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Stability Analysis of Models Simulating Global Processes in the Biosphere

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Abstract

The system under consideration is written in a special uniform vector-matrix form. An algorithm for evaluation of the stability region in the phase space of the zero equilibrium system with a quadratic right-hand part is given. We study the stability of the stationary state of the system with delay. The mathematical instruments of investigation are Lyapunov's second method of quadratic functions and B. S. Razumikhin's condition.

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1 Introduction

Simulation models of global processes in the biosphere are models that describe changes of ecosystem components (biogeochemical cycles) under the action of anthropogenic factors on the scale of the biosphere. One of the first authors of models of global changes in the biosphere and atmosphere climate model was V. A. Kostitsyn (1935). D. V. Krapivin has considered models involving biogeochemical cycles models. The invention of computers enabled us to consider very difficult problems that are important to all humanity. Emerging science globalism, based on a study using computers, allows modeling global problems. In the late 70s and early 80s, a version of the global biosphere model, called "System of Gaia", was created in the Computing Center of the USSR under the direction of President N. N. Moiseev. It was used to analyze the scenario of a local nuclear conflict, and it describes the effect of "nuclear winter" and the

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forecast of global changes in the biosphere. In modeling the functioning of the biosphere, it turned out that after large-scale effects of the biosphere, it never returns to its original state. Each time it will be a new biosphere, and its parameters, as a rule, rule out the possibility of further human development.

Now we briefly discuss some models of the Club of Rome. The methodological basis of forecasting global change was mathematical modeling techniques and, above all, the methods of dynamical systems. J. Forrester built a primitive but fairly comprehensive mathematical model that could roughly simulate the development of the world situation by means of the five main interdependent variables: Population size, capital investment, the use of nonrenewable resources, food production, and pollution. Model "WORLD-1" consisted of 42 nonlinear equations that describe the relationship between selected variables. It shows that an increased population results in an accelerated growth of all other indicators. The forecast is an ecological disaster between the first 30 and 50 years of the 21st century. Professor D. Meadows with a group of scientists has refined this model of the world to obtain the model "WORLD-2" or "stabilization model of society": After 1975, the population growth is assumed to be zero and the re-use of resources and nonwaste technology is introduced. The forecast is stabilization, in which the level of output per capital is three times the world average of 1970. Model "World 3" assumes the same measures as in the model of stabilization, but after 2000 instead of 1975. The forecast is that the balance is not achieved and that the lack of food will be felt before 2100. By the mid-1980s, there were more than 15 global forecasts bearing by the title of "models of the world". Some of the most famous and interesting ones are "World Dynamics" (J. Forrester), "The Limits to Growth" (D. Meadows et al), "Humanity is at a turning point" (M. Mesarovic and E. Pestel), "The future of the world economy" (Wassily Leontief), and others. The principles of sustainable development key challenges in achieving the goal of sustainable development are: 1. Population growth. 2. The problem of food production, protection of resources owls and the environment. 3. Saving soil. 4. Protection of water resources of the Earth. 5. The protection of forests. 6. Protecting the Earth's atmosphere. 7. Waste generated in human activity. 8. Efficient use of energy. 9. Development of industry and the greening of technology. 10. Sustainability of ecosystems. 11. Storage of biological diversity. 12. The responsibility and the value of individuals for environmental choices.

In this paper, we propose a mathematical instrument that may aid to tackle some of the problems explained above. We consider a nonlinear system of differential equations with quadratic nonlinearity and an asymptotically stable matrix in the linear approximation. We propose an algorithm that estimates the region of stability with quadratic right-hand side. The algorithm is based on using Lyapunov's second method of quadratic functions of the form [7]. The resulting estimates can be viewed as the analysis of conditions for stable functioning of ecosystems and develop a strategy for the survival of humanity [2, 10, 11].

2 Systems without Delay

2.1 Problem Statement

Let us consider the nonlinear system with quadratic right-hand side (written in vector-matrix form [1,3,6])

$$\dot{x}(t) = Ax(t) + X^T(t)Bx(t), \tag{2.1}$$

where $B = \{B_1, B_2, \dots, B_n\}^T$ is a rectangular $n^2 \times n$ matrix consisting of symmetric $n \times n$ matrices

$$B_i := \begin{bmatrix} b_{11}^i & b_{12}^i & \dots & b_{1n}^i \\ b_{21}^i & b_{22}^i & \dots & b_{2n}^i \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1}^i & b_{n2}^i & \dots & b_{nn}^i, \end{bmatrix}, \quad i = 1, 2, \dots, n,$$

$X^T = \{X_1(t), X_2(t), \dots, X_n(t)\}$ is a rectangular $n^2 \times n$ matrix consisting of square $n \times n$ matrices $X_i(t)$, in which the ith rows are the vectors $x(t)$ and the other elements are zero, i.e.,

$$\begin{aligned} X_1(t) \quad &:= \quad \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0, \end{bmatrix}, \\ &\vdots \\ X_n(t) \quad &:= \quad \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ x_1(t) & x_2(t) & \dots & x_n(t), \end{bmatrix}. \end{aligned}$$

Vector and matrix norms are defined by

$$\mid x(t) \mid = \left\{ \sum_{i=1}^{n} x_i^2(t) \right\}^{\frac{1}{2}} \text{ and } \mid B \mid = \left\{ \lambda_{\max}(B^T B) \right\}^{\frac{1}{2}},$$

and we use the notation $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ to denote the extreme eigenvalues of the corresponding symmetric matrix.

Assume that the matrix of the linear part of (2.1) is asymptotically stable. Then, as it follows from the theory of the stability of the linear approximation [4], the zero solution of a nonlinear system is also asymptotically stable. If we take a quadratic form $V(x) = x^T H x$ of the Lyapunov function, then its derivative with respect to the system (2.1) has the form

$$\frac{dV(x(t))}{dt} = x^T(t) \left[(A^T H + HA) + (B^T X(t) H + H X^T B) \right] x(t). \tag{2.2}$$

If the matrix A is asymptotically stable, then for any positive definite matrix C, the Lyapunov matrix equation

$$A^T H + HA = -C$$

has a unique solution H which is positive definite [4]. Taking in account that H is the solution of the this Lyapunov equation, we obtain

$$\frac{dV(x(t))}{dt} = -x^T(t)\left[C - (B^T X(t)H + HX^T(t)B)\right]x(t).$$

The stability domain of the zero equilibrium is the interior surface of the level of the Lyapunov function $V(x) = r$, $r > 0$ which lies within the area

$$G_0 = \left\{x \in \mathbb{R}^n : C - B^T XH - HX^T B > \Theta\right\},$$

where

$$C - B^T XH - HX^T B > \Theta$$

denotes a positive definite matrix. Let us replace this condition with a more "rough" one. Since, by the selected matrix and vector norms, it will be implemented that $\mid X(t) \mid = \mid x(t) \mid$, the total derivative of the Lyapunov function satisfies

$$\frac{dV(x(t))}{dt} \leq -\left[\lambda_{min}(C) - 2 \mid H \mid\mid B \mid\mid x(t) \mid\right] \mid x(t) \mid^2 .$$

Let us denote

$$G_0 = \left\{x \in \mathbb{R}^n : \mid x \mid < \frac{\lambda_{\min}(C)}{2\lambda_{\max}(H) \mid B \mid}\right\}.$$

Then the regions of "guaranteed" stability have the form

$$G_{r_0} = \max\left\{G_r : G_r \subset G_0\right\},$$
$$G_r = \left\{x \in \mathbb{R}^n : x^T Hx < r^2\right\}.$$

It follows from this dependence that in order to determine the "maximum" of stainability, it should be placed inside a sphere of radius

$$R = \frac{\lambda_{\min}(C)}{2 \mid H \mid\mid B \mid}$$

an ellipse $x^T Hx = r^2$ and "stretched" by $r \to \infty$ as long as the ellipse touches the sphere. We obtain an estimate of the convergence of solutions, the initial position of which is to "guarantee stability".

Theorem 2.1. *Suppose the matrix of the linear part of system* (2.1) *is asymptotically stable. Then the trivial solution of this system is asymptotically stable, and for the solutions with initial conditions*

$$|x(0)| < \frac{\gamma(H)}{2|B|\varphi(H)}, \quad \varphi(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \gamma(H) = \frac{\lambda_{\min}(C)}{\lambda_{\max}(H)},$$

we have

$$|x(t)| \leq \frac{\gamma(H)\sqrt{\lambda_{\min}(H)}|x(0)|}{[\gamma(H) - 2 \mid B \mid| \varphi(H) \mid |x(0)|]\, e^{0.5\gamma(H)t} + 2 \mid B \mid \varphi(H)|x(0)|}. \tag{2.3}$$

2.2 Model Problem

Let us consider the scalar equations

$$\dot{x}(t) = -ax(t) + bx^2(t)$$

with solutions

$$x(t) = \frac{ax(0)e^{-at}}{a - bx(0)\left[1 - e^{-at}\right]}.$$

Let us consider the use of the Lyapunov function $V(x) = x^2$. For this function, $\lambda_{\min}(H) = \lambda_{\max}(H) = 1$. The total derivative has the form

$$\frac{dV(x(t))}{dt} = -2ax^2(t).$$

The estimate (2.3) for solutions with initial conditions $x(0) < \dfrac{a}{b}$ has the form

$$x(t) \leq \frac{ax(0)}{[a - bx(0)]\, e^{at} + bx(0)} \to 0.$$

Therefore, for these scalar equations, the exact solutions coincide with the estimates of the given quadratic Lyapunov functions.

2.3 Systems in the Plane

More positive results of the assessment of convergence of systems with quadratic right-hand side can be obtained by considering a system of the form (2.1) on the plane. It has the form

$$\begin{aligned} \dot{x}_1(t) &= a_{11}x_2(t) + a_{12}x_2(t) + b^1_{11}x_1^2(t) + 2b^1_{12}x_1x_2 + b^1_{22}x_2^2(t), \\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + b^2_{11}x_1^2(t) + 2b^2_{12}x_1x_2 + b^2_{22}x_2^2(t). \end{aligned}$$

By introducing

$$A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B_1 := \begin{bmatrix} b^1_{11} & b^1_{12} \\ b^1_{21} & b^1_{22}, \end{bmatrix}, \quad B_2 := \begin{bmatrix} b^2_{11} & b^2_{12} \\ b^2_{21} & b^2_{22}, \end{bmatrix},$$

the last system can be rewritten in the vector-matrix form

$$\dot{x}(t) = Ax(t) + X^T(t)Bx(t).$$

Then the total derivative of the Lyapunov function has the form

$$\frac{dV(x(t))}{dt} \le -\left[\lambda_{\min}(C) - 2 \mid H \mid\mid B \mid\mid x(t) \mid\right] \mid x(t) \mid^2,$$

where

$$\mid H \mid = \lambda_{\max}(H) = \frac{1}{2}\left\{(h_{11} + h_{22} + \sqrt{(h_{11} - h_{22}) + 4h_{12}^2}\right\},$$
$$\lambda_{\min}(C) = \frac{1}{2}\left\{(c_{11} + c_{22} + \sqrt{(c_{11} - c_{22}) + 4c_{12}^2}\right\},$$
$$\mid B \mid = \left\{\lambda_{\max}(B^T B)\right\}.$$

The guaranteed stability region will be the interior of the ellipse

$$h_{11}x^2 + 2h_{12}xy + h_{22}y^2 \le r_0^2.$$

2.4 Systems with Linear Part

In general, the system (2.1) with the linear part can be written as [5]

$$\dot{x}_i(t) = \left[a_i - \sum_{j=1}^{n} b_{ij}x_j(t)\right] x_i(t),$$

where A is the square diagonal matrix with constant coefficients $A = \{a_{ii}\}$, $B = \{B_1, B_2, \ldots, B_n\}^T$ is the rectangular matrix consisting of symmetric square matrices B_i, in which the ith column is the vector $b_i^T = (b_{i1}, b_{i2}, \ldots, b_{in})$, and

$$X^T = \{X_1(t), X_2(t), \ldots, X_n(t)\}$$

is the rectangular $n \times n^2$ matrix, consisting of the matrices $X_i(t)$, in which the ith rows are the vectors $x(t)$ and the other elements zero. Let us suppose that $\det B_0 \neq 0$ and

$$B := \begin{bmatrix} b_{11} & b_{21} & \ldots & b_{n1} \\ b_{12} & b_{22} & \ldots & b_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n} & b_{2n} & \ldots & b_{nn}, \end{bmatrix}.$$

Then, as a rule, the interest is to search for a singular point $x_0^T = (x_1^0, x_2^0, \ldots, x_n^0)$ that is the solution of the algebraic equation

$$B_0 x = a, \quad a^T = (a_1, a_2, \ldots, a_n)$$

and located in the first quadrant, i.e., $x_i^0 > 0$. After replacing $x(t) = y(t) + x_0$ and transformation, we get the perturbed system

$$\dot{y}(t) = \bar{A}y(t) + Y(t)^T By(t).$$

Let us suppose that the matrix $\bar{A}$ is asymptotically stable, i.e., $\mathrm{Re}\lambda_i(\bar{A}) < 0$. Then the singular point $x_0^T = (x_1^0, x_2^0, \ldots, x_n^0)$ is asymptotically stable and the region of it's stability can be assessed using a quadratic Lyapunov function $V(y) = y^T Hy$, in which H is a symmetric positive definite solution of the Lyapunov equation [2]

$$\bar{A}^T H + H\bar{A} = C.$$

Here C is an arbitrary, symmetric, positive definite matrix. Taking the total derivative of the Lyapunov function, we obtain

$$\frac{dV(y(t))}{dt} \leq -\left[\lambda_{\min}(C) - 2 \mid H \mid\mid B \mid\mid y(t) \mid\right] \mid y(t) \mid^2,$$

and, in the case of asymptotic stability of the matrix $\bar{A}$, the guaranteed stability of the equilibrium area of the singular point is inside the ellipse $y^T Hy = r^2$ which is inside the sphere $\mid y \mid = R$. Denoting

$$G_0 = \left\{ y \in \mathbb{R}^n : \mid y \mid < \frac{\lambda_{\min}(C)}{2 \mid H \mid\mid B \mid} \right\},$$

we find that the area of "guaranteed" stability has the form [5]

$$G_{r_0} = \max\left\{G_r : G_r \subset G_0\right\},$$
$$G_r = \left\{y \in \mathbb{R}^n : x^T Hx < r^2\right\}.$$

Thus, the ellipse $y^T Hy = r^2$ should be placed inside a sphere of radius

$$R = \frac{\lambda_{\min}(C)}{2 \mid H \mid\mid B \mid}$$

and "stretched" $r \to \infty$ as long as the ellipse touches the sphere. We obtain estimates of solutions whose initial state are inside the "guaranteed" stability regions.

3 Systems with Delay

Typically, models of the economy and the environment exhibit a certain time lag. Therefore, using systems of functional differential equations with delay [1, 9] leads to more appropriate mathematical models. One of the first mathematical models described by differential equations with constant delay was the equation by Hutchison and Voltaire [7, 10, 11].

3.1 Verhulst Equation with Delay

The Verhulst equation displays the dynamics of population growth with saturation. There is limited growth due to "internal competition". Note the following factor, specifying the model of Verhulst. Competition generally occurs between the new population and the population born with retardation. In this case, population dynamics is determined by the equation of Hutchison (1948), which has the form of a differential equation with delay

$$\frac{dx(t)}{dt} = ax(t)\left(1 - \frac{x(t-\tau)}{k}\right). \tag{3.1}$$

The dynamical system described by equation (3.1) has two equilibria $x(t) \equiv 0$ and $x(t) \equiv k$. It is easy to see that the linear approximation is given by the equation at the point $x = 0$ and points to the instability of the zero equilibrium. Consider the second point of rest $x(t) \equiv k$. Draw the linearisation

$$\frac{dx(t)}{dt} = f(x(t), x(t-\tau))$$

in the neighbourhood $x(t) \equiv k$. After transformation and substitution of the corresponding values, we get

$$\frac{dx(t)}{dt} = -a\left[x(t-\tau) - k\right].$$

In the neighbourhood of the singular point $x(t) = k$ of the equation of the linear approximation, we obtain

$$\frac{dy(t)}{dt} = -a\left[y(t-\tau) - k\right],$$

where $y(t) = x(t) - k$. The characteristic equation is

$$\lambda + ae^{\lambda\tau} = 0.$$

As a consequence of [1, 9], assuming

$$0 < a\tau < \frac{\pi}{2},$$

the equilibrium $x(t) = k$ is locally asymptotically stable.

Now we estimate the stability region in the phase space of the equilibrium position $x(t) = k$ of the original nonlinear system (3.1). After the transformation $x(t) = y(t) + k$, the point $x(t) = k$ moves to the origin, and we obtain the equation

$$\frac{dy(t)}{dt} = -a\left[y(t) + k\right] y(t-\tau).$$

We use the quadratic Lyapunov function $V(y) = \frac{1}{2}y^2$. Since we consider the delay equation, we use B. S. Razumikhin's condition [4, 5, 8] to evaluate the total derivative.

This condition means geometrically that the total derivative is calculated by an approach from the inside surface of the level of the Lyapunov function. For the function $V(y) = \frac{1}{2}y^2$, we find

$$| y(t-\tau) | < | y(t) |,$$

and the total derivative of the Lyapunov function in view of (2.2) has the form

$$\frac{dV(x(t))}{dt} \leq -a\left[1 - \frac{1}{k} \mid y(t) \mid\right] \mid y(t) \mid^2 + \frac{a}{k} \mid y(t) \mid [\mid y(t) \mid +k] \mid y(t) - y(t-\tau) \mid .$$

We estimate the phase coordinates with delay and without. Rewrite the previous inequality in the form

$$\frac{dV(x(t))}{dt} \leq -a\left[1 - \frac{1}{k} \mid y(t) \mid\right] \mid y(t) \mid^2 + \frac{a}{k} \mid y(t) \mid [\mid y(t) \mid +k] \mid y(t) - y(t-\tau) \mid .$$

Thus, when

$$a\left[1 - \frac{1}{k} \mid y(t) \mid\right] > \left(\frac{a}{k}\right)^2 [\mid y(t) \mid]^2 \tau,$$

the total derivative of the Lyapunov function is negative definite. Thus, the stability conditions are determined by the inequalities

$$\mid y(t) \mid < k, \quad \tau < \frac{k[k- \mid y(t) \mid]}{a[\mid y(t) \mid +k]^2}.$$

3.2 General Quadratic Model with Delay

In the universal vector-matrix form, the quadratic model with delay is written as

$$\dot{x}(t) = \left[Ax(t) + X^T(t-\tau)\right] Bx(t),$$

Suppose, as in the case without delay, $x_0^T = (x_1^0, x_2^0, \ldots, x_n^0)$ is the solution of the algebraic equation

$$B_0 x = a, \quad a^T = (a_1, a_2, \ldots, a_n)^T.$$

Making the substitution $x(t) = y(t) + k$, we obtain the perturbed system which, after transformation, takes the form

$$\dot{y}(t) = \bar{A}y(t) + Y^T(t-\tau)By(t), \tag{3.2}$$

where

$$\bar{A} := \begin{bmatrix} b_{11}x_1^0 & b_{21}x_1^0 & \ldots & b_{2n}x_1^0 \\ b_{21}x_2^0 & b_{22}x_2^0 & \ldots & b_{2n}x_2^0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n}x_n^0 & b_{1n}x_n^0 & \ldots & b_{nn}x_n^0 \end{bmatrix}.$$

We study stability of the zero equilibrium state of the system (3.2) using the method of Lyapunov functions in the quadratic form $V(y) = y^T Hy$. In evaluating the total derivative, we use B. S. Razumikhin's condition [6]. For the function $V(y)$, it has the form

$$| y(t-\tau) | \leq \sqrt{\varphi(H)} \mid y(t) \mid .$$

We denote the set of points $y \in \mathbb{R}^n$ that are within the level surfaces $V(y) = \alpha$ of the Lyapunov function $V(y) = y^T Hy$ by V^α, and its boundary by ∂V^α, i.e.,

$$V^\alpha = \{y \in \mathbb{R}^n : V(y) < \alpha\} .$$

Theorem 3.1. *Let the solution y of* (3.2) *satisfy $y(T) \in \partial V^\alpha$ at the time $t = T > \tau$, and $y(t) \in V^\alpha$ at $-\tau \leq t < T$. Then we have*

$$|y(T) - y(T-\tau)| \leq \left[\bar{A} + B\sqrt{\varphi(H)} y(T)\right] \sqrt{\varphi(H)} y(T)\tau.$$

Theorem 3.2. *Let the initial condition φ for the solution y satisfy $\varphi(t) < \delta$ for all $-\tau \leq t \leq 0$. Then we have*

$$|y(t)| \leq \delta \exp\left[\bar{A} + B\delta\right] \tau \text{ for all } 0 \leq t \leq \tau.$$

Theorem 3.3. *Let the matrix $\bar{A}$ be asymptotically stable, i.e., $\mathrm{Re}\lambda_i(\bar{A}) < 0$. Then for $\tau < \tau_0$, where*

$$\tau_0 = \frac{\lambda_{\min}(C)}{2HB\sqrt{\varphi(H)}},$$

the equilibrium is asymptotically stable.

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