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Elliptic Problems for Pseudo Differential Equations in a Polyhedral Cone

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Abstract

The author develops the theory of pseudo differential equations and boundary value problems in nonsmooth domains. A model pseudo differential equation in a special cone is reduced to a certain integral equation.

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1 Introduction

We consider a model elliptic pseudo differential equation in a cone because it is very important to obtain invertibility conditions for such equation according to freezing coefficients principle [1]. It was shown from the mathematical point of view in the papers by M. I. Vishik and G. I. Eskin, and it was established that for the invertibility one needs some additional conditions (for example, boundary conditions), but it is not enough, and these conditions must correlate with the initial equation (so-called Shapiro–Lopatinskii conditions). A manifold with nonsmooth boundary can have different kinds of singularities, and the basic conclusion of this paper is the following: each type of singularity generates a different type of general solution. Thus, we need different types of boundary conditions for different singularities and consequently we will obtain different solvability conditions.

In this paper we consider a special polyhedral cone.

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2 Key Concept

Let us study solvability of pseudo differential equations [3–5]

$$(Au_+)(x) = f(x),\ x \in C_+^a, \tag{2.1}$$

in the space $H^s(C_+^a)$, where C_+^a is the m-dimensional cone

$$\left\{x \in \mathbb{R}^m : x = (x_1, \dots, x_{m-1}, x_m), x_m > \sum_{k=1}^{m-1} a_k|x_k|,\ a_k > 0,\ k = 1, 2, \dots, m-1\right\},$$

A is a pseudo differential operator with the symbol $A(\xi)$, satisfying the condition

$$c_1 \le |A(\xi)(1 + |\xi|)^{-\alpha}| \le c_2. \tag{2.2}$$

Such symbols are elliptic and have the order $\alpha \in \mathbb{R}$ at infinity.

By definition, the space $H^s(C_+^a)$ consists of distributions from $H^s(\mathbb{R}^m)$ [1] whose support belongs to $\overline{C_+^a}$. The norm in the space $H^s(C_+^a)$ is induced by the norm of $H^s(\mathbb{R}^m)$. The right-hand side f is chosen from the space $H_0^{s-\alpha}(C_+^a)$, which is a space of distributions $S'(C_+^a)$, admitting the continuation on $H^{s-\alpha}(\mathbb{R}^m)$. The norm in the space $H_0^{s-\alpha}(C_+^a)$ is defined by

$$||f||_{s-\alpha}^+ = \inf ||lf||_{s-\alpha},$$

where the *infimum* is chosen from all continuations l.

Further, let us define a special multi-dimensional singular integral by the formula

$$(G_m u)(x) = (2i)^{m-1} \lim_{\tau \to 0+} \int_{\mathbb{R}^m} \prod_{j=1}^{m-1} \frac{a_j(x_m - y_m + i\tau)^{m-2}}{(x_j - y_j)^2 - a_j^2(x_m + y_m + i\tau)^2} u(y)dy$$

(we omit some details, see, for example, [3]). Let us recall that this operator is the multi-dimensional analogue of a one-dimensional Cauchy type integral, or Hilbert transform.

Before giving the next definition, we need some notations. The symbol $\overset{*}{C_+^a}$ denotes a conjugate cone for C_+^a:

$$\overset{*}{C_+^a} = \{x \in \mathbb{R}^m : x \cdot y > 0,\ y \in C_+^a\},$$

$C_-^a \equiv -C_+^a$, $T(C_+^a)$ denotes the radial tube domain over the cone C_+^a, i.e., a domain in the multidimensional complex space $\mathbf{C}^m$ of type $\mathbb{R}^m + iC_+^a$.

In order to describe the solvability picture for the equation (2.1), we will introduce the following definition.

Definition 2.1. A *wave factorization* for the symbol $A(\xi)$ is its representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where the factors $A_{\neq}(\xi), A_{=}(\xi)$ must satisfy the following conditions:

1) $A_{\neq}(\xi), A_{=}(\xi)$ are defined for all admissible values $\xi \in \mathbb{R}^m$, with the possible exception of the points $\{\xi \in \mathbb{R}^m : |\xi'|^2 = a^2\xi_m^2\}$;

2) $A_{\neq}(\xi), A_{=}(\xi)$ admit an analytical continuation to the radial tube domains $T(\overset{*}{C}{}^a_+)$, $T(\overset{*}{C}{}^a_-)$, respectively with estimates

$$|A_{\neq}^{\pm 1}(\xi + i\tau)| \le c_1(1 + |\xi| + |\tau|)^{\pm æ},$$
$$|A_{=}^{\pm 1}(\xi - i\tau)| \le c_2(1 + |\xi| + |\tau|)^{\pm(\alpha - æ)},\ \forall \tau \in \overset{*}{C}{}^a_+ .$$

The number $æ \in \mathbb{R}$ is called the *index of the wave factorization*.

Everywhere below we will suppose that the mentioned wave factorization exists, and the sign $\sim$ denotes the Fourier transform.

3 Main Result

Now we consider the equation (2.1) only for the case $æ - s = n + \delta, n \in \mathbf{N}, |\delta| < 1/2$.

Let T be the bijection operator transferring ∂C_+^a into the hyperplane $x_m = 0$, more precisely, it is a transformation $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ of the following type

$$\begin{cases} t_1 = x_1, \\ \dots\dots\dots\dots \\ t_{m-1} = x_{m-1}, \\ t_m = x_m - \sum\limits_{k=1}^{m-1} a_k|x_k|, \end{cases}$$

and we introduce the operator

$$FTF^{-1} \equiv V_{\mathbf{a}}, \tag{3.1}$$

where $\mathbf{a} = (a_1, a_2, \dots, a_{m-1})$, and further one can construct the general solution for our pseudo differential equation (2.1).

Theorem 3.1. *A general solution of the equation* (2.1) *in Fourier image is given by the formula*

$$\begin{aligned} \tilde{u}_+(\xi) &= A_{\neq}^{-1}(\xi)Q(\xi)G_m Q^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{lf}(\xi) \\ &\quad + A_{\neq}^{-1}(\xi)V_{-\mathbf{a}}F\left(\sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m)\right), \end{aligned}$$

where $c_k(x') \in H^{s_k}(\mathbb{R}^{m-1})$ are arbitrary functions, $s_k = s - æ + k - 1/2,\ k = 1, 2, \dots, n$, lf is an arbitrary continuation f on $H^{s-\alpha}(\mathbb{R}^m)$, $Q(\xi)$ is an arbitrary polynomial satisfying (2.2) *for $\alpha = n$.*

Proof. The general solution is constructed in the following way. Let us continue the distribution f on the whole space $\mathbb{R}^m$, denote this continuation by lf, further put

$$u_-(x) = (Au_+)(x) - lf(x),$$

and rewrite the last identity in the form

$$(Au_+)(x) + u_-(x) = lf(x).$$

After wave factorization for symbol with preliminary Fourier transform, we write

$$A_{\neq}(\xi)\tilde{u}_+(\xi) + A_{=}^{-1}(\xi)\tilde{u}_-(\xi) = A_{=}^{-1}(\xi)\tilde{lf}(\xi).$$

One can see that $A_{=}^{-1}(\xi)\tilde{lf}(\xi)$ belongs to the space $\tilde{H}^{s-æ}(\mathbb{R}^m)$, and if we choose the polynomial $Q(\xi)$ satisfying the condition

$$|Q(\xi)| \sim (1+|\xi|)^n,$$

then $Q^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{lf}(\xi)$ belongs to the space $\tilde{H}^{-\delta}(\mathbb{R}^m)$.

Further, according to the theory of the multi-dimensional Riemann problem [3], we can decompose the last function into two summands (jump problem):

$$Q^{-1}A_{=}^{-1}\tilde{lf} = f_+ + f_-,$$

where $f_+ \in \tilde{H}(C_+^a)$, $f_- \in \tilde{H}(\mathbb{R}^m \setminus C_+^a)$, and

$$f_+ = G_m Q^{-1}A_{=}^{-1}\tilde{lf} \tag{3.2}$$

So, we have

$$Q^{-1}A_{\neq}\tilde{u}_+ + Q^{-1}A_{=}^{-1}\tilde{u}_- = f_+ + f_-,$$

or

$$Q^{-1}A_{\neq}\tilde{u}_+ - f_+ = f_- - Q^{-1}A_{=}^{-1}\tilde{u}_-.$$

In other words,

$$A_{\neq}\tilde{u}_+ - Qf_+ = Qf_- - A_{=}^{-1}\tilde{u}_-.$$

The left-hand side of the equality belongs to the space $\tilde{H}^{-n-\delta}(C_+^a)$, and the right-hand side belongs to $\tilde{H}^{-n-\delta}(\mathbb{R}^m \setminus C_+^a)$. Hence

$$F^{-1}(A_{\neq}\tilde{u}_+ - Qf_+) = F^{-1}(Qf_- - A_{=}^{-1}\tilde{u}_-),$$

where the left-hand side belongs to $H^{-n-\delta}(C_+^a)$, and the right-hand side belongs to $H^{-n-\delta}(\mathbb{R}^m \setminus C_+^a)$, from which we conclude immediately that it is a distribution supported on ∂C_+^a. Then the function

$$TF^{-1}(A_{\neq}\tilde{u}_+ - Qf_+)$$

is supported on the hyperplane $t_m = 0$ and belongs to $H^{-n-\delta}(\mathbb{R}^m)$. Such distribution belongs to the subspace generated by a Dirac mass-function and its derivatives [2], and it can be written as

$$\sum_{k=0}^{n-1} c_k(t')\delta^{(k)}(t_m).$$

Therefore

$$TF^{-1}(A_{\neq}\tilde{u}_+ - Qf_+) = \sum_{k=0}^{n-1} c_k(t')\delta^{(k)}(t_m).$$

Further we apply the Fourier transform

$$FTF^{-1}(A_{\neq}\tilde{u}_+ - Qf_+) = F\left(\sum_{k=0}^{n-1} c_k(t')\delta^{(k)}(t_m)\right),$$

taking into account (3.1), and obtain

$$A_{\neq}(\xi)\tilde{u}_+(\xi) - Q(\xi)f_+(\xi) = V_{-\mathbf{a}}F\left(\sum_{k=0}^{n-1} c_k(t')\delta^{(k)}(t_m)\right),$$

from which, according to (3.2), we have

$$\begin{aligned}\tilde{u}_+(\xi) &= A_{\neq}^{-1}(\xi)Q(\xi)G_mQ^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{lf}(\xi)\\ &\quad + A_{\neq}^{-1}(\xi)V_{-\mathbf{a}}F\left(\sum_{k=1}^{n} c_k(x')\delta^{(k-1)}(x_m)\right).\end{aligned}$$

This completes the proof. □

In order to explain the formula (3.1), we write

$$\begin{aligned}(FTu)(\xi) &= \int_{\mathbb{R}^m} e^{-ix\cdot\xi}u(x_1,\ldots,x_{m-1},x_m - a|x'|)dx\\ &= \int_{\mathbb{R}^m} e^{-iy'\xi'}e^{-i(y_m+a|y'|)\xi_m}u(y_1,\ldots,y_{m-1},y_m)dy\\ &= \int_{\mathbb{R}^{m-1}} e^{-ia|y'|\xi_m}e^{-iy'\xi'}\hat{u}(y_1,\ldots,y_{m-1},\xi_m)dy',\end{aligned}$$

where $\hat{u}$ denotes the Fourier transform on the last variable, and the Jacobian is

$$\frac{D(x_1,x_2,\ldots,x_m)}{D(y_1,y_2,\ldots,y_m)} = \begin{vmatrix} 1 & 0 & \cdots 0 & 0\\ 0 & 1 & \cdots 0 & 0\\ 0 & 0 & \cdots 1 & 0\\ & \ldots\ldots\ldots\ldots\ldots & & \\ -a\mathrm{sgn}(y_1) & -a\mathrm{sgn}(y_2) & \ldots\ldots - a\mathrm{sgn}(y_{m-1}) & 1\end{vmatrix} = 1.$$

If we define a pseudo differential operator by the formula

$$(Au)(x) = \int_{\mathbb{R}^m} e^{ix\xi} A(\xi)\tilde{u}(\xi)d\xi$$

and the direct Fourier transform by

$$\tilde{u}(\xi) = \int_{\mathbb{R}^m} e^{-ix\xi} u(x)dx,$$

then we have, at least formally,

$$(FTu)(\xi) = \int_{\mathbb{R}^{m-1}} e^{-i(a_1|y_1|+\ldots+a_{m-1}|y_{m-1}|)\xi_m} e^{-iy'\xi'} \hat{u}(y_1,\ldots,y_{m-1},\xi_m)dy'. \tag{3.3}$$

In other words, if we denote the $(m-1)$-dimensional Fourier transform ($y' \to \xi'$ in distribution sense) of the function $e^{-i(a_1|y_1|+\ldots+a_{m-1}|y_{m-1}|)\xi_m}$ by $E_a(\xi',\xi_m)$, then the formula (3.3) reads

$$(FTu)(\xi) = (E_a * \tilde{u})(\xi),$$

where the sign $*$ denotes a convolution for the first $m-1$ variables, and the multiplier for the last variable ξ_m. Thus, $V_{\mathbf{a}}$ is a combination of a convolution operator and the multiplier with the kernel $E_a(\xi',\xi_m)$. It is a very good operator, and it is bounded in Sobolev–Slobodetskii spaces $H^s(\mathbb{R}^m)$.

Notice that distributions supported on conical surface and their Fourier transforms were considered in [2], but the author did not find the multi-dimensional analogue of the theorem on a distribution supported in a single point in all issues of this book.

4 Simple Boundary Value Problem

Here we consider the very simple case when $f \equiv 0,\ a_1 = \ldots = a_{m-1} = 1,\ n = 1$. Then the formula from Theorem 3.1 takes the form

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi').$$

We consider separately the following construction. According to the Fourier transform, our solution is

$$u_+(x) = F^{-1}\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\}.$$

(Here we write V_1 in the case $a_1 = \ldots = a_{m-1} = 1$).

Let us suppose we choose the Dirichlet boundary condition on ∂C_+^1 for unique identification of an unknown function c_0, i.e.,

$$(Pu)(y) = g(y),$$

where g is a given function on ∂C_+^1, P is a restriction operator on a boundary, so we know the solution on the boundary ∂C_+^1. Thus,

$$Tu(x) = TF^{-1}\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\},$$

so we have

$$FTu(x) = FTF^{-1}\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\} = V_1\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\}, \tag{4.1}$$

and we know $(P'Tu)(x') \equiv v(x')$, where P' is the restriction operator on the hyperplane $x_m = 0$. The relation between the operators P' and F is well known [1]:

$$(FP'u)(\xi') = \int_{-\infty}^{+\infty} \tilde{u}(\xi', \xi_m) d\xi_m.$$

Returning to the formula (4.1), we obtain

$$\tilde{v}(\xi') = \int_{-\infty}^{+\infty} \{V_1\{A_{\neq}^{-1}(\xi)V_{-1}\tilde{c}_0(\xi')\}\}(\xi', \xi_m) d\xi_m, \tag{4.2}$$

where $\tilde{v}(\xi')$ is a given function. Hence, the equation (4.2) is an integral equation for determining $c_0(x')$.

One can consider other types of boundary conditions by the same way, but one needs to take into account that the analogue of the equation (4.2) will not be as simple.

5 Integral Equation

Let us consider the particular case $f \equiv 0, n = 1$. The formula for a general solution of the equation (2.1) takes the form

$$\tilde{u}_+(\xi)) = A_{\neq}^{-1}(\xi)V_{-1}F\{c_0(x')\delta^{(0)}(x_m)\},$$

and further after Fourier transform (for simplicity we write $\tilde{c}$ instead of $V_{-1}\tilde{c}_0$)

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)\tilde{c}(\xi'), \tag{5.1}$$

or, equivalently, the solution is

$$u_+(x) = F^{-1}\{A_{\neq}^{-1}(\xi)\tilde{c}(\xi')\}.$$

Then we apply the operator T to formula (5.1)

$$(Tu_+)(t) = T_aF^{-1}\{A_{\neq}^{-1}(\xi)\tilde{c}(\xi')\}$$

and the Fourier transform

$$(FTu_+)(\xi) = FTF^{-1}\{A_{\neq}^{-1}(\xi)\tilde{c}(\xi')\}.$$

If the boundary values of our solution u_+ are known on ∂C_+^a, then it means that the following function is given:

$$\int\limits_{-\infty}^{+\infty} (FTu_+)(\xi)d\xi_m.$$

So, if we denote

$$\int\limits_{-\infty}^{+\infty} (FTu_+)(\xi)d\xi_m \equiv \tilde{g}(\xi'),$$

then for determining $\tilde{c}(\xi')$, we have

$$\int\limits_{-\infty}^{+\infty} (FTF^{-1})\{A_{\neq}^{-1}(\xi)\tilde{c}(\xi')\}d\xi_m = \tilde{g}(\xi'), \tag{5.2}$$

This is an equation like a convolution, and evaluating the inverse Fourier transform $\xi' \to x'$, we obtain the conical analogue of layer potential.

Now we will try to determine the form of the operator FT_aF^{-1} (see Section 3). We write

$$(FTF^{-1}\tilde{u})(\xi) = (FTu)(\xi) = \int\limits_{\mathbb{R}^{m-1}} e^{-i(a_1|y_1|+\ldots+a_{m-1}|y_{m-1}|)\xi_m} e^{-iy'\cdot\xi'}\hat{u}(y',\xi_m)dy', \tag{5.3}$$

where $y' = (y_1, \ldots, y_{m-1})$, $\hat{u}$ is the Fourier transform of u on the last variable y_m.

Let us denote the convolution operator with symbol $A_{\neq}^{-1}(\xi)$ by letter a, so that by definition

$$(a * u)(x) = \int\limits_{\mathbb{R}^m} a(x-y)u(y)dy,$$

or, for Fourier images,

$$F(a * u)(\xi) = A_{\neq}^{-1}(\xi)\tilde{u}(\xi).$$

As above, we denote by $\hat{a}(x', \xi_m)$ the Fourier transform of the convolution kernel $a(x)$ on the last variable x_m. The integral in (5.2) takes the form (according to (5.3))

$$\int\limits_{\mathbb{R}^{m-1}} e^{-i(a_1|y_1|+\ldots+a_{m-1}|y_{m-1}|\xi_m} e^{-iy'\cdot\xi'}(\hat{a} * c)(y',\xi_m)dy'.$$

Taking into account the properties of convolution operator and the Fourier transform, we have the representation (see Section 3)

$$E_a * (A_{\neq}^{-1}(\xi)\tilde{c}(\xi')),$$

or, in enlarged notation,

$$\int\limits_{\mathbb{R}^{m-1}} E_a(\xi' - \eta', \xi_m) A_{\neq}^{-1}(\eta', \xi_m) \tilde{c}(\eta') d\eta'.$$

Then the equation (5.2) will take the form

$$\int\limits_{\mathbb{R}^{m-1}} W_a(\eta', \xi' - \eta') \tilde{c}(\eta') d\eta' = \tilde{g}(\xi'), \tag{5.4}$$

where $W_a(\eta', \xi' - \eta') = \int\limits_{-\infty}^{+\infty} \frac{E_a(\xi' - \eta', \xi_m) d\xi_m}{A_{\neq}(\eta', \xi_m)}$. The equation (5.4) is an integral equation for determining of the unknown function c. If we solve this integral equation, then we can find the solution of our boundary value problem by the formula (5.1).

Thus, we obtain the following result.

Theorem 5.1. *The boundary value problem consisting of the equation* (2.1) *and the Dirichlet condition is equivalent to the integral equation* (5.4).

5.1 Possible Simplification

In case when the kernel of the integral equation is degenerated, one can obtain some simplifications. Let

$$W_a(\eta', \xi' - \eta') \equiv b(\eta') K(\xi' - \eta').$$

Further, we apply the inverse Fourier transform to (5.4) and obtain

$$k(x') \int\limits_{\mathbb{R}^{m-1}} B(x' - y') c(y') dy' = g(x'), \tag{5.5}$$

and if $k(x') \neq 0, \ \forall x' \in \mathbb{R}^{m-1}$, then

$$\int\limits_{\mathbb{R}^{m-1}} B(x' - y') c(y') dy' = k^{-1}(x') g(x'),$$

where $B(x')$ is the Fourier transform for $b(\eta')$, and $k(x')$ is the Fourier transform for $K(\eta')$. The equation (5.5) is easily solvable.

If the restriction of u_+ on ∂C_+^a is given as

$$u_{+|\partial C_+^a} = v(x', x_m),$$

then it means that the function

$$T_a(u_{+|\partial C_+^a}) = v_1(x')$$

is given.

The equation (5.4) and the formula (5.5) can be rewritten after change of variables as integral over ∂C^a_+ (because the integral is considered over $\mathbb{R}^{m-1}$). It seems, it will be the special analogue of classical double layer potential. More precisely, the Fourier transform in the formula (5.5) gives

$$b(\xi')\tilde{c}(\xi') = (K^{-1} * \tilde{g})(\xi'),$$

or

$$\tilde{c}(\xi') = b^{-1}(\xi')(K^{-1} * \tilde{g})(\xi'), \tag{5.6}$$

if $b(\xi') \neq 0,\ \forall \xi' \in \mathbb{R}^{m-1}$. Substituting (5.6) into (5.1), we obtain

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)b^{-1}(\xi')(K^{-1} * \tilde{g})(\xi'),$$

Further, if we denote $A_{\neq}^{-1}(\xi)b^{-1}(\xi') \equiv d(\xi)$, and $D(x)$ the kernel of convolution operator with the symbol $d(\xi)$, then the last formula can be rewritten as

$$\tilde{u}_+(\xi) = d(\xi)(K^{-1} * \tilde{g})(\xi'),$$

and finally,

$$u_+(x', x_m) = \int\limits_{\mathbb{R}^{m-1}} D(x' - y', x_m)k^{-1}(y')g(y')dy'.$$

This is a certain analogue of the double layer potential, more precisely, Poisson integral for the half-space.

6 Conclusion

Earlier the author considered a plane case, and for general homogeneous boundary conditions obtained an equivalent system of linear difference equations [6]. This system is very complicated, and even for simplest boundary conditions it is very hard to solve [7]. It may be the case that the integral equations approach will be more convenient for solving these boundary value problems, at least by numerical methods.

References

[1] G. Eskin, Boundary value problems for elliptic pseudo differential equations. AMS, Providence, 1981.

[2] I. M. Gel'fand and G. E. Shilov, Distributions and operations with them. Fizmatgiz, Moscow, 1959. (in Russian)

[3] V. B. Vasil'ev, Wave factorization of elliptic symbols: Theory and applications. Introduction to the theory of boundary value problems in non-smooth domains. Kluwer Academic publishers. Dordrecht-Boston-London. 2000.

[4] V. B. Vasilyev, Multipliers of Fourier integrals, pseudo differential equations, wave factorization, boundary value problems. Editorial URSS, Moscow, 2nd edition, 2010. (in Russian)

[5] V. B. Vasilyev, Elliptic equations and boundary value problems in non-smooth domains. Pseudo Differential Operators: Analysis, Applications and Computations. Eds. Rodino L., Wong M.W., Zhu H. Operator Theory: Advances and Applications, 2011, V.213. Birkhäuser, Basel. P. 105–121.

[6] V. B. Vasilyev, General boundary value problems for pseudo differential equations and related difference equations. Advances in Difference Equations 2013, 2013:289. 7 pp.

[7] V. B. Vasilyev, On some difference equations of first order. Tatra Mt. Math. Publ. 2013. V. 54. P. 165–181.