t)\right)(1-i\mu)\right)\right)\right\rangle.$$

3 Main Results

The classical Noether's theorem is valid along extremals q which are C^2-differentiable. The biggest class where a Noether type theorem has been proved for the classical problem of the calculus of variations is the class of Lipschitz functions [25]. In this work we prove a more general Noether type theorem, valid for nondifferentiable scale extremals.

In [14] the calculus of variations with scale derivatives is introduced and respective Euler–Lagrange equations derived without the dependence of ϵ. In this section we obtain a formulation of Noether's theorem for the scale calculus of variations. The proof of our Noether's theorem is done in two steps: first we extend the DuBois–Reymond condition to problems with scale derivatives (Theorem 3.7); then, using this result, we obtain the scale/quantum Noether's theorem (Theorem 3.8). The problem of the calculus of variations with scale derivatives is defined as

$$I[q(\cdot)] = \int_a^b L\left(t, q(t), \Box q(t)\right) dt \longrightarrow \min \tag{3.1}$$

under given boundary conditions $q(a) = q_a$ and $q(b) = q_b$, $(q(\cdot), \Box q(\cdot)) \in H^{2\alpha}$, $0 < \alpha < 1$. The Lagrangian L is assumed to be a C^1-function with respect to all its arguments.

Remark 3.1. In the case of admissible differentiable functions $q(\cdot)$, functional $I[q(\cdot)]$ in (3.1) reduces to the classical variational functional of the fundamental problem of the calculus of variations:

$$I[q(\cdot)] = \int_a^b L\left(t, q(t), \dot{q}(t)\right) dt.$$

Theorem 3.2 (Nondifferentiable Euler–Lagrange equations [14])**.** *Let* $0 < \alpha, \beta < 1$ *with* $\alpha + \beta > 1$*. If* $q \in H^\alpha\left(I, \mathbb{R}^d\right)$ *satisfies* $\Box q \in H^\alpha\left(I, \mathbb{R}^d\right)$ *and* $L\left(t, q(t), \Box q(t)\right) \cdot h(t)$ *satisfies condition* (2.3) *for all* $h \in H^\beta\left(I, \mathbb{R}^d\right)$*, then function* q *satisfies the following nondifferentiable Euler–Lagrange equation:*

$$\partial_2 L\left(t, q(t), \Box q(t)\right) - \Box \partial_3 L\left(t, q(t), \Box q(t)\right) = 0. \tag{3.2}$$

It is worth to mention that the Euler–Lagrange equation (3.2) can be generalized in many different ways: see [3] for the cases when the Lagrangian L contains multiple scale derivatives, depends on a parameter, or contains higher-order scale derivatives.

Definition 3.3 (Nondifferentiable extremals)**.** The solutions $q(\cdot)$ of the nondifferentiable Euler–Lagrange equation (3.2) are called *nondifferentiable extremals*.

Definition 3.4. Functional (3.1) is said to be invariant under the s-parameter group of infinitesimal transformations

$$\begin{cases} \bar{t} = t + s\tau(t, q) + o(s), \\ \bar{q}(t) = q(t) + s\xi(t, q) + o(s), \end{cases} \tag{3.3}$$

if

$$0 = \frac{d}{ds} \int_{\bar{t}(I)} L\left[t + s\tau(t, q(t)), q(t) + s\xi(t, q(t)), \frac{\Box q(t) + s\Box \xi(t, q(t))}{1 + s\Box \tau(t, q(t))}\right] \left(1 + s\Box \tau(t, q(t))\right) dt \bigg|_{s=0} \tag{3.4}$$

for any subinterval $I \subseteq [a, b]$, where $\tau, \xi \in H^\alpha$.

Lemma 3.5 establishes a necessary condition of invariance for (3.1). Condition (3.5) will be used in the proof of our Noether type theorem.

Lemma 3.5 (Necessary condition of invariance)**.** *If functional* (3.1) *is invariant under the one-parameter group of transformations* (3.3)*, then*

$$\int_{t_a}^{t_b} \Big[\partial_1 L\left(t, q(t), \Box q(t)\right) \tau + \partial_2 L\left(t, q(t), \Box q(t)\right) \cdot \xi + \partial_3 L\left(t, q(t), \Box q(t)\right) \cdot \left(\Box \xi - \Box q(t) \Box \tau\right) + L\left(t, q(t), \Box q(t)\right) \Box \tau\Big] dt = 0. \tag{3.5}$$

Proof. Without loss of generality, we take $I = [t_a, t_b]$. Equality (3.5) follows directly from condition (3.4). □

Definition 3.6 (Nondifferentiable constants of motion)**.** A quantity $C(t, q(t), \Box q(t))$ is a *nondifferentiable constant of motion* if $C(t, q(t), \Box q(t))$ is constant along all the nondifferentiable extremals $q(\cdot) \in H^\alpha(I, \mathbb{R}^d)$, $\frac{1}{2} \leq \alpha < 1$ (cf. Definition 3.3).

Theorem 3.7 generalizes the classical DuBois–Reymond optimality condition

$$\partial_1 L\left(t, q(t), \dot{q}(t)\right) = \frac{d}{dt}\left\{L\left(t, q(t), \dot{q}(t)\right) - \partial_3 L\left(t, q(t), \dot{q}(t)\right) \cdot \dot{q}(t)\right\}$$

for Cresson's quantum problems of the calculus of variations.

Theorem 3.7 (Nondifferentiable DuBois–Reymond necessary optimality condition)**.** *Let* $\frac{1}{2} \leq \alpha < 1$*. If* $q \in H^\alpha\left(I, \mathbb{R}^d\right)$ *with* $\Box q \in H^\alpha\left(I, \mathbb{R}^d\right)$*, then any nondifferentiable extremal* q *satisfies the following DuBois–Reymond necessary condition:*

$$\frac{\Box}{\Box t}\left\{L\left(t, q, \frac{\Box q}{\Box t}\right) - \partial_3 L\left(t, q, \frac{\Box q}{\Box t}\right) \cdot \frac{\Box q}{\Box t}\right\} = \partial_1 L\left(t, q, \frac{\Box q}{\Box t}\right). \qquad (3.6)$$

Proof. Using the linearity of the quantum derivative operator, Theorems 2.9 and 2.12, and the nondifferentiable Euler–Lagrange equation (3.2), we can write that

$$\begin{aligned}
\Box\Big\{L(t,q,\Box q) - \partial_3 L(t,q,\Box q) \cdot \Box q\Big\}& \\
&= \partial_1 L(t,q,\Box q) + \partial_2 L(t,q,\Box q) \cdot \Box q + \partial_3 L(t,q,\Box q) \cdot \Box\Box q \\
&\qquad - \Box\partial_3 L(t,q,\Box q) \cdot \Box q - \partial_3 L(t,q,\Box q) \cdot \Box\Box q \\
&= \partial_1 L(t,q,\Box q) + \Box q \cdot \left(\partial_2 L(t,q,\Box q) - \Box\partial_3 L(t,q,\Box q)\right) \\
&= \partial_1 L(t,q,\Box q).
\end{aligned}$$

This concludes the proof. □

Our main result is the following.

Theorem 3.8 (Nondifferentiable Noether's theorem)**.** *If functional* (3.1) *is invariant in the sense of Definition 3.4, then*

$$\begin{aligned}
C(t, q(t), \Box q(t)) = \partial_3 L(t,q,\Box q)) \cdot \xi(t,q)& \\
+ \Big(L(t,q,\Box q) - \partial_3 L(t,q,\Box q) \cdot \Box q\Big)\tau(t,q)& \qquad (3.7)
\end{aligned}$$

is a nondifferentiable constant of motion (cf. Definition 3.6).

Proof. Noether's nondifferentiable constant of motion (3.7) follows by using the scale DuBois–Reymond condition (3.6), the nondifferentiable Euler–Lagrange equation (3.2) and Theorem 2.9, into the necessary condition of invariance (3.5):

$$\begin{aligned}0 &= \int_{t_a}^{t_b} \Big[\partial_1 L\left(t, q(t), \Box q(t)\right)\tau + \partial_2 L\left(t, q(t), \Box q(t)\right)\cdot \xi \\ &\qquad + \partial_3 L\left(t, q, \Box q\right)\cdot\left(\Box\xi - \Box q \Box \tau\right) + L\Box\tau\Big] dt \\ &= \int_{t_a}^{t_b} \Big[\tau\Box(L\left(t, q, \Box q\right) - \partial_3 L\left(t, q, \Box q\right)\cdot\Box q) \\ &\qquad + \left(L\left(t, q, \Box q\right) - \partial_3 L\left(t, q, \Box q\right)\cdot\Box q\right)\Box\tau \\ &\qquad + \xi\cdot\Box\partial_3 L\left(t, q, \Box q\right) + \partial_3 L\left(t, q, \Box q\right)\cdot\Box\xi\Big] dt \\ &= \int_{t_a}^{t_b} \frac{\Box}{\Box t}\Big\{\partial_3 L\left(t, q, \Box q\right)\cdot\xi + \left(L\left(t, q, \Box q\right) - \partial_3 L\left(t, q, \Box q\right)\cdot\Box q\right)\tau\Big\} dt.\end{aligned} \tag{3.8}$$

Using formula (2.4) and having in mind that (3.8) holds for an arbitrary $[t_a, t_b] \subseteq [a, b]$, we conclude that $L\left(t, q, \Box q\right)\cdot\xi + \Big(L\left(t, q, \Box q\right) - \partial_3 L\left(t, q, \Box q\right)\cdot\Box q\Big)\tau$ is constant. $\Box$

If the admissible functions q are differentiable, then the nondifferentiable constant of motion (3.7) reduces to classical Noether's constant of motion

$$C(t, q, \dot{q}) = \partial_3 L\left(t, q, \dot{q}\right)\cdot\xi(t, q) + \left(L(t, q, \dot{q}) - \partial_3 L\left(t, q, \dot{q}\right)\cdot\dot{q}\right)\tau(t, q).$$

For this reason, the term $\partial_3 L(t, q, \Box q))$ can be seen as the *momentum* while the term $L(t, q, \Box q) - \partial_3 L(t, q, \Box q)\cdot\Box q$ can be interpreted as *energy*.

4 An Application

In [14, §3], a linear Schrödinger equation, with particular interest in quantum mechanics, is studied. It is proved that, under certain conditions, solutions of the linear Schrödinger equation coincide with the extremals of a certain functional (3.1) of Cresson's quantum calculus of variations. In this section we use our nondifferentiable Noether's theorem to find constants of motion for the problem studied in [14, §3]. Precisely, consider the following linear Schrödinger equation:

$$i\bar{h}\frac{\partial\Psi(t, q)}{\partial t} + \frac{\bar{h}^2}{2m}\sum_{j=1}^{d}\frac{\partial^2\Psi(t, q)}{\partial q_j^2} = U(q)\Psi(t, q), \tag{4.1}$$

where $\bar{h} = \dfrac{h}{2\pi}$, h is the Planck constant, $m > 0$ the mass of particle, $U : \mathbb{R} \longrightarrow \mathbb{R}$, $\Psi : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{C}$ is the wave function associated to the particle on $\mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}, \mathbb{C})$, subject to the condition

$$\frac{\Box q_k(t)}{\Box t} = -i2\gamma\frac{\partial\ln(\Psi(t, q))}{\partial q_k}, \quad k = 1, \ldots, d,$$

with $\gamma = \frac{\bar{h}}{2m} \in \mathbb{R}$. In [14, Theorem 9] it is shown that the solutions $q(\cdot)$ of (4.1) coincide with Euler–Lagrange extremals of functional (3.1) with Lagrangian

$$L(t, q, \Box q) = \frac{1}{2} m (\Box q)^2 - U(q).$$

The functional

$$I[q(\cdot)] = \frac{1}{2} \int_a^b \left[m \left(-i2\gamma \sum_{k=1}^{d} \frac{\partial \ln(\Psi(t, q))}{\partial q_k} \right)^2 - 2U(q) \right] dt$$

is invariant in the sense of Definition 3.4 under the symmetries $(\tau, \xi) = (c_k, 0)$, where c_k is an arbitrary constant. It follows from our Theorem 3.8 that

$$2m \left(\gamma \sum_{k=1}^{d} \frac{\partial \ln(\Psi(t, q))}{\partial q_k} \right)^2 + U(q) = \frac{1}{8m} \left(\frac{h}{\pi} \sum_{k=1}^{d} \frac{\partial \ln(\Psi(t, q))}{\partial q_k} \right)^2 + U(q) \quad (4.2)$$

is a nondifferentiable constant of motion: (4.2) is preserved along all solutions $q(t)$ of the linear Schrödinger equation (4.1).

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