Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 9, Number 2, pp. 149–159 (2014)
http://campus.mst.edu/adsa

Structure of the Fractional Lyapunov Spectrum for Linear Fractional Differential Equations

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Abstract

In this paper we build on the spectral theory for linear fractional differential equations and prove that the fractional Lyapunov spectrum of solutions starting from a unit sphere is the union of a compact interval in $\mathbb{R}_{<0}$ and at most d distinct fractional Lyapunov exponents.

AMS Subject Classifications: 34A08, 34D08.
Keywords: Lyapunov exponents, linear fractional differential equations.

1 Introduction

Lyapunov's first and second methods for ordinary differential equations offer two ways to study the stability of solutions. Whereas his second (or direct) method was originally developed to study the stability of a fixed point of an autonomous or nonautonomous equation, his first method linearizes a nonlinear equation along an orbit and the resulting linear variational equation is studied by means of characteristic numbers (exponential

Received April 4, 2014; Accepted May 22, 2014
Communicated by Sandra Pinelas

growth rates of solutions, today known as Lyapunov exponents [1, 11]). The stability of the linear equation is then transferred to the nonlinear equation. In order to be able to develop Lyapunov's first method for fractional differential equations, it is pivotal to extend the concept of Lyapunov exponents to linear nonautonomous fractional differential equations in such a way, such that the (fractional) Lyapunov exponents characterize the stability properties and, in a second step, those stability properties can be transferred locally to a nonlinearly perturbed fractional differential equation (see also [8] for the autonomous fractional case).

Even in the ordinary differential equations case, the investigation of Lyapunov stability of nonautonomous systems is more difficult than for autonomous systems (even if they are linear). Using Lyapunov exponents as a tool, one can show that the state space can be decomposed into a direct sum of subspaces in which each subspace is given as the set of all solutions corresponding to a characteristic Lyapunov exponent. The dimensions of these subspaces are called multiplicities of the corresponding Lyapunov exponents. The Lyapunov exponents together with their multiplicities form the Lyapunov spectrum of this system. The qualitative behavior of a linear system, such as stability, attractivity and hyperbolicity, can be characterized completely by its Lyapunov spectrum and the corresponding decomposition of the state space.

For a qualitative theory of fractional differential equations, the linear theory can also considered as the most fundamental step for the development of the other parts of the qualitative theory such as the invariant manifold theory and the linearization theory of nonlinear systems. Here, we would like to distinguish two distinct cases: linear autonomous and linear nonautonomous fractional differential equations. The behavior of solutions of linear autonomous fractional differential equations is well developed, see [5]. Based on these results, stability of nonlinear fractional differential equations in a neighborhood of a fixed point is also established and we refer the reader to [9] for a survey on this topic.

In contrast to the well developed linear autonomous theory, the corresponding research for linear nonautonomous fractional differential equations is still in its infancy. Lyapunov exponents are first discussed in [7] in which the authors use the asymptotic behavior (in comparison with the exponential function) of the fundamental matrix to define the Lyapunov spectrum. This notion of Lyapunov spectrum is used to investigate chaotic behavior in a class of fractional differential systems, see e.g., [4, 10].

Surprisingly, it is shown in [3] that the classical Lyapunov exponent of any solution of a bounded linear nonautonomous fractional differential equation is always nonnegative. As a consequence, many properties such as stability and attractivity of linear fractional differential equations cannot be characterized by its Lyapunov spectrum.

Using the Mittag–Leffler function $E_\alpha(\lambda t^\alpha)$, a meaningful notion of fractional Lyapunov spectrum is developed in [3]. In comparison to the classical Lyapunov spectrum, the fractional Lyapunov spectrum enables us to characterize the stability of linear fractional differential equations.

It is worth mentioning that a scaling of the initial condition of a solution with negative fractional Lyapunov exponent leads to a different fractional Lyapunov exponent (cp. Remark 2.4). As a consequence, in contrast to the Lyapunov spectrum, the fractional Lyapunov spectrum is, in general, different from the set of fractional Lyapunov exponents of all solutions starting from a unit sphere. Our aim in this paper is to go one step further than in [3] and to fully understand the second set.

The paper is organized as follows: In Section 2, we recall basic knowledge on fractional analysis and linear fractional differential equations developed in [3] and references therein. Section 3 is devoted to our main result on the structure of the set of fractional Lyapunov exponents of solutions starting from the unit sphere.

To conclude this introductory section, we introduce notation which is used throughout this paper. Let $\mathbb{R}_{\geq 0}, \mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$ denote the set of all nonnegative, positive and negative real numbers, respectively.

2 Preliminaries

This section is devoted to recall an abstract framework of fractional calculus and the corresponding linear fractional differential equations. We refer the reader to some textbooks about fractional differential equations, e.g., [5, 6], for more details of this theory.

2.1 Linear Fractional Differential Equations

For $\alpha \in (0,1)$, the *Riemann–Liouville integral operator of order* α for an integrable function $f : [a,b] \to \mathbb{R}$, where $[a,b] \subset \mathbb{R}$, is defined by

$$I^{\alpha}_{a+}f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \qquad \text{for } t > a,$$

where the *Gamma function* $\Gamma : (0,\infty) \to \mathbb{R}$ is defined as

$$\Gamma(\alpha) := \int_0^\infty \tau^{\alpha-1} \exp(-\tau)\, d\tau,$$

see, e.g., [5]. The corresponding Riemann–Liouville fractional derivative is given by

$$D^{\alpha}_{a+}f(t) := (D^m I^{m-\alpha}_{a+} f)(t),$$

where $D = \dfrac{d}{dt}$ is the usual derivative and $m := \lceil \alpha \rceil$. On the other hand, the *Caputo fractional derivative* ${}^{C}D^{\alpha}_{a+}f$ of a function $f \in C^m([a,b])$, which was introduced by Caputo (see e.g., [5]), is defined by

$${}^{C}D^{\alpha}_{a+}f(t) := (I^{m-\alpha}_{a+} D^m f)(t), \qquad \text{for } t > a.$$

The Caputo derivative of a d–dimensional vector function $f(t) = (f_1(t), \dots, f_d(t))^{\mathrm{T}}$ is defined component-wise as

$$^{C}D^{\alpha}_{0+}f(t) = (^{C}D^{\alpha}_{0+}f_1(t), \dots, ^{C}D^{\alpha}_{0+}f_d(t))^{\mathrm{T}}.$$

We refer the readers to [5, Chapter 2 & 3] for a discussion on differences between building models with Caputo in comparison to Riemann–Liouville derivatives, and also on some advantages of Caputo derivatives over Riemann–Liouville derivatives. In this paper, we consider linear fractional differential equations involving Caputo fractional derivative

$$^{C}D^{\alpha}_{0+}x(t) = A(t)x(t), \tag{2.1}$$

where $\alpha \in (0,1)$ and $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ is a continuous matrix-valued function. For any initial value $x_0 \in \mathbb{R}^d$, the integral equation which corresponds to the initial value problem of system (2.1), $x(0) = x_0$, is given by

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}A(s)x(s)\,ds, \quad x_0 \in \mathbb{R}^d, \tag{2.2}$$

see, e.g., [5, Lemma 6.2, p. 86]. Using the integral form (2.2) and fixed point arguments (see e.g., [2]), it is easy to show that the initial value problem of system (2.1), $x(0) = x_0$, has a unique solution $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ which is denoted by $\Phi(t, x_0)$. Furthermore, by linearity of (2.2) the map $\Phi(t, \cdot)$ is linear for all $t \in \mathbb{R}_{\geq 0}$, i.e.,

$$\Phi(t, ax_1 + bx_2) = a\Phi(t, x_1) + b\Phi(t, x_2) \quad \text{for } a, b \in \mathbb{R}, x_1, x_2 \in \mathbb{R}^d.$$

2.2 Fractional Lyapunov Exponent

In this subsection, we recall the notion of fractional Lyapunov exponent for an arbitrary function in [3]. Let $\|\cdot\|$ be an arbitrary but fixed norm on $\mathbb{R}^d$.

Definition 2.1. The function $E_\alpha : \mathbb{C} \to \mathbb{C}$ which is defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}$$

is called *Mittag–Leffler* function.

For any $\alpha \in (0,1)$, the restriction $E_\alpha : \mathbb{R} \to \mathbb{R}$ of the Mittag–Leffler function is strictly monotonically increasing. Thus, the restriction of the Mittag–Leffler function E_α on $\mathbb{R}$ is strictly monotonically increasing. Furthermore, using [12, Theorem 1.3, p. 32] and [12, Theorem 1.4, p. 33], we obtain that

$$\lim_{z\to\infty} E_\alpha(z) = \infty \quad \text{and} \quad \lim_{z\to-\infty} E_\alpha(z) = 0.$$

Consequently, $E_\alpha(\mathbb{R}) = \mathbb{R}_{>0}$ and due to continuity and monotonicity of E_α the inverse function of the restriction function $E_\alpha : \mathbb{R} \to \mathbb{R}_{>0}$, which is denoted by $\log^M_\alpha : \mathbb{R}_{>0} \to \mathbb{R}$, exists.

Definition 2.2 (Fractional Lyapunov Exponent, [3]). Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ be an arbitrary function. The *fractional Lyapunov exponent* of order α of f is defined as

$$\chi_\alpha(f) = \limsup_{t\to\infty} \frac{1}{t^\alpha} \log_\alpha^M \|f(t)\|.$$

The following theorem provides a practical formulation of $\chi_\alpha(f)$.

Theorem 2.3. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ be an arbitrary function. The following statements hold:*

(i) $\chi_\alpha(f) > 0$ *if and only if* $\chi(f) > 0$ *and in this case we get*

$$\chi_\alpha(f) = \chi(f)^\alpha = \limsup_{t\to\infty} \left(\frac{1}{t}\log\|f(t)\|\right)^\alpha, \tag{2.3}$$

where $\chi(f) := \limsup_{t\to\infty} \frac{1}{t}\log\|f(t)\|$.

(ii) $\chi_\alpha(f) < 0$ *if and only if* $\limsup_{t\to\infty} t^\alpha\|f(t)\| < \infty$ *and in this case we get*

$$\chi_\alpha(f) = -\frac{1}{\Gamma(1-\alpha)\limsup_{t\to\infty} t^\alpha\|f(t)\|}. \tag{2.4}$$

(iii) $\chi_\alpha(f) = 0$ *if and only if*

$$\chi(f) \leq 0 \quad \textit{and} \quad \limsup_{t\to\infty} t^\alpha\|f(t)\| = \infty,$$

where $\chi(f)$ *is the Lyapunov exponent of* f.

Proof. See [3, Theorem 9]. □

Remark 2.4. Note that in contrast to classical Lyapunov exponents, fractional Lyapunov exponents of functions do not remain constant if these functions are multiplied by a nonzero number, i.e., in general $\chi_\alpha(f) \neq \chi_\alpha(cf)$ where $c \neq 0$ (cp. also [3, Lemma 3.3(i)]).

2.3 Fractional Lyapunov Spectral Theorem for Linear Fractional Differential Equations

We recall the result on the spectral theory for linear fractional differential equations in [3].

Theorem 2.5. *Suppose that* $\|A(t)\| \le M$ *for all* $t \in \mathbb{R}_{\ge 0}$. *The* fractional Lyapunov spectrum *of* (2.1) *is defined by*

$$\Sigma_\alpha := \Big\{\chi_\alpha(\Phi(\cdot, x_0)) : x_0 \in \mathbb{R}^d \setminus \{0\}\Big\}.$$

Then, the following assertions hold:

(i) $\Sigma_\alpha \subset (-\infty, M]$.

(ii) The set $\Sigma_\alpha \cap \mathbb{R}_{\ge 0}$ *consists of at most* d *distinct elements* $\lambda_j < \lambda_{j-1} < \cdots < \lambda_1$.

(iii) If $\Sigma_\alpha \cap \mathbb{R}_{<0} \neq \emptyset$, *then* $\mathbb{R}_{<0} \subset \Sigma_\alpha$.

Moreover, the following sets

$$S := \{x_0 \in \mathbb{R}^d : \chi_\alpha(\Phi(\cdot, x_0)) < 0\} \tag{2.5}$$

and for $i = 1, \dots, j$

$$E_i := \{x_0 \in \mathbb{R}^d : \chi_\alpha(\Phi(\cdot, x_0)) \le \lambda_i\}$$

are linear subspaces of $\mathbb{R}^d$. *Furthermore, we have a filtration* $S =: E_{j+1} \subsetneq E_j \subsetneq E_{j-1} \subsetneq \cdots \subsetneq E_1$ *satisfying that for* $i = 1, \dots, j$

$$\chi_\alpha(\Phi(\cdot, x_0)) = \lambda_i \quad \textit{if and only if} \quad x_0 \in E_i \setminus E_{i+1}. \tag{2.6}$$

Proof. See [3, Theorem 12]. □

3 Structure of the Set of Fractional Lyapunov Exponents of Solutions starting from a Unit Sphere

Throughout this section, let $\|\cdot\|$ be an arbitrary but fixed norm on $\mathbb{R}^d$ and $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$. Suppose that

$$\|A(t)\| \le M \qquad \text{for all } t \in \mathbb{R}_{\ge 0}. \tag{3.1}$$

Our aim in this section is to better understand the structure of the set of all fractional Lyapunov exponents of the solution of (2.1) starting from the unit sphere. This object is defined by

$$\Lambda_\alpha := \Big\{\chi_\alpha(\Phi(\cdot, x_0)) : x_0 \in \mathbb{R}^d \text{ and } ||x_0|| = 1\Big\}.$$

Using [3, Lemma 3.4], we obtain that

$$\Lambda_\alpha \subset [-M, M].$$

In this section, we deal with the structure of the set Λ_α. For this purpose, we need the following preparatory result.

Proposition 3.1. *Let S be the subspace given as in* (2.5) *in Theorem 2.5. Suppose that $S \neq \emptyset$ and define*

$$a = \inf\{\chi_\alpha(\Phi(\cdot, x)) : x \in S \cap \mathbb{S}^{d-1}\}, \quad b = \sup\{\chi_\alpha(\Phi(\cdot, x)) : x \in S \cap \mathbb{S}^{d-1}\}.$$

Then the following statement holds:

(i) $b < 0$.

(ii) The map $\lambda_\alpha : S \cap \mathbb{S}^{d-1} \to \mathbb{R}$ defined by

$$\lambda_\alpha(x) = \chi_\alpha(\Phi(\cdot, x)) \qquad \textit{for } x \in S \cap \mathbb{S}^{d-1},$$

is Lipschitz continuous.

(iii) $\lambda_\alpha(S \cap \mathbb{S}^{d-1}) = [a, b]$.

Proof. (i) Let $u_1, \dots, u_\ell$ be an orthonormal basis of S. We define the map $g : \mathbb{R}^\ell \to \mathbb{R}_{<0}$ by

$$g(\gamma_1, \dots, \gamma_\ell) := \chi_\alpha\bigg(\Phi\Big(\cdot, \sum_{i=1}^{\ell} \gamma_i u_i\Big)\bigg).$$

We prove that g is continuous. Indeed, for any $(\gamma_1, \dots, \gamma_\ell), (\widehat{\gamma}_1, \dots, \widehat{\gamma}_\ell) \in \mathbb{R}^\ell$ we have

$$\Phi\Big(t, \sum_{i=1}^{\ell} \gamma_i u_i\Big) - \Phi\Big(t, \sum_{i=1}^{\ell} \widehat{\gamma}_i u_i\Big) = \sum_{i=1}^{\ell} (\gamma_i - \widehat{\gamma}_i)\Phi(t, u_i),$$

which implies that

$$\Big\|\Phi\Big(t, \sum_{i=1}^{\ell} \gamma_i u_i\Big)\Big\| \leq \Big\|\Phi\Big(t, \sum_{i=1}^{\ell} \widehat{\gamma}_i u_i\Big)\Big\| + \sum_{i=1}^{\ell} |\gamma_i - \widehat{\gamma}_i| \|\Phi(t, u_i)\|.$$

Therefore,

$$\begin{aligned} \limsup_{t\to\infty} t^\alpha \Big\|\Phi\Big(t, \sum_{i=1}^{\ell} \gamma_i u_i\Big)\Big\| \;&\leq\; \limsup_{t\to\infty} t^\alpha \Big\|\Phi\Big(t, \sum_{i=1}^{\ell} \widehat{\gamma}_i u_i\Big)\Big\| + \\ &\quad \sum_{i=1}^{\ell} |\gamma_i - \widehat{\gamma}_i| \limsup_{t\to\infty} t^\alpha \|\Phi(t, u_i)\|. \end{aligned}$$

Using Theorem 2.3, we obtain that

$$-\frac{1}{\Gamma(1-\alpha) g(\gamma_1, \dots, \gamma_\ell)} \;\leq\; -\frac{1}{\Gamma(1-\alpha) g(\widehat{\gamma}_1, \dots, \widehat{\gamma}_\ell)} - \sum_{i=1}^{\ell} \frac{|\gamma_i - \widehat{\gamma}_i|}{\Gamma(1-\alpha)\chi_\alpha(\Phi(\cdot, u_i))}.$$

Exchanging $(\gamma_1, \ldots, \gamma_\ell)$ and $(\widehat{\gamma}_1, \ldots, \widehat{\gamma}_\ell)$ in the above inequality, we get that

$$\left| \frac{1}{g(\gamma_1, \ldots, \gamma_\ell)} - \frac{1}{g(\widehat{\gamma}_1, \ldots, \widehat{\gamma}_\ell)} \right| \leq - \sum_{i=1}^{\ell} \frac{|\gamma_i - \widehat{\gamma}_i|}{\chi_\alpha(\Phi(\cdot, u_i))},$$

which proves the continuity of the function $(\gamma_1, \ldots, \gamma_\ell) \mapsto \dfrac{1}{g(\gamma_1, \ldots, \gamma_\ell)}$ and therefore the function $g(\gamma_1, \ldots, \gamma_\ell)$ is also continuous. Note that all norms on $\mathbb{R}^d$ are equivalent. Thus, there exists $m > 0$ such that

$$\mathbb{S}^{d-1} \subset \left\{ x \in \mathbb{R}^d : \|x\|_2 \leq m \right\},$$

where $\| \cdot \|_2$ is the standard Euclidean norm. Consequently, we get

$$\begin{aligned} \sup \left\{ \chi_\alpha(\Phi(\cdot, x)) : x \in S \cap \mathbb{S}^{d-1} \right\} &\leq \sup \left\{ \chi_\alpha(\Phi(\cdot, x)) : x \in S, \|x\|_2 \leq m \right\} \\ &= \sup \left\{ \chi_\alpha \Big(\Phi \Big(\cdot, \sum_{i=1}^{\ell} \gamma_i u_i \Big) \Big) : \sum_{i=1}^{\ell} \gamma_i^2 \leq m \right\} \\ &= \sup \left\{ g(\gamma_1, \ldots, \gamma_\ell) : \sum_{i=1}^{\ell} \gamma_i^2 \leq m \right\}. \qquad (3.2) \end{aligned}$$

Since the set $\{(\gamma_1, \ldots, \gamma_\ell) \in \mathbb{R}^\ell : \sum_{i=1}^{\ell} \gamma_i^2 \leq m\}$ is compact in $\mathbb{R}^\ell$, it follows together with the continuity of g that

$$\sup \left\{ g(\gamma_1, \ldots, \gamma_\ell) : \sum_{i=1}^{\ell} \gamma_i^2 \leq m \right\} < 0,$$

which together with (3.2) proves that

$$b = \sup\{\chi_\alpha(\Phi(\cdot, x)) : x \in S \cap \mathbb{S}^{d-1}\} < 0.$$

(ii) Let $x, y \in S \cap \mathbb{S}^{d-1}$, $x \neq y$ be arbitrary. Using the triangle inequality, we obtain

$$||\Phi(t, x)|| + ||\Phi(t, y - x)|| \geq ||\Phi(t, y)||,$$

which implies that

$$\limsup_{t \to \infty} t^\alpha \, ||\Phi(t, x)|| + ||x - y|| \limsup_{t \to \infty} t^\alpha ||\Phi(t, \tfrac{x-y}{\|x-y\|})|| \geq \limsup_{t \to \infty} t^\alpha \, ||\Phi(t, y)|| \, .$$

By virtue of Theorem 2.3, we have

$$- \frac{1}{\Gamma(1 - \alpha)\chi_\alpha(\Phi(\cdot, x))} - \frac{||x - y||}{\Gamma(1 - \alpha)\chi_\alpha(\Phi(\cdot, \frac{x-y}{||x-y||}))} \geq - \frac{1}{\Gamma(1 - \alpha)\chi_\alpha(\Phi(\cdot, y))}.$$

This together with (i) implies that

$$\frac{1}{\chi_\alpha(\Phi(\cdot,x))}-\frac{||x-y||}{|b|}\leq\frac{1}{\chi_\alpha(\Phi(\cdot,y))}\,.$$

Exchanging the role of x and y in the above inequality, we get that

$$\left|\frac{1}{\chi_\alpha(\Phi(\cdot,x))}-\frac{1}{\chi_\alpha(\Phi(\cdot,y))}\right|\leq\frac{\|x-y\|}{|b|}.$$

Note that $\chi_\alpha(\Phi(\cdot,x)),\chi_\alpha(\Phi(\cdot,y))\in[-M,b]$. Hence, we get that

$$|\lambda_\alpha(x)-\lambda_\alpha(y)|\leq\frac{M^2}{|b|}\|x-y\|,$$

which completes the proof of this part.

(iii) Due to the Lipschitz continuity of the map λ_α and the definitions of a,b, there exist $x,y\in S\cap\mathbb{S}^{d-1}$ such that

$$\lambda_\alpha(x)=a,\qquad\lambda_\alpha(y)=b$$

It remains to show that $(a,b)\subset\lambda_\alpha(S\cap\mathbb{S}^{d-1})$. We define the following map $h:[0,1]\to\mathbb{R}$ by

$$h(u)=\lambda_\alpha\left(\frac{ux+(1-u)y}{\|ux+(1-u)y\|}\right).$$

Due to the continuity of λ_α, the map h is also continuous. On the other hand, $h(0)=a,h(1)=b$, and by the mean value theorem we get that

$$[a,b]\subset h([0,1])\subset\lambda_\alpha(S\cap\mathbb{S}^{d-1}),$$

which completes the proof of this proposition. □

We are now in a position to state and prove the main result in this section.

Theorem 3.2. *The set Λ_α of all fractional Lyapunov exponents of solutions starting from the unit sphere is given as follows*

$$\Lambda_\alpha=[a,b]\cup\bigcup_{i=1}^{j}\{\lambda_i\},$$

where a,b is defined as in Proposition 3.1 and $\lambda_1,\dots,\lambda_j$ is given as in Theorem 2.5.

Proof. According to Theorem 2.5, we get that

$$\begin{aligned}\Lambda_\alpha \cap \mathbb{R}_{<0} &= \{\chi_\alpha(\Phi(\cdot,x)) : x \in S \cap \mathbb{S}^{d-1}\}\\ &= \lambda_\alpha(S \cap \mathbb{S}^{d-1}),\\ &= [a,b],\end{aligned}$$

where λ_α is defined as in Proposition 3.1(ii). Meanwhile, the fact that

$$\Lambda_\alpha \cap \mathbb{R}_{\geq 0} = \bigcup_{i=1}^{j} \{\lambda_i\}$$

follows from Theorem 2.5(ii) and the proof is complete. □

Acknowledgement

The work of the first, the second and the third author is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED).

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