

Perturbation Method for a Class of Singularly Perturbed Systems

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Abstract

In the present work, we study a class of nonautonomous singularly perturbed discrete systems formulated as initial value problems. We develop a convergent iterative method to obtain asymptotic solutions. This method generalizes the *perturbation method* that is valid for perturbed difference equations and improves the *singular perturbation method*.

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1 Introduction

Many physical problems are described by discrete dynamical models, as in chemistry or electrical engineering where many small components or parasitic parameters increase the dynamic order of their mathematical representation and complicate their stability analysis. A system in which the suppression of a small parameter involves the degeneration of its dimension is said to be singularly perturbed. Such a system generates error accumulation in computing numerical solutions and claims a large computation time because of its high dimension. The first characteristic of these systems is the lack of a single model characterizing all the possible mechanisms of order reduction. Several model structures can be considered in the study of singularly perturbed discrete systems, the *C*-model and the *R*-model where the small parameter is located, respectively, on a column and on a row of the system matrix, see [8–10, 12], and the *standard* model or *fast sampling* model that results from a discretization of a general continuous-time singularly perturbed system, see [1,3]. Many researchers studied these kinds of systems for

the autonomous case, they used the *singular perturbation* approach for stability investigations by reducing the order of the problems and separating the time scale. This formal technique is similar to the method for continuous-time systems, it consists of obtaining the approximate solution in terms of an *outer* solution and a *boundary layer correction* solution, see [2, 5–7, 9, 11]. The *perturbation method* used in [13] and [14] is straightforward and does not need *boundary layer correction* terms, moreover it is valid for nonautonomous (time-varying) systems and the asymptotic approximations converge to the exact solution. This paper is aimed at exploiting this approach in an effective new way for the analysis of nonautonomous *standard* singularly perturbed discrete system formulated as initial value problem. We consider the system

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} I + \varepsilon A_{11}(k) & \varepsilon A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad k = 0, \dots, N-1, \quad (1.1)$$

where $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^m$ are the state vectors at the k th discrete time, and the overall system is of dimension $n + m$; $A_{ij}(k)$, $i, j = 1, 2$, $k = 0, \dots, N-1$, are constant matrices with appropriate dimensions, and ε is a small real parameter. We suppose given the initial values

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad (1.2)$$

with α and β in \mathbb{R}^n and \mathbb{R}^m , respectively. We denote the problem (1.1)–(1.2) by $P(\varepsilon)$. The autonomous case of problem $P(\varepsilon)$, i.e., the matrix system does not depend on the discrete time k , was studied in [1] and [3] using the *singular perturbation method*.

The paper is organized as follows. In Section 2, we give the main result. We develop the *perturbation method* for problem $P(\varepsilon)$. We write its solution as a convergent series in the small parameter ε and we give an algorithm to find its coefficients. Section 3 is devoted to show how we extend our procedure to the case of a nonhomogeneous system said *shift-invariant singularly perturbed system* [4].

2 Main Result

In this section, we develop the *perturbation method* for problem $P(\varepsilon)$ to obtain asymptotic approximate solutions at all orders. This iterative method converges to the exact solution, it offers a time scale decomposition with decoupled state variables satisfying subsystems of reduced order. Moreover, it is straightforward and gives considerable reduction and simplicity in computation compare to the *singular perturbation method* because it does not need to compute *boundary layer correction* terms.

2.1 Formal Asymptotic Solution

The solution $(x_k(\varepsilon), y_k(\varepsilon))', k = 0, \dots, N$, of problem $P(\varepsilon)$ is assumed as a power series in the small parameter ε :

$$x_k = \sum_{j=0}^{\infty} \varepsilon^j x_k^{(j)}, \quad y_k = \sum_{j=0}^{\infty} \varepsilon^j y_k^{(j)}, \quad k = 0, \dots, N. \tag{2.1}$$

Using this expansion in (1.1)–(1.2), and equating the terms of order ε^0 , we find the equations

$$x_0^{(0)} = \alpha, \quad x_{k+1}^{(0)} = x_k^{(0)}, \quad k = 0, \dots, N - 1, \tag{2.2}$$

it means the state $x_k^{(0)}$ remains fixed at the initial value α for all $k = 0, \dots, N$, then equations (2.2) are equivalent to

$$x_k^{(0)} = \alpha, \quad k = 0, \dots, N. \tag{2.3}$$

The resulting equations for the state $y_k^{(0)}$ are

$$y_0^{(0)} = \beta, \tag{2.4}$$

and

$$y_{k+1}^{(0)} = A_{21}(k)x_k^{(0)} + A_{22}(k)y_k^{(0)}, \quad k = 0, \dots, N - 1. \tag{2.5}$$

Substituting (2.3) in (2.5) yields

$$y_{k+1}^{(0)} = A_{21}(k)\alpha + A_{22}(k)y_k^{(0)}, \quad k = 0, \dots, N - 1. \tag{2.6}$$

Equating the terms of order $\varepsilon^j, j \geq 1$, we have the equations

$$x_0^{(j)} = 0, \tag{2.7}$$

$$x_{k+1}^{(j)} = A_{11}(k)x_k^{(j-1)} + A_{12}(k)y_k^{(j-1)}, \quad k = 0, \dots, N - 1, \tag{2.8}$$

and

$$y_0^{(j)} = 0, \tag{2.9}$$

$$y_{k+1}^{(j)} = A_{21}(k)x_k^{(j)} + A_{22}(k)y_k^{(j)}, \quad k = 0, \dots, N - 1. \tag{2.10}$$

The system described by (2.3) and (2.4)–(2.6) is the *reduced problem* of $P(\varepsilon)$, it results by suppressing the small parameter ε in (1.1)–(1.2). The equations (2.4)–(2.6) define an initial value problem for the state $y_k^{(0)}$ regardless of the state $x_k^{(0)}$ which remains fixed to its initial value. It is seen that the *reduced problem* offers the advantage of order reduction, time scale separation and decoupled state variables.

For the j^{th} -order approximation, the equation (2.7) sets the value of $x_0^{(j)}$ and the algebraic equation (2.8) defines the coefficients $x_k^{(j)}, k = 1, \dots, N$, where the values $x_k^{(j-1)}$ and $y_k^{(j-1)}, k = 0, \dots, N - 1$ are determined from the previous step, i.e., the approximation of order $j - 1$. The initial value problem (2.9)–(2.10) determines the sequence $y_k^{(j)}, k = 0, \dots, N$ where the terms $x_k^{(j)}, k = 1, \dots, N$, are previously computed from (2.8).

2.2 Convergence of the Asymptotic Solution

In this section, we present the main result of this paper, we prove the convergence of the series (2.1). The following theorem includes this result. Suppose

$$v = (x_0, y_0, x_1, y_1, \dots, x_N, y_N)'. \tag{2.11}$$

We consider the norm in $\mathbb{R}^{(n+m)(N+1)}$

$$\|v\| = \max(|x_0|, |y_0|, |x_1|, |y_1|, \dots, |x_N|, |y_N|),$$

and for a matrix $A = (a_{ij})$, the associated matrix norm

$$\|A\| = \sup_{\|v\|=1} \|Av\| = \max_{k=0, \dots, (n+m)(N+1)} \left(\sum_{j=0}^{(n+m)(N+1)} |a_{kj}| \right).$$

Theorem 2.1. *There exists a positive real number ε_0 , for all ε such that $|\varepsilon| < \varepsilon_0$, the solution $(x_k(\varepsilon), y_k(\varepsilon))'$, $k = 0, \dots, N$, of problem $P(\varepsilon)$ satisfies (2.1) uniformly for $0 \leq k \leq N$, where $x_k^{(0)}, y_k^{(0)}, x_k^{(j)}, y_k^{(j)}, k = 0, \dots, N$, are the solutions of (2.3), (2.4)–(2.6), (2.7)–(2.8) and (2.9)–(2.10), respectively. More precisely, for all $k = 0, \dots, N$, we have*

$$\begin{cases} \left| x_k(\varepsilon) - \sum_{j=0}^n \varepsilon^j x_k^{(j)} \right| \leq C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \\ \left| y_k(\varepsilon) - \sum_{j=0}^n \varepsilon^j y_k^{(j)} \right| \leq C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \end{cases} \tag{2.12}$$

where C is a positive constant.

Proof. We write the system (2.2)–(2.4)–(2.5) in the matrix form

$$A_0 v^{(0)} = f, \tag{2.13}$$

where $v^{(0)}$ and f are defined in $\mathbb{R}^{(n+m)(N+1)}$ by

$$v^{(0)} := (x_0^{(0)}, y_0^{(0)}, x_1^{(0)}, y_1^{(0)}, \dots, x_N^{(0)}, y_N^{(0)})', \tag{2.14}$$

$$f := (\alpha, \beta, 0, 0, \dots, 0, 0)', \tag{2.15}$$

and the matrix A_0 is the triangular matrix

$$\begin{pmatrix} I_n & 0 & 0 & 0 & \dots & & & 0 \\ 0 & I_m & 0 & 0 & & & & \\ I_n & 0 & -I_n & 0 & & & & \\ A_{21}(0) & A_{22}(0) & 0 & -I_m & & & & \\ \vdots & & & & \ddots & & & \vdots \\ & & & & & I_n & 0 & -I_n & 0 \\ 0 & & & \dots & A_{21}(N-1) & A_{22}(N-1) & 0 & -I_m \end{pmatrix}.$$

The matrix A_0 is nonsingular, we denote

$$\varepsilon_0 := \frac{1}{\|UA_0^{-1}\|}, \quad C := \|A_0^{-1}\| \|f\|. \tag{2.16}$$

We write the system (2.7)–(2.8)–(2.9)–(2.10) in the form

$$A_0 v^{(j)} = -U v^{(j-1)}; \quad v^{(j)} := \left(x_0^{(j)}, y_0^{(j)}, x_1^{(j)}, y_1^{(j)}, \dots, x_N^{(j)}, y_N^{(j)} \right)', \tag{2.17}$$

where U is the matrix

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & & & & & 0 \\ 0 & 0 & 0 & 0 & & & & & & \\ A_{11}(0) & 0 & A_{12}(0) & 0 & & & & & & \\ 0 & 0 & 0 & 0 & & & & & & \\ 0 & 0 & A_{11}(1) & A_{12}(1) & & & & & & \\ \vdots & & & & \ddots & & & & & \vdots \\ & & & & & A_{11}(N-1) & A_{12}(N-1) & 0 & 0 & \\ 0 & & & \dots & & 0 & 0 & 0 & 0 & \end{pmatrix}.$$

Hence the problem $P(\varepsilon)$ defined by (1.1)–(1.2) can be written in the matrix form

$$A_\varepsilon v = f, \tag{2.18}$$

where v and f are given by (2.11) and (2.15), respectively, and A_ε is defined by the combination

$$A_\varepsilon = A_0 + \varepsilon U.$$

Since $|\varepsilon| < \varepsilon_0$, from (2.16) we have $\|\varepsilon U A_0^{-1}\| < 1$, and we can write

$$A_0^{-1} \sum_{j=0}^{\infty} (-\varepsilon U A_0^{-1})^j = A_0^{-1} (I + \varepsilon U A_0^{-1})^{-1} = A_\varepsilon^{-1}. \tag{2.19}$$

The condition $|\varepsilon| < \varepsilon_0$ ensures the existence and uniqueness of the solution $v(\varepsilon)$ of system (2.18), it is given by

$$v(\varepsilon) = A_\varepsilon^{-1} f. \tag{2.20}$$

From (2.19) and (2.20), we have

$$v(\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j v^{(j)}; \quad v^{(j)} = A_0^{-1} (-U A_0^{-1})^j f. \tag{2.21}$$

From (2.13) and (2.17), for $j \geq 1$, we infer from (2.14) and (2.15) that the components $x_k^{(0)}, y_k^{(0)}, x_k^{(j)}$, and $y_k^{(j)}$ are the solutions of the problems (2.2), (2.4), (2.5), (2.7)–(2.8)

and (2.9)–(2.10), respectively. Notice that the equations (2.2) and (2.5) are equivalent to (2.3) and (2.6), respectively, which concludes the first part of the proof.

Now we evaluate the remainder of the series. We have

$$\begin{aligned} & \left\| A_\varepsilon^{-1} - A_0^{-1} \sum_{j=0}^{\infty} (-\varepsilon U A_0^{-1})^{(j)} \right\| \leq \|A_0^{-1}\| \sum_{j=n+1}^{\infty} \|\varepsilon U A_0^{-1}\|^j \\ & = \frac{\|A_0^{-1}\| \|\varepsilon U A_0^{-1}\|^n}{1 - \|\varepsilon U A_0^{-1}\|} \leq \|A_0^{-1}\| \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}. \end{aligned} \quad (2.22)$$

From (2.19) and (2.22), it follows

$$\begin{aligned} \|v(\varepsilon) - \sum_{j=0}^{\infty} \varepsilon^j v^{(j)}\| & \leq \|A_\varepsilon^{-1} - A_0^{-1} \sum_{j=0}^n (-\varepsilon U A_0^{-1})^j\| \|f\| \\ & \leq \|A_0^{-1}\| \|f\| \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}. \end{aligned} \quad (2.23)$$

The chosen norm and (2.16) give us the result (2.12). This concludes the proof. \square

2.3 Algorithm

An algorithm is given to indicate the sequence of various steps involved in the actual working of the *perturbation method*.

Zeroth-order solution

- Step 1. Fix $x_k^{(0)} = \alpha$, for all $k = 0, \dots, N$. Set $y_0^{(0)} = \beta$, and solve (2.6) to obtain $y_k^{(0)}$ for $k = 1, \dots, N$, hence determine $(x_k^{(0)}, y_k^{(0)})'$, for $k = 0, \dots, N$.

First-order solution

- Step 2. Fix $x_0^{(1)} = 0$, then determine $x_k^{(1)}$ from (2.8) for $k = 1, \dots, N$. Set $y_0^{(1)} = 0$, and solve (2.10) to obtain $y_k^{(1)}$ for $k = 1, \dots, N$, hence determine $(x_k^{(0)}, y_k^{(0)})' + \varepsilon (x_k^{(1)}, y_k^{(1)})'$, for $k = 0, \dots, N$.

Jth-order solution

- Step 3. Fix $x_0^{(j)} = 0$, then determine $x_k^{(j)}$ from (2.8) all $k = 1, \dots, N$. Set $y_0^{(j)} = 0$, and solve (2.10) to obtain $y_k^{(j)}$ for $k = 1, \dots, N$, hence determine $(x_k^{(0)}, y_k^{(0)})' + \varepsilon (x_k^{(1)}, y_k^{(1)})' + \dots + \varepsilon^j (x_k^{(j)}, y_k^{(j)})'$, for $k = 0, \dots, N$.

3 Shift-Invariant Singularly Perturbed System

Consider the linear shift-invariant singularly perturbed discrete system proposed in [4]

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} I + \varepsilon A_{11} & \varepsilon A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} \varepsilon B_1 \\ B_2 \end{pmatrix} u_k, \tag{3.1}$$

where $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^m$ are the state vectors, $u_k \in \mathbb{R}^r$ is the control vector or the input, $k = 0, \dots, N - 1$; A_{ij} , B_i , $i, j = 1, 2$, are constant matrices with appropriate dimensions, and ε is a small real parameter. We suppose given the initial values

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3.2}$$

The system (3.1) is known as the *fast sampling model* and results from the discretization of the singularly perturbed continuous-time system. In paper [4], a zeroth order approximation of the solution is expressed using a *boundary layer method* or the singular perturbation method. In this section, we give an algorithm to determine the asymptotic solutions of the initial value problem (3.1)–(3.2). We show how to compute the coefficients of the power series

$$x_k = \sum_{j=0}^{\infty} \varepsilon^j x_k^{(j)}, \quad y_k = \sum_{j=0}^{\infty} \varepsilon^j y_k^{(j)}, \quad k = 0, \dots, N. \tag{3.3}$$

Suppose the coefficients of ε^0 in the series (3.3) satisfy the following problems. The algebraic equation

$$x_k^{(0)} = \alpha, \quad k = 0, \dots, N, \tag{3.4}$$

and the initial value problem

$$y_0^{(0)} = \beta, \tag{3.5}$$

$$y_{k+1}^{(0)} = A_{21}\alpha + A_{22}y_k^{(0)} + B_2u_k, \quad k = 0, \dots, N - 1. \tag{3.6}$$

For the coefficients of ε , they satisfy the initial value problems

$$x_0^{(1)} = 0, \tag{3.7}$$

$$x_{k+1}^{(1)} = x_k^{(1)} + A_{11}\alpha + A_{12}y_k^{(0)} + B_1u_k, \quad k = 0, \dots, N - 1, \tag{3.8}$$

and

$$y_0^{(1)} = 0,$$

$$y_{k+1}^{(1)} = A_{21}x_k^{(1)} + A_{22}y_k^{(1)}, \quad k = 0, \dots, N - 1. \tag{3.9}$$

The coefficients of ε^j , $j \geq 2$, satisfy the algebraic equations

$$x_0^{(j)} = 0, \tag{3.10}$$

$$x_{k+1}^{(j)} = x_k^{(j)} + A_{11}x_k^{(j-1)} + A_{12}y_k^{(j-1)}, \quad k = 0, \dots, N - 1, \tag{3.11}$$

and the initial value problem

$$y_0^{(j)} = 0, \tag{3.12}$$

$$y_{k+1}^{(j)} = A_{21}x_k^{(j)} + A_{22}y_k^{(j)}, \quad k = 0, \dots, N - 1. \tag{3.13}$$

It is easy to prove the following result.

Theorem 3.1. *There exists a positive real number ε_0 , for all ε such that $|\varepsilon| < \varepsilon_0$, the solution $(x_k(\varepsilon), y_k(\varepsilon))', k = 0, \dots, N$ of problem (3.1)–(3.2) satisfies (3.3) uniformly for $0 \leq k \leq N$, where $x_k^{(0)}, y_k^{(0)}, x_k^{(j)}, y_k^{(j)}$ are the solutions of (3.10), (3.11), (3.5), (3.6), (3.12) and (3.13), respectively. More precisely, for all $k = 0, \dots, N$, we have*

$$\begin{aligned} \left| x_k(\varepsilon) - \sum_{j=0}^n \varepsilon^j x_k^{(j)} \right| &\leq C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}, \\ \left| y_k(\varepsilon) - \sum_{j=0}^n \varepsilon^j y_k^{(j)} \right| &\leq C \frac{(|\varepsilon|/\varepsilon_0)^{n+1}}{1 - |\varepsilon|/\varepsilon_0}. \end{aligned} \tag{3.14}$$

Proof. We only give a brief proof because it is similar to that of Theorem 2.1. We write the problem (3.1)–(3.2) in the matrix form

$$A_\varepsilon v = f(\varepsilon),$$

where

$$f(\varepsilon) = (\alpha, \beta, -\varepsilon B_1 u_0, -\varepsilon B_2 u_0, \dots, -\varepsilon B_1 u_{N-1}, -\varepsilon B_2 u_{N-1})', \tag{3.15}$$

and the value of A_ε is easy to find. We write $A_\varepsilon = A_0 + \varepsilon U$, and

$$f(\varepsilon) = f(0) + \varepsilon \bar{f} \tag{3.16}$$

where

$$\begin{aligned} f(0) &= (\alpha, \beta, 0, 0, \dots, 0, 0)', \\ \bar{f} &= (0, 0, -B_1 u_0, -B_2 u_0, \dots, -B_1 u_{N-1}, -B_2 u_{N-1})'. \end{aligned} \tag{3.17}$$

The remainder of the proof is routine and left to the reader. □

3.1 Algorithm

Zeroth-order approximation

- Step 1. Fix $x_k^{(0)} = \alpha$, for all $k = 0, \dots, N$. Set $y_0^{(0)} = \beta$, and solve (3.6) to obtain $y_k^{(0)}$ for $k = 1, \dots, N$, hence determine $(x_k^{(0)}, y_k^{(0)})'$, for $k = 0, \dots, N$.

First-order approximation

- Step 2. Fix $x_0^{(1)} = 0$, then determine $x_k^{(1)}$ from (3.8) for $k = 1, \dots, N$. Set $y_0^{(1)} = 0$, and solve (3.9) to obtain $y_k^{(1)}$ for $k = 1, \dots, N$, hence determine $(x_k^{(0)}, y_k^{(0)})' + \varepsilon (x_k^{(1)}, y_k^{(1)})'$, for $k = 0, \dots, N$.

Jth-order approximation

- Step 3. Fix $x_0^{(j)} = 0$, then determine $x_k^{(j)}$ from (3.11) for $k = 1, \dots, N$. Set $y_0^{(j)} = 0$, and solve (3.13) to obtain $y_k^{(j)}$ for $k = 1, \dots, N$, hence determine $(x_k^{(0)}, y_k^{(0)})' + \varepsilon (x_k^{(1)}, y_k^{(1)})' + \dots + \varepsilon^j (x_k^{(j)}, y_k^{(j)})'$, for $k = 0, \dots, N$.

4 Conclusion

A method has been developed for a class of nonautonomous (homogeneous) discrete singularly perturbed systems called *standard models*. Besides the advantage of removing the time scale and the decomposition in subsystems of reduced order with decoupled state variables, there is no need to compute the *boundary layer correction* terms as for the *singular perturbation method*. We extended the proposed method for a nonhomogeneous system known in the theory of control as the linear shift-invariant singularly perturbed system. For both problems, convergent algorithms have been provided showing the sequence of the various steps of the actual application of the method. Notice that the *perturbation method* can be extended to all important classes of linear singularly perturbed problems resulting from optimal control. This will be indicated separately.

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