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Fixed Points and Exponential Stability for Stochastic Partial Integro-Differential Equations with Delays

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Abstract

In this paper, we study the existence and asymptotic stability in the pth-moment of mild solutions of nonlinear impulsive stochastic partial functional integro-differential equations with delays. We suppose that the linear part possesses a resolvent operator in the sense given by Grimmer in [9], and the nonlinear terms are assumed to be Lipschitz continuous. A fixed point approach is used to achieve the required result. An example is provided to illustrate the abstract results in this work.

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1 Introduction

In recent years, existence, uniqueness, stability, and other quantitative and qualitative properties of solutions to stochastic partial differential equations have been extensively investigated by several authors. Many important results have been reported, for instance, in [5, 8]. In particular, Caraballo extended in [3] the results from Haussmann [11]; Mao proved in [18] the mean-square exponential stability for the strong solutions of linear stochastic delay equations with finite constant delay, by using the method developed in [11, 12]. Following the ideas of Haussmann [11] and Ichikawa [12], Caraballo and Real [5] considered the stability of the strong solutions of semilinear stochastic delay evolution equations.

In the case of delay differential equations, in particular when we are concerned with mild solutions of stochastic partial differential equations, the Lyapunov second method, despite being a powerful technique in proving stability theorems, is not as appropriate as in the non-delay case. A difficulty is that mild solutions do not have stochastic differentials, so that one cannot apply Ito's formula in a straightforward way. Following Ichikawa [12], Liu [16] solved this problem by introducing approximating systems and then using a limiting argument. Caraballo and Liu [4] have also solved the problem by using the properties of the stochastic convolution integral, a method employed in Yor [21] and Khas'minskii [14] to study the exponential stability of the mild solutions of semilinear stochastic evolution equations. Very recently, Burton has successfully utilized in [2] the fixed-point theory to investigate the stability of deterministic systems; Luo in [17] and Appleby in [1] have applied this valuable method for the stability of stochastic differential equations. Following the ideas of Burton [2], Luo [17] and Appleby [1], by employing the contraction mapping principle and stochastic analysis, some sufficient conditions ensuring the exponential stability in pth-moment $(p \geq 2)$ and almost sure exponential stability for mild solution of stochastic partial differential equations with delays were obtained in [17], which did not comprise the monotone decreasing behavior of the delays.

Motivated by the facts stated in the above discussion, in this paper we aim to study the stability problem for a class of stochastic partial integro-differential equations with delays. We prove that the mild solution to a class of stochastic partial integro-differential equations with delays exists, is unique and also pth-moment exponentially stable, by using a fixed point argument. Due to the presence of the integro-differential term in our equation, we need to use the theory of resolvent operators as developed by Grimmer [9] instead of using strongly continuous semigroups. The advantage of using this method is that one can prove at the same time not only the existence and uniqueness of solution of the problem, but also the exponential stability in the pth moment. It is worth noticing that for this reason, the set of assumptions that we have to impose may seem more restrictive than the ones which might be sufficient to ensure the existence and uniqueness of solution, but as we are interested in the stability of solutions we prefer to state all the assumptions needed for that at one stage.

The paper is organized as follows. In Section 2, we summarize several important and helpful working tools on the Wiener process and deterministic integro-differential equations that will be used to develop our results. Section 3 is devoted to the existence and exponential stability of mild solutions. An example is provided in Section 4 to illustrate our main abstract results.

2 Preliminaries

2.1 Wiener Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\mathbb{H}, \mathbb{K}$ be two real separable Hilbert spaces and we denote by $\langle .,.\rangle_{\mathbb{H}}$, $\langle .,.\rangle_{\mathbb{K}}$ their inner products and by $\|.\|_{\mathbb{H}}$, $\|.\|_{\mathbb{K}}$ their vector norms, respectively. We denote by $\mathcal{L}(\mathbb{K}, \mathbb{H})$ the set of linear bounded operators from $\mathbb{K}$ into H, equipped we the usual operator norm $\|.\|$. In this paper, we will always use the same symbol $\|.\|$ to denote norms of operators regardless of the spaces potentially involved, when no confusion may arise. Let $\tau > 0$ and let $D := D\left([-\tau, 0]\,; \mathbb{H}\right)$ denote the family of all right-continuous functions with left-hand limits φ from $[-\tau, 0]$ to $\mathbb{H}$. The space $D\left([-\tau, 0]\,; \mathbb{H}\right)$ is equipped with the norm $\|\varphi\|_D = \sup\limits_{-\tau\leq\theta\leq 0} \|\varphi(\theta)\|_{\mathbb{H}}$. We also use the space $D^b_{\mathcal{F}_0}\big([-\tau, 0]; \mathbb{H}\big)$ to denote the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $D\left([-\tau, 0]\,; \mathbb{H}\right)$-valued random variables.

Let $\{W(t), t \geq 0\}$ be a $\mathbb{K}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Wiener process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance operator Q, i.e., $\mathbb{E}\langle W(t), x\rangle_{\mathbb{K}}\langle W(t), y\rangle_{\mathbb{K}} = (t \wedge s)\langle Qx, y\rangle_{\mathbb{K}}$ for all $x, y \in \mathbb{K}$, where Q is a positive, self-adjoint, trace class operator on $\mathbb{K}$. In particular, we shall call such $W(t)$, $t \geq 0$, a $\mathbb{K}$-valued Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

In order to define stochastic integrals with respect to the Q-Wiener process $W(t)$, we introduce the subspace $K_0 = Q^{1/2}(\mathbb{K})$ of $\mathbb{K}$, which, endowed with the inner product

$$\langle u, v\rangle_{K_0} = \langle Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v\rangle_{\mathbb{K}}$$

is a Hilbert space. Let $\mathcal{L}_2^0 := \mathcal{L}_2(K_0, \mathbb{H})$ denote the space of all Hilbert-Schmidt operators from K_0 into $\mathbb{H}$ which turns out to be a separable space, equipped with the norm

$$\|\Psi\|^2_{\mathcal{L}_2^0} = tr((\Psi Q^{1/2})(\Psi Q^{1/2})^*)$$

for any $\Psi \in \mathcal{L}_2^0$. Clearly, for any bounded operator $\Psi \in \mathcal{L}(\mathbb{K}, \mathbb{H})$, this norm reduces to $\|\Psi\|^2_{\mathcal{L}_2^0} = tr(\Psi Q \Psi^*)$.

For arbitrary given $T \geq 0$, let $J(t, \omega)$, $t \in [0, T]$, be an $\mathcal{F}_t$-adapted, $\mathcal{L}_2^0$-valued process, and we define the following norm for arbitrary $t \in [0, T]$,

$$|J|_t = \Big\{\mathbb{E}\int_0^t \mathrm{tr}((J(s,\omega)Q^{\frac{1}{2}})(J(s,\omega)Q^{\frac{1}{2}})^*)ds\Big\}^{1/2}$$

In particular, we denote by $\mathcal{U}^2([0,T];\mathcal{L}_2^0)$ the set of all $\mathcal{L}_2^0$-valued predictable processes J satisfying $|J|_T < \infty$. The stochastic integral $\int_0^t J(s,\omega)dW(s) \in \mathbb{H},\ t \geq 0$, can be defined for all $J(s,\omega) \in \mathcal{U}^2([0,T];\mathcal{L}_2^0)$ by

$$\int_0^t J(s,\omega)dW(s) = L^2 - \lim_{n\to+\infty}\sum_{i=1}^{n}\int_0^t \sqrt{\lambda_i}J(s,\omega)e_i dB_s^i,\ t\in[0,T],$$

where $W(t) = \sum_{i=1}^{+\infty}\sqrt{\lambda_i}B_t^i e_i$. Here $\{\lambda_i \geq 0, i \in \mathbb{N}\}$ are the eigenvalues of Q and $\{e_i, i\in\mathbb{N}\}$ are the corresponding eigenvectors, $\left\{B_t^i, i\in\mathbb{N}\right\}$ are independent standard real-valued Brownian motions. The reader is referred to [10] for a systematic theory about stochastic integral of this type.

2.2 Partial Integro-Differential Equations in Banach Spaces

In the present section, we recall some definitions, notations and properties needed in the sequel. Let Z_1 and Z_2 denote two Banach spaces. We denote by $\mathcal{L}(Z_1, Z_2)$ the Banach space of bounded linear operators from Z_1 into Z_2 endowed with the operator norm and we abbreviate this notation to $\mathcal{L}(Z_1)$ when $Z_1 = Z_2$.

In what follows, $\mathbb{H}$ will denote a Banach space, A and $B(t)$ are closed linear operators on $\mathbb{H}$. Y represents the Banach space $D(A)$, the domain of operator A, equipped with the graph norm

$$|y|_Y := |Ay| + |y| \quad \text{for } \ y\in Y.$$

The notation $C([0,+\infty);Y)$ denotes the space of all continuous functions from $[0,+\infty)$ into Y. We then consider the following Cauchy problem

$$\begin{cases} v'(t) &= Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds \ \text{ for } \ t\geq 0, \\ v(0) &= v_0 \in \mathbb{H}. \end{cases} \tag{2.1}$$

Definition 2.1 (See [9]). A resolvent operator for Eq. (2.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{H})$ for $t \geq 0$, satisfying the following properties:

(i) $R(0) = I$ and $|R(t)| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For each $x \in \mathbb{H}$, $R(t)x$ is strongly continuous for $t \geq 0$.

(iii) For $x \in Y,\ R(\cdot)x \in C^1([0,+\infty);\mathbb{H}) \cap C([0,+\infty);Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \qquad \text{for } \ t\geq 0. \end{aligned}$$

For additional details on resolvent operators, we refer the reader to [9, 19]. The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants formula for nonlinear systems. For this reason, we need to know when the linear system (2.1) possesses a resolvent operator. Theorem 2.2 below provides a satisfactory answer to this problem.

In what follows we suppose the following assumptions:

(H_1) A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{H}$.

(H_2) For all $t \geq 0$, $B(t)$ is a continuous linear operator from $(Y, |\cdot|_Y)$ into $(\mathbb{H}, |\cdot|_{\mathbb{H}})$. Moreover, there exists an integrable function $c : [0, +\infty) \to \mathbb{R}^+$ such that for any $y \in Y$, $y \mapsto B(t)y$ belongs to $W^{1,1}([0, +\infty), \mathbb{H})$ and

$$\left|\frac{d}{dt}B(t)y\right|_{\mathbb{H}} \leq c(t)|y|_Y \quad \text{for} \quad y \in Y \quad \text{and} \quad t \geq 0.$$

Theorem 2.2 (See [7])**.** *Assume that hypotheses* (H_1) *and* (H_2) *hold. Then Eq.* (2.1) *admits a resolvent operator* $(R(t))_{t\geq 0}$.

Theorem 2.3 (See [15])**.** *Assume that hypotheses* (H_1) *and* (H_2) *hold. Let* $T(t)$ *be a compact operator for* $t > 0$*. Then, the corresponding resolvent operator* $R(t)$ *of Eq.* (2.1) *is continuous for* $t > 0$ *in the operator norm, namely, for all* $t_0 > 0$*, it holds that* $\lim_{h\to 0} \|R(t_0 + h) - R(t_0)\| = 0$.

In the sequel, we recall some results on the existence of solutions for the following integro-differential equation

$$\begin{cases} v'(t) &= Av(t) + \displaystyle\int_0^t B(t-s)v(s)ds + q(t) \quad \text{for} \quad t \geq 0, \\ v(0) &= v_0 \in \mathbb{H}, \end{cases} \tag{2.2}$$

where $q : [0, +\infty[\to \mathbb{H}$ is a continuous function.

Definition 2.4 (See [9])**.** A continuous function $v : [0, +\infty) \to \mathbb{H}$ is said to be a strict solution of Eq. (2.2) if

(i) $v \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); Y)$,

(ii) v satisfies Eq. (2.2) for $t \geq 0$.

Remark 2.5. From this definition we deduce that $v(t) \in D(A)$, and the function $B(t - s)v(s)$ is integrable, for all $t > 0$ and $s \in [0, +\infty)$.

Theorem 2.6 (See [9])**.** *Assume that* (H_1)–(H_2) *hold. If* v *is a strict solution of Eq.* (2.2)*, then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad \textit{for} \quad t \geq 0. \tag{2.3}$$

Accordingly, we can establish the following definition.

Definition 2.7 (See [9])**.** A function $v : [0, +\infty) \to \mathbb{H}$ is called a mild solution of (2.2), for $v_0 \in \mathbb{H}$, if v satisfies the variation of constants formula (2.3).

The next theorem provides sufficient conditions ensuring the regularity of solutions of Eq. (2.2).

Theorem 2.8 (See [9])**.** *Let* $q \in C^1([0, +\infty); \mathbb{H})$ *and* v *be defined by* (2.3). *If* $v_0 \in D(A)$*, then* v *is a strict solution of Eq.* (2.2).

Consider the following semilinear stochastic partial integro-differential equation with delays

$$\begin{cases} dx(t) = \left[Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + F(t, x(t-\rho(t)))\right] dt \\ \qquad\qquad +G(t, x(t-\delta(t)))dW(t) \ \text{ for } \ t \geq 0, \\ x(\theta) = \phi(\theta) \text{ for } \theta \in [-\tau, 0], \text{ where } \phi \in D^b_{\mathcal{F}_0}([-\tau, 0]; H), \quad \tau > 0. \end{cases} \tag{2.4}$$

The mappings $F : \mathbb{R}^+ \times D([-\tau, 0]; \mathbb{H}) \to \mathbb{H}$, $G : \mathbb{R}^+ \times D([-\tau, 0]; \mathbb{H}) \to \mathcal{L}_2^0(\mathbb{K}, \mathbb{H})$ are both Borel measurable, $\rho : \mathbb{R}_+ \to [0, \tau]$, $\delta : \mathbb{R}_+ \to [0, \tau]$ are continuous.

Definition 2.9. A stochastic process $\{x(t), t \in [0, T]\}$, ($0 < T < +\infty$), is called a mild solution of (2.4) if

(i) $x(t)$ is $\mathcal{F}_t$-adapted;

(ii) $x(t) \in \mathbb{H}$, possesses càdlàg paths on $t \in [0, T]$ almost surely, and for arbitrary $0 \leq t \leq T$,

$$\begin{aligned} x(t) &= R(t)\phi(0) + \int_0^t R(t-s)F(s, x(s-\rho(s)))ds \\ &\quad + \int_0^t R(t-s)G(s, x(s-\delta(s))dW(s) \end{aligned} \tag{2.5}$$

(iii) and $x(\theta) = \phi(\theta)$ for $\theta \in [-\tau, 0]$, where $\phi \in D^b_{\mathcal{F}_0}([-\tau, 0]; H)$.

For our stability interest, we can assume, without loss of generality, that

$$F(t, 0) \equiv 0 \text{ and } G(t, 0) \equiv 0 \text{ for any } t \geq 0. \tag{2.6}$$

Then (2.4) obviously possesses the trivial solution when the initial value is $\phi \equiv 0$.

Definition 2.10. Let $p \geq 2$ be an integer. The trivial solution of Eq. (2.4) is said to be exponentially stable in the pth mean, if for any initial value $\phi \in D^b_{\mathcal{F}_0}\big([-\tau, 0]; \mathbb{H}\big)$, there exists a pair of positive constants $\lambda, C > 0$ (λ independent of ϕ) such that

$$\mathbb{E}\|x(t)\|^p_{\mathbb{H}} \leq C \sup_{-\tau \leq s \leq 0} \mathbb{E}\|\phi(s)\|^p_{\mathbb{H}} e^{-\lambda t}, \ t \geq 0. \tag{2.7}$$

In order to set our problem, we always assume that the following conditions hold:

$$\begin{aligned} \|R(t)\|_{\mathbb{H}} &\leq Me^{-\gamma t} \ \forall t \geq 0, \text{ where } M \geq 1 \text{ and } \gamma > 0, & (2.8)\\ \|F(t,x) - F(t,y)\|_{\mathbb{H}} &\leq C_1\|x - y\|_{\mathbb{H}} \ \forall t \geq 0, x, y \in H, \text{ where } C_1 > 0, & (2.9)\\ \|G(t,x) - G(t,y)\|_{\mathcal{L}^0_2} &\leq C_2\|x - y\|_{\mathbb{H}} \ \forall t \geq 0, x, y \in H, \text{ where } C_2 > 0. & (2.10) \end{aligned}$$

3 Main Results

In this section, we will consider existence, uniqueness and exponential stability in the pth mean of mild solutions of (2.4) by means of the fixed-point theory. As we already emphasized at the Introduction, one advantage of using this method is that one can prove at the same time the existence, uniqueness and the exponential stability in the pth moment of the solutions. For this reason we impose the whole set of assumptions needed for the complete analysis. An alternative way could be to prove first the existence and uniqueness of solutions under less restrictive assumptions, and later on to add some additional ones ensuring the stability. But, as our primary interest is the stability analysis we prefer to do everything at the same time.

Now we can finally state and prove our main result.

Theorem 3.1. *Assume that conditions* (2.8)–(2.10) *hold and that, for* $p \geq 2$,

$$3^{p-1}M^p(C_1^p\gamma^{1-p} + C_2^p(p(p-1)/2)^{p/2}(2\gamma(p-1)/(p-2))^{1-p/2}) < \gamma. \tag{3.1}$$

Then, for any initial value $\phi \in D^b_{\mathcal{F}_0}\big([-\tau, 0], \mathbb{H}\big)$, *there exists a unique mild solution to Equation* (2.5) *defined for* $t \geq 0$, *and the trivial solution to this equation is exponentially stable in the* p*th mean.*

Proof. Fix an initial value $\phi \in D^b_{\mathcal{F}_0}\big([-\tau, 0], \mathbb{H}\big)$, and denote by $\mathcal{S}$ the space of all $\mathcal{F}_t$-adapted and càdlàg processes $x(t,\omega) : [-\tau, +\infty) \times \Omega \to \mathbb{H}$ satisfying $x(s,\omega) = \phi(s,\omega)$ for $s \in [-\tau, 0], \omega \in \Omega$, and $e^{\alpha t}\mathbb{E}\|x(t,\omega)\|^p_{\mathbb{H}} \to 0$ as $t \to +\infty$, where α is a positive constant such that $0 < \alpha < \gamma$ and

$$3^{p-1}M^p(C_1^p\gamma^{1-p} + C_2^p(p(p-1)/2)^{p/2}(2\gamma(p-1)/(p-2))^{1-p/2})/(e^{\alpha\tau}/(\gamma - \alpha)) < 1. \tag{3.2}$$

Observe that such an α exists thanks to condition (3.1).

The space $\mathcal{S}$ endowed with the norm $\|x\|^p_{\mathcal{S}} := \sup_{t \geq 0} \mathbb{E}\,\|x(t)\|^p_{\mathbb{H}}$ is a Banach space (see [6] for more details).

Define now a mapping π on $\mathcal{S}$ by

$$\pi(x)(t) = \begin{cases} \phi(t) \ \text{ for } \ t \in [-\tau, 0] \\ \\ R(t)\phi(0) + \displaystyle\int_0^t R(t-s)F(s, x(s-\rho(s)))ds \\ + \displaystyle\int_0^t R(t-s)G(s, x(s-\delta(s)))dW(s) \ \text{ for } \ t \geq 0, \\ \\ := \displaystyle\sum_{i=1}^{3} I_i(t). \end{cases} \tag{3.3}$$

In order to obtain our result, it is enough to show that the operator π has a fixed point in $\mathcal{S}$. First, is not difficult to see that the right hand side of (3.3) defines an $\mathcal{F}_t$-adapted and càdlàg process. Arguing in a similar way as it is done in Luo [17], to prove the exponential stability it is enough to show that the operator π possesses a fixed point in $\mathcal{S}$. To this end, we use the contraction mapping principle (see [20]), what requires that first we verify the continuity in the pth mean of π on $[0, +\infty)$.

Let $x \in \mathcal{S}, t_1 \geq 0$, and $|r|$ be sufficiently small, then

$$\mathbb{E}\|(\pi x)(t_1 + r) - (\pi x)(t_1)\|_{\mathbb{H}}^p \leq 3^{p-1} \sum_{i=1}^{3} \mathbb{E}\|I_i(t_1 + r) - I_i(t_1)\|_{\mathbb{H}}^p.$$

It is easy to check that

$$\mathbb{E}\|I_i(t_1 + r) - I_i(t_1)\|_{\mathbb{H}}^p \to 0, i = 1, 2, \text{ as } r \to 0.$$

Furthermore, by using Hölder's inequality and [10, Lemma 7.7], we obtain

$$\begin{aligned} &\|I_3(t_1 + r) - I_3(t_1)\|_{\mathbb{H}}^p \\ &\leq \ 2^{p-1}\mathbb{E}\left\|\int_0^{t_1} (R(t_1 + r - s) - R(t_1 - s))G(s, x(s - \delta(s)))dW(s)\right\|_{\mathbb{H}}^p \\ &\quad +2^{p-1}\mathbb{E}\left\|\int_{t_1}^{t_1+r} R(t_1 + r - s)G(s, x(s - \delta(s)))dW(s)\right\|_{\mathbb{H}}^p \\ &\leq \ 2^{p-1}c_p\left(\int_0^{t_1} (\mathbb{E}\|(R(t_1 + r - s) - R(t_1 - s))G(s, x(s - \delta(s)))\|_{\mathcal{L}_2^0}^p)^{2/p}ds\right)^{p/2} \\ &\quad +2^{p-1}c_p\left(\int_{t_1}^{t_1+r} (\mathbb{E}\|R(t_1 + r - s)G(s, x(s - \delta(s)))\|_{\mathcal{L}_2^0}^p)^{2/p}ds\right)^{p/2} \to 0 \end{aligned}$$

as $r \to 0$, where $c_p = (p(p-1)/2)^{p/2}$. Thus, π is indeed continuous in pth mean on

$[0, +\infty)$. Next, we show that $\pi(\mathcal{S}) \subset \mathcal{S}$. It follows from (3.3) that

$$\begin{aligned} e^{\alpha t}\mathbb{E}\|(\pi x)(t)\|_{\mathbb{H}}^p &\le 3^{p-1}e^{\alpha t}\mathbb{E}\|R(t)\phi(0)\|_{\mathbb{H}}^p \\ &+3^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t R(t-s)F(s,x(s-\rho(s)))ds\right\|_{\mathbb{H}}^p \\ &+3^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t R(t-s)G(s,x(s-\delta(s)))dW(s)\right\|_{\mathbb{H}}^p. \quad (3.4) \end{aligned}$$

Now we estimate the terms on the right-hand side of (3.4). First, from condition (2.8) we obtain

$$3^{p-1}e^{\alpha t}\mathbb{E}\|R(t)\phi(0)\|_{\mathbb{H}}^p \le 3^{p-1}M^p e^{-p\gamma}e^{\alpha t}\sup_{-\tau\le s\le 0}\mathbb{E}\|\phi(s)\|_{\mathbb{H}}^p \to 0 \text{ as } t\to+\infty. \quad (3.5)$$

Secondly, Hölder's inequality and (2.8) yield

$$\begin{aligned} &3^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t R(t-s)F(s,x(s-\rho(s)))ds\right\|_{\mathbb{H}}^p \\ \le\ & 3^{p-1}e^{\alpha t}\mathbb{E}\left[\int_0^t \|R(t-s)F(s,x(s-\rho(s)))\|_{\mathbb{H}}ds\right]^p \\ \le\ & 3^{p-1}e^{\alpha t}\mathbb{E}\left[\int_0^t Me^{-\gamma(t-s)}\|F(s,x(s-\rho(s)))\|_{\mathbb{H}}ds\right]^p \\ \le\ & 3^{p-1}M^pC_1^pe^{\alpha t}\mathbb{E}\left[\int_0^t e^{-\gamma(t-s)}\|x(s-\rho(s))\|_{\mathbb{H}}ds\right]^p \\ =\ & 3^{p-1}M^pC_1^pe^{\alpha t}\mathbb{E}\left[\int_0^t e^{-(\gamma(p-1)/p)(t-s)}e^{-(\gamma/p)(t-s)}\|x(s-\rho(s))\|_{\mathbb{H}}ds\right]^p \\ \le\ & 3^{p-1}M^pC_1^pe^{\alpha t}\left[\int_0^t e^{-\gamma(t-s)}ds\right]^{p-1}\left[\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^pds\right] \\ \le\ & 3^{p-1}M^pC_1^p(1/\gamma)^{p-1}e^{\alpha t}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^pds \\ =\ & 3^{p-1}M^pC_1^p(1/\gamma)^{p-1}e^{\alpha t}\int_0^t e^{-\gamma(t-s)}e^{-\alpha(s-\rho(s))}e^{\alpha(s-\rho(s)}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^pds \\ \le\ & 3^{p-1}M^pC_1^p\gamma^{1-p}e^{\alpha\tau}e^{-(\gamma-\alpha)t}\int_0^t e^{(\gamma-\alpha)s}e^{\alpha(s-\rho(s))}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^pds. \quad (3.6) \end{aligned}$$

For any $x \in \mathcal{S}$ and any $\epsilon > 0$, there exists a t_1 such that $e^{\alpha(s-\rho(s))}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^p < \epsilon$

for $s \geq t_1$. Thus, from (3.6),

$$\begin{aligned}
&3^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t R(t-s)F(s,x(s-\rho(s)))ds\right\|_{\mathbb{H}}^p \\
&\leq 3^{p-1}M^pC_1^p\gamma^{1-p}e^{\alpha\tau}e^{-(\gamma-\alpha)t}\int_0^{t_1} e^{(\gamma-\alpha)s}e^{\alpha(s-\rho(s))}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^p ds \\
&\quad +3^{p-1}M^pC_1^p\gamma^{1-p}e^{\alpha\tau}e^{-(\gamma-\alpha)t}\int_{t_1}^{t} e^{(\gamma-\alpha)s}e^{\alpha(s-\rho(s))}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^p ds \\
&\leq 3^{p-1}M^pC_1^p\gamma^{1-p}e^{\alpha\tau}e^{-(\gamma-\alpha)t}\int_0^{t_1} e^{(\gamma-\alpha)s}e^{\alpha(s-\rho(s))}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^p ds \\
&\quad +3^{p-1}M^pC_1^p\gamma^{1-p}(e^{\alpha\tau}/(\gamma-\alpha))\epsilon. \qquad (3.7)
\end{aligned}$$

As $e^{-(\gamma-\alpha)t} \to 0$ as $t \to +\infty$, thanks to condition (2.7), we can claim that there exists $t_2 \geq t_1$ such that, for any $t \geq t_2$, we have

$$\begin{aligned}
&3^{p-1}M^pC_1^p\gamma^{1-p}e^{\alpha\tau}e^{-(\gamma-\alpha)t}\int_0^{t_1} e^{(\gamma-\alpha)s}e^{\alpha(s-\rho(s))}\mathbb{E}\|x(s-\rho(s))\|_{\mathbb{H}}^p ds \\
&\quad \leq \epsilon - 3^{p-1}M^pC_1^p\gamma^{1-p}(e^{\alpha\tau}/(\gamma-\alpha))\epsilon. \qquad (3.8)
\end{aligned}$$

From the above arguments and (3.7) we obtain, for any $t \geq t_2$,

$$3^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t R(t-s)F(s,x(s-\rho(s)))ds\right\|_{\mathbb{H}}^p < \epsilon.$$

In other words,

$$3^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t R(t-s)F(s,x(s-\rho(s)))ds\right\|_{\mathbb{H}}^p \to 0 \text{ as } t \to +\infty. \qquad (3.9)$$

As for the third term on the right-hand side of (3.4), for any $x(t) \in \mathcal{S}, t \in [-\tau, +\infty)$, we have, for $p > 2$,

$$\begin{aligned}
&3^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t R(t-s)G(s,x(s-\delta(s)))dW(s)\right\|_{\mathbb{H}}^p \qquad (3.10) \\
&\leq 3^{p-1}e^{\alpha t}c_pM^p\Big\{\int_0^t (e^{-\gamma p(t-s)}\mathbb{E}\|G(s,x(s-\delta(s)))\|_{\mathcal{L}_2^0}^p)^{2/p}ds\Big\}^{p/2} \\
&\leq 3^{p-1}e^{\alpha t}c_pM^pC_2^p\Big\{\int_0^t (e^{-\gamma p(t-s)}\mathbb{E}\|x(s-\delta(s)))\|_{\mathbb{H}}^p)^{2/p}ds\Big\}^{p/2} \\
&= 3^{p-1}e^{\alpha t}c_pM^pC_2^p\Big\{\int_0^t (e^{-\gamma(p-1)(t-s)}e^{-\gamma(t-s)}\mathbb{E}\|x(s-\delta(s)))\|_{\mathbb{H}}^p)^{2/p}ds\Big\}^{p/2} \\
&\leq 3^{p-1}e^{\alpha t}c_pM^pC_2^p\Big\{\int_0^t e^{-\frac{2(p-1)\gamma(t-s)}{p-2}}\Big\}^{\frac{p}{2}-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\delta(s))\|_{\mathbb{H}}^p ds \\
&\leq 3^{p-1}c_pM^pC_2^p(2\gamma(p-1)/(p-2))^{1-p/2}e^{\alpha t}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\delta(s)))\|_{\mathbb{H}}^p ds
\end{aligned}$$

where $c_p = (p(p-1)/2)^{p/2}$. We remark that if $p = 2$, then inequality (3.10) also holds with $0^0 := 1$. Hence we have for $p \geq 2$,

$$\begin{aligned} & 3^{p-1} e^{\alpha t} \mathbb{E} \left\| \int_0^t R(t-s) G(s, x(s-\delta(s))) dW(s) \right\|_{\mathbb{H}}^p \qquad (3.11) \\ & \quad \leq 3^{p-1} c_p M^p C_2^p (2\gamma(p-1)/(p-2))^{1-p/2} e^{\alpha t} \int_0^t e^{-\gamma(t-s)} \mathbb{E} \|x(s-\delta(s)))\|_{\mathbb{H}}^p ds. \end{aligned}$$

Similar to the proof of (3.9), from (3.11) we obtain

$$3^{p-1} e^{\alpha t} \mathbb{E} \left\| \int_0^t R(t-s) G(s, x(s-\delta(s))) dW(s) \right\|_{\mathbb{H}}^p \to 0 \text{ as } t \to +\infty. \qquad (3.12)$$

Thus, from (3.4), (3.5), (3.9) and (3.12) we deduce that $e^{\alpha t} \mathbb{E} \|(\pi x)(t)\|_{\mathbb{H}}^p \to 0$ as $t \to +\infty$. Since the $\mathcal{F}_t$-measurability of $(\pi x)(t)$ is easily verified, it follows that π is well defined. Thus, we conclude that $\pi(\mathcal{S}) \subset \mathcal{S}$.

Finally, we will show that π is a contraction. For $x, y \in \mathcal{S}$, and proceeding as we did previously, we can obtain

$$\begin{aligned} & \sup_{t \in [0,T]} \mathbb{E} \|(\pi x)(t) - (\pi y)(t)\|_{\mathbb{H}}^p \qquad (3.13) \\ \leq \; & 2^{p-1} \sup_{t \in [0,T]} \mathbb{E} \| \int_0^t R(t-s)(F(s, x(s-\rho(s))) - F(s, x(s-\rho(s)))) ds \|_{\mathbb{H}}^p \\ & +2^{p-1} \sup_{t \in [0,T]} \mathbb{E} \left\| \int_0^t R(t-s)(G(s, x(s-\delta(s))) - G(s, x(s-\delta(s)))) dW(s) \right\|_{\mathbb{H}}^p \\ \leq \; & \sup_{t \in [0,T]} \mathbb{E} \|x(t) - y(t)\|_{\mathbb{H}}^p \times \\ & \times 2^{p-1} M^p \left[C_1^p \gamma^{1-p} + C_2^p \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} \left(\frac{2\gamma(p-1)}{p-2} \right)^{1-\frac{p}{2}} \right] (e^{\alpha\tau}/(\gamma-\alpha)). \end{aligned}$$

As thanks to (3.1) we managed to choose α small enough such that the constant appearing in the last line of (3.13) is less than one, then π is a contraction mapping and, by the contraction mapping principle in [20], π possesses a unique fixed point $x(t)$ in $\mathcal{S}$, which is a solution of (2.5) with $x(s) = \phi(s)$ on $[-\tau, 0]$ and $e^{\alpha t} \mathbb{E} \|x(t)\|_{\mathbb{H}}^p \to 0$ as $t \to \infty$, moreover $x(t)$ is exponentially stable in the pth-moment. This completes the proof. □

4 Application

To illustrate our theory, we consider the following model

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}u(t,\xi) + \displaystyle\int_0^t b(t-s)\dfrac{\partial^2}{\partial \xi^2}u(s,\xi)ds \\ \qquad\qquad +f(t,u(t-\tau_1,\xi))dt + g(t,u(t-\tau_2,\xi))dW(t) \;\text{ for }\; t \geq 0 \\ u(t,0) = 0 \;\text{ for }\; t \geq 0 \\ u(t,\pi) = 0 \;\text{ for }\; t \geq 0 \\ u(\theta,\xi) = \phi(\theta,\xi) \;\text{ for }\; -\tau < \theta \leq 0 \;\text{ and }\; 0 \leq \xi \in \pi, \end{cases} \tag{4.1}$$

where $\tau = \max(\tau_1, \tau_2)$, $f, g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, $\phi : [-\tau, 0] \times [0, \pi] \times \Omega \to \mathbb{R}$ is a given càdlàg function such that $\phi(\cdot)$ is $\mathcal{F}_0$-measurable and satisfies $E\|\phi\|^2 < \infty$.

Let $\mathbb{H} = L^2([0,\pi])$ and let $e_n := \sqrt{\frac{2}{\pi}}\sin(nx)$ $(n = 1, 2, 3, \cdots)$ denote the complete orthonormal basis in $\mathbb{H}$. Let $W(t) := \sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n$ $(\lambda_n > 0)$, where $\beta_n(t)$ are one dimensional standard Brownian motions mutually independent on a usual complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Define $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $A = \dfrac{\partial^2}{\partial z^2}$, with domain

$$D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi]).$$

It is well known that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{H}$, which is given by

$$T(t)\phi = \sum_{n=1}^{\infty} e^{-n^2 t} < \phi, e_n > e_n, \quad \phi \in D(A).$$

Let $B : D(A) \subset \mathbb{H} \to \mathbb{H}$ be the operator defined by

$$B(t)(z) = b(t)Az \;\text{ for }\; t \geq 0 \;\text{ and }\; z \in D(A).$$

We suppose that

(i) For $t \geq 0$, $f(t,0) = g(t,0) = 0$.

(ii) There exist positive constants C_1, C_2 such that

$$|f(t,\zeta_1) - f(t,\zeta_2)| \leq C_1|\zeta_1 - \zeta_2| \quad \text{for } \; t \geq 0 \; \text{ and } \; \zeta_1, \zeta_2 \in \mathbb{R}. \tag{4.2}$$

$$|g(t,\zeta_1) - g(t,\zeta_2)| \leq C_2|\zeta_1 - \zeta_2| \quad \text{for } \; t \geq 0 \; \text{ and } \; \zeta_1, \zeta_2 \in \mathbb{R}. \tag{4.3}$$

Let $D = D([-\tau, 0], \mathbb{H})$ and define the operators $F, G \; : \; \mathbb{R}^+ \times \mathbb{H} \to \mathbb{H}$ by

$$F(t,u)(\xi) = f(t, u(t-\tau_1, \xi)) \; \text{ for } \; \xi \in [0,\pi] \; \text{ and } \; u \in \mathbb{H},$$

$$G(t,u)(\xi) = g(t, u(t-\tau_1, \xi)) \; \text{ for } \; \xi \in [0,\pi] \; \text{ and } \; u \in \mathbb{H}.$$

We put

$$\begin{cases} x(t)(\xi) &= x(t,\xi) \; \text{ for } \; t \geq 0 \; \text{ and } \; \xi \in [0,\pi] \\ x(\theta)(\xi) &= \phi(\theta, \xi) \; \text{ for } \; \theta \in [-\tau, 0] \; \text{ and } \; \xi \in [0,\pi]. \end{cases}$$

Then Eq. (4.1) takes the following abstract form

$$\begin{cases} dx(t) = \left[Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + F(t, x(t-\tau_1)) \right] dt \\ \qquad\quad + G(t, x(t-\tau_2))dW(t) \; \text{ for } \; t \geq 0, \\ x_0(\cdot) = \phi \in D^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{H}). \end{cases}$$

Moreover, if b is bounded and a continuously differentiable function such that b' is bounded and uniformly continuous, then (H_1) and (H_2) are satisfied and hence, by Theorem 2.2, Eq. (4.1) possesses a resolvent operator $(R(t))_{t\geq 0}$ on $\mathbb{H}$. As a consequence of the continuity of f and g and assumption (i) it follows that F and G are continuous. By assumption (ii), one can see, for all $u_1, u_2 \in \mathbb{H}$,

$$\|F(t,u_1) - F(t,u_2)\| \leq C_1 \|u_1 - u_2\|,$$

and

$$\|G(t,u_1) - G(t,u_2)\| \leq C_2 \|u_1 - u_2\|.$$

Moreover, if we suppose that b is small enough, then one can ensure (see [9]) that there exists $a > 0$ and $N \geq 1$ such that

$$\|R(t)\| \leq Ne^{-at} \; \text{ for } \; t \geq 0.$$

Then all the assumptions of Theorem 3.1 are fulfilled. Therefore, Equation (4.1) possesses a unique mild solution which is exponentially stable in pth-moment ($p \geq 2$) provided that

$$3^{p-1}N^p(C_1^p a^{1-p} + C_2^p(p(p-1)/2)^{p/2}(2a(p-1)/(p-2))^{1-p/2}) < a.$$

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