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Periodic Solutions in Shifts $\delta_{\pm}$ for a Dynamic Equation on Time Scales

Erbil Çetin and **F. Serap Topal**
Department of Mathematics, Faculty of Science
Ege University, 35100, Bornova, Izmir, Turkey
erbil.cetin@ege.edu.tr
f.serap.topal@ege.edu.tr

Abstract

Let $\mathbb{T} \subset \mathbb{R}$ be a periodic time scale in shifts $\delta_{\pm}$ associated with the initial point $t_0 \in \mathbb{T}^*$. We use Brouwer's fixed point theorem to show that the initial value problem

$$x^{\Delta}(t) = p(t)x(t) + q(t), \ \ t \in \mathbb{T}, \quad x(t_0) = x_0$$

has a periodic solution in shifts $\delta_{\pm}$. We extend and unify periodic differential, difference, h-difference and especially q-difference equations and more by a new periodicity concept on time scales.

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1 Introduction

The existence problem of periodic solutions is an important topic in qualitative analysis of ordinary differential equations. The time scales approach unifies differential, difference, h-difference, q-differences equations and more under dynamic equations on time scales. The theory of dynamic equations on time scales was introduced by Stefan Hilger in his Ph.D. thesis in 1988 [8].

In 1950, Massera [13] proved the relationship between the boundedness of the solutions and the existence of periodic solutions to ordinary differential equations. Since then, many researchers obtained existence results on periodic solutions; see [4,11,14,15] and references therein. There is only a few results concerning periodic solutions of dynamic equations on time scales such as in [9, 12]. In these papers, authors considered

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the existence of periodic solutions for dynamic equation on time scales satisfying the condition

$$\text{“there exists a } \omega > 0 \text{ such that } t \pm \omega \in \mathbb{T} \text{ for all } t \in \mathbb{T}\text{”}. \tag{1.1}$$

Under this condition all periodic time scales are unbounded above and below. However, there are many time scales such as $\overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\sqrt{\mathbb{N}} = \{\sqrt{n} : n \in \mathbb{N}\}$ which do not satisfy the condition (1.1). M. Adıvar introduced a new periodicity concept on time scales which does not oblige the time scale to be closed under the operation $t \pm \omega$ for a fixed $\omega > 0$. He defined this concept with the aid of shift operators $\delta_\pm$ which are first defined in [2] and then generalized in [3].

Let $t_0 \in \mathbb{T}$ and $\mathbb{T}$ be a periodic time scale in shifts $\delta_\pm$ with period $P \in (t_0, \infty)_{\mathbb{T}}$. In this paper we are concerned with the existence of periodic solutions in shifts $\delta_\pm$ for the linear dynamic equation on time scales

$$x^{\Delta}(t) = p(t)x(t) + q(t), \quad t \in \mathbb{T}, \tag{1.2}$$

with the initial condition

$$x(t_0) = x_0, \tag{1.3}$$

where $p, q : \mathbb{T} \to \mathbb{R}$ are Δ-periodic functions in shifts $\delta_\pm$ with the period $T \in [P, \infty)_{\mathbb{T}}$, $p \in \mathcal{R}$ and q is rd-continuous.

Hereafter, we use the notation $[a, b]_{\mathbb{T}}$ to indicate the time scale interval $[a, b] \cap \mathbb{T}$. The intervals $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$ and $(a, b)_{\mathbb{T}}$ are similarly defined.

In Section 2, we will state some facts about exponential function on time scales, the new periodicity concept for time scales and some important theorems which will be needed to show the existence of a periodic solution in shifts $\delta_\pm$. In Section 3, we will give some lemmas about the exponential function and the graininess function with shift operators. Finally, we present our main result in Section 4 by using Brouwer's fixed point theorem.

2 Preliminaries

In this section, we mention some definitions, lemmas and theorems from calculus on time scales which can be found in [5, 6]. Next, we state some definitions, lemmas and theorems about the shift operators and the new periodicity concept which can be found in [1].

Definition 2.1 (See [5]). A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$.

Let $p \in \mathcal{R}$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \zeta_{\mu(r)}(p(r))\Delta r\right) \tag{2.1}$$

where $\zeta_{\mu(s)}$ is the cylinder transformation given by

$$\zeta_{\mu(r)}(p(r)) := \begin{cases} \dfrac{1}{\mu(r)}\log(1+\mu(r)p(r)), & \text{if } \mu(r) > 0; \\ p(r), & \text{if } \mu(r) = 0. \end{cases} \tag{2.2}$$

The exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.2 (See [5]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$*;*

(iii) $\dfrac{1}{e_p(t,s)} = e_\ominus(t,s)$*, where* $\ominus p(t) = -\dfrac{p(t)}{1+\mu(t)p(t)}$*;*

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

(vii) $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$*;*

(viii) $\left(\dfrac{1}{e_p(\cdot,s)}\right)^\Delta = -\dfrac{p(t)}{e_p^\sigma(\cdot,s)}$.

Theorem 2.3 (See [5]). *Let $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$ and assume that q is rd-continuous, and that $p \in \mathcal{R}$. The unique solution of the initial value problem*

$$y^\Delta(t) = p(t)y(t) + q(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t,t_0)y_0 + \int_{t_0}^t e_p(t,\sigma(s))q(s)\Delta s.$$

The following definitions, lemmas, corollaries and examples are about the shift operators and new periodicity concept which can be found in [1].

Definition 2.4 (See [1,2])**.** Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_\pm : [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:

(P.1) The functions $\delta_\pm$ are strictly increasing with respect to their second arguments, i.e., if

$$(T_0,t),(T_0,u) \in \mathcal{D}_\pm := \{(s,t) \in [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\mp(s,t) \in \mathbb{T}^*\},$$

then

$$T_0 \le t < u \text{ implies } \delta_\pm(T_0,t) < \delta_\pm(T_0,u),$$

(P.2) If $(T_1,u),(T_2,u) \in \mathcal{D}_-$ with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$, and if $(T_1,u),(T_2,u) \in \mathcal{D}_+$ with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$,

(P.3) If $t \in [t_0,\infty)_\mathbb{T}$, then $(t,t_0) \in \mathcal{D}_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in \mathcal{D}_+$ and $\delta_+(t_0,t) = t$ holds,

(P.4) If $(s,t) \in \mathcal{D}_\pm$, then $(s,\delta_\pm(s,t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s,\delta_\pm(s,t)) = t$, respectively,

(P.5) If $(s,t) \in \mathcal{D}_\pm$ and $(u,\delta_\pm(s,t)) \in \mathcal{D}_\mp$, then $(s,\delta_\mp(u,t)) \in \mathcal{D}_\pm$ and $\delta_\mp(u,\delta_\pm(s,t)) = \delta_\pm(s,\delta_\mp(u,t))$, respectively.

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be backward and forward shift operators on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0,\infty)_\mathbb{T}$ in $\delta_\pm(s,t)$ is called the shift size. The value $\delta_+(s,t)$ and $\delta_-(s,t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets $\mathcal{D}_\pm$ are the domains of the shift operator $\delta_\pm$, respectively. Hereafter, $\mathbb{T}^*$ is the largest subset of the time scale $\mathbb{T}$ such that the shift operators $\delta_\pm : [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ exist (see [1]).

Example 2.5 (See [1])**.** We give different time scales with their corresponding shift operators.

1. $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{R}$, $\delta_-(s,t) = t - s$ and $\delta_+(s,t) = t + s$.
2. $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{Z}$, $\delta_-(s,t) = t - s$ and $\delta_+(s,t) = t + s$.
3. $\mathbb{T} = q^\mathbb{Z} \cup \{0\}$, $t_0 = 1$, $\mathbb{T}^* = q^\mathbb{Z}$, $\delta_-(s,t) = \dfrac{t}{s}$ and $\delta_+(s,t) = ts$.
4. $\mathbb{T} = \mathbb{N}^{\frac{1}{2}}$, $t_0 = 0$, $\mathbb{T}^* = \mathbb{N}^{\frac{1}{2}}$, $\delta_-(s,t) = \sqrt{t^2 - s^2}$ and $\delta_+(s,t) = \sqrt{t^2 + s^2}$.

Definition 2.6 (Periodicity in shifts, see [1])**.** Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$ if there exists a $p \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t) \in \mathcal{D}_\pm$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$P := \inf\{p \in (t_0,\infty)_{\mathbb{T}^*} : (p,t) \in \mathcal{D}_\pm,\ \forall t \in \mathbb{T}^*\} \ne t_0,$$

then P is called the period of the time scale $\mathbb{T}$.

Example 2.7 (See [1])**.** The following time scales are not periodic in the sense of the condition (1.1) but periodic with respect to the notion of shift operators given in Definition 2.6.

1. $\mathbb{T}_1 = \{\pm n^2 : n \in \mathbb{Z}\}$, $\delta_\pm(P,t) = \begin{cases} (\sqrt{t} \pm \sqrt{P})^2, & t > 0; \\ \pm P, & t = 0; \\ -(\sqrt{-t} \pm \sqrt{P})^2, & t < 0; \end{cases}$ $P = 1, t_0 = 0$.

2. $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = q$.

3. $\mathbb{T}_3 = \overline{\cup_{n\in\mathbb{Z}}[2^{2n}, 2^{2n+1}]}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = 4$, $t_0 = 1$.

4. $\mathbb{T}_4 = \left\{ \dfrac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z} \right\} \cup \{0,1\}$.

$$\delta_\pm(P,t) = \frac{q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}{1 + q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}, \quad P = \frac{q}{1+q}.$$

Notice that the time scale $\mathbb{T}_4$ is bounded above and below and

$$\mathbb{T}_4^* = \left\{ \frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z} \right\}.$$

Remark 2.8 (See [1]). Let $\mathbb{T}$ be a time scale that is periodic in shifts with the period P. Thus, by P.4 of Definition 2.4 the mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ defined by $\delta_+^P(t) = \delta_+(P,t)$ is surjective. On the other hand, by P.1 of Definition 2.4 shift operators $\delta_\pm$ are strictly increasing in their second arguments. That is, the mapping $\delta_+^P(t) = \delta_+(P,t)$ is injective. Hence, δ_+^P is an invertible mapping with the inverse $(\delta_+^P)^{-1} = \delta_-^P$ defined by $\delta_-^P(t) := \delta_-(P,t)$.

We assume that $\mathbb{T}$ is a periodic time scale in shifts $\delta_\pm$ with period P. The operators $\delta_\pm^P : \mathbb{T}^* \to \mathbb{T}^*$ are commutative with the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ given by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. That is, $(\delta_\pm^P \circ \sigma)(t) = (\sigma \circ \delta_\pm^P)(t)$ for all $t \in \mathbb{T}^*$.

Lemma 2.9 (See [1])**.** *The mapping $\delta_+^P : \mathbb{T}^* \to \mathbb{T}^*$ preserves the structure of the points in $\mathbb{T}^*$. That is,*

$$\sigma(t) = t \ \textit{implies} \ \sigma(\delta_+(P,t)) = \delta_+(P,t),$$

$$\sigma(t) > t \ \textit{implies} \ \sigma(\delta_+(P,t)) > \delta_+(P,t).$$

Corollary 2.10 (See [1])**.** $\delta_+(P,\sigma(t)) = \sigma(\delta_+(P,t))$ *and* $\delta_-(P,\sigma(t)) = \sigma(\delta_-(P,t))$ *for all* $t \in \mathbb{T}^*$.

Definition 2.11 (Periodic function in shifts $\delta_\pm$, see [1]). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real value function f defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T, t) \in \mathcal{D}_\pm \text{ and } f(\delta_\pm^T(t)) = f(t) \text{ for all } t \in \mathbb{T}^* \tag{2.3}$$

where $\delta_\pm^T := \delta_\pm(T, t)$. The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ such that (2.3) holds is called the period of f.

Definition 2.12 (Δ-periodic function in shifts $\delta_\pm$, see [1]). Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. We say that a real-valued function f defined on $\mathbb{T}^*$ is Δ-periodic in shifts $\delta_\pm$ if there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that

$$(T, t) \in \mathcal{D}_\pm \text{ for all } t \in \mathbb{T}^*, \tag{2.4}$$

$$\text{the shifts } \delta_\pm^T \text{ are } \Delta\text{-differentiable with rd-continuous derivatives} \tag{2.5}$$

and

$$f(\delta_\pm^T(t))\delta_\pm^{\Delta T} = f(t) \text{ for all } t \in \mathbb{T}^*, \tag{2.6}$$

where $\delta_\pm^T := \delta_\pm(T, t)$. The smallest number $T \in [P, \infty)_{\mathbb{T}^*}$ such that (2.4)–(2.6) hold is called the period of f.

Notice that Definition 2.11 and Definition 2.12 give the classic periodicity definition on time scales whenever $\delta_\pm^T := t \pm T$ are the shifts satisfying the assumptions of Definition 2.11 and Definition 2.12.

Now, we give three theorems concern the composition of two functions. The first theorem is the chain rule on time scales.

Theorem 2.13 (Chain Rule, see [5]). *Assume that $\upsilon : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \upsilon(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^\Delta(t)$ and $w^{\widetilde{\Delta}}$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(w \circ \upsilon)^\Delta = (w^{\widetilde{\Delta}} \circ \nu)\nu^\Delta.$$

Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$. If we take $\nu(t) = \delta_\pm(T, t)$, then we have $\nu(\mathbb{T}) = \mathbb{T}$ and $[f(\nu(t))]^\Delta = (f^\Delta \circ \nu(t))\nu^\Delta(t)$.

The second theorem is the substitution rule on time scales.

Theorem 2.14 (Substitution, see [5]). *Assume $\upsilon : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \upsilon(\mathbb{T})$ is a time scale. If $g : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and υ is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$*

$$\int_a^b g(s)\upsilon^\Delta(s)\Delta s = \int_{\upsilon(a)}^{\upsilon(b)} g(\upsilon^{-1}(s))\widetilde{\Delta}s. \tag{2.7}$$

The third theorem is an application of the substitution rule where $\upsilon(t) = \delta_\pm^T(t)$.

Theorem 2.15 (See [1]). *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with period $P \in [t_0, \infty)_{\mathbb{T}^*}$ and f a Δ-periodic function in shifts $\delta_\pm$ with the period $T \in [P, \infty)_{\mathbb{T}^*}$. Suppose that $f \in \mathcal{C}_{rd}(\mathbb{T})$, then*

$$\int_{t_0}^{t} f(s)\Delta s = \int_{\delta_\pm^T(t_0)}^{\delta_\pm^T(t)} f(s)\Delta s. \tag{2.8}$$

Our work is mainly based on the following theorem.

Theorem 2.16 (Brouwer's Fixed Point Theorem, see [7]). *Let $D \subset \mathbb{R}$ be a nonempty compact convex set and $F : D \to D$ be continuous. Then F has a fixed point.*

3 Some Lemmas

In this section, we show some interesting properties of the exponential function $e_p(t, t_0)$ and shift operators on time scales.

Lemma 3.1. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^T$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$. Then the graininess function $\mu : \mathbb{T} \to [0, \infty)$ satisfies*

$$\mu(\delta_\pm^T(t)) = \delta_\pm^{\Delta T}(t)\mu(t).$$

Proof. Since $\delta_\pm^T$ are Δ-differentiable at t, from the calculus on time scales we know

$$\mu(t)\delta_\pm^{\Delta T}(t) = \delta_\pm^T(\sigma(t)) - \delta_\pm^T(t).$$

Then by using Corollary 2.10 we have

$$\begin{aligned} \mu(t)\delta_\pm^{\Delta T}(t) &= \sigma(\delta_\pm^T(t)) - \delta_\pm^T(t) \\ &= \mu(\delta_\pm^T(t)). \end{aligned}$$

Thus, the proof is complete. □

Lemma 3.2. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period P. Suppose that the shifts $\delta_\pm^T$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_\pm$ with the period T. Then*

$$e_p(\delta_\pm^T(t), \delta_\pm^T(t_0)) = e_p(t, t_0)\ for\ t, t_0 \in \mathbb{T}^*.$$

Proof. Assume that $\mu(\tau) \neq 0$. Set $f(\tau) = \dfrac{1}{\mu(\tau)} \log(1 + p(\tau)\mu(\tau))$. Using Lemma 3.1 and Δ-periodicity of p in shifts $\delta_{\pm}$ we get

$$\begin{aligned}
f(\delta_{\pm}^{T}(\tau))\delta_{\pm}^{\Delta T}(\tau) &= \frac{\delta_{\pm}^{\Delta T}(\tau)}{\mu(\delta_{\pm}^{T}(\tau))} \log(1 + p(\delta_{\pm}^{T}(\tau))\mu(\delta_{\pm}^{T}(\tau))) \\
&= \frac{\delta_{\pm}^{\Delta T}(\tau)}{\mu(\delta_{\pm}^{T}(\tau))} \log(1 + p(\delta_{\pm}^{T}(\tau))\delta_{\pm}^{\Delta T} \frac{1}{\delta_{\pm}^{\Delta T}} \mu(\delta_{\pm}^{T}(\tau))) \\
&= \frac{1}{\mu(\tau)} \log(1 + p(\tau)\mu(\tau)) \\
&= f(\tau).
\end{aligned}$$

Thus, f is Δ-periodic in shifts $\delta_{\pm}$ with the period T. By using Theorem 2.15 we have

$$\begin{aligned}
e_p(\delta_{\pm}^{T}(t), \delta_{\pm}^{T}(t_0)) &= \begin{cases} \exp\left(\displaystyle\int_{\delta_{\pm}^{T}(t_0))}^{\delta_{\pm}^{T}(t))} \frac{1}{\mu(\tau)} \log(1 + p(\tau)\mu(\tau))\Delta\tau\right), & \text{if } \mu(\tau) \neq 0; \\ \exp\left(\displaystyle\int_{\delta_{\pm}^{T}(t_0))}^{\delta_{\pm}^{T}(t))} p(\tau)\Delta\tau\right), & \text{if } \mu(\tau) = 0, \end{cases} \\
&= \begin{cases} \exp\left(\displaystyle\int_{t_0}^{t} \frac{1}{\mu(\tau)} \log(1 + p(\tau)\mu(\tau))\Delta\tau\right), & \text{if } \mu(\tau) \neq 0; \\ \exp\left(\displaystyle\int_{t_0}^{t} p(\tau)\Delta\tau\right), & \text{if } \mu(\tau) = 0, \end{cases} \\
&= e_p(t, t_0).
\end{aligned}$$

□

Lemma 3.3. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P. Suppose that the shifts $\delta_{\pm}^{T}$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_{\pm}$ with the period T. Then*

$$e_p((\delta_{\pm}^{T})^n(t_0), t_0) = (e_p(\delta_{\pm}^{T}(t_0), t_0))^n \ for\ n \in \mathbb{N}\ and\ t_0 \in \mathbb{T}^*.$$

Proof. From Lemma 1.2 (v.) and Lemma 3.2, we see that

$$\begin{aligned}
e_p(\delta_{\pm}^{T}(\delta_{\pm}^{T}(t_0)), t_0) &= e_p(\delta_{\pm}^{T}(\delta_{\pm}^{T}(t_0)), \delta_{\pm}^{T}(t_0))e_p(\delta_{\pm}^{T}(t_0), t_0) \\
&= e_p(\delta_{\pm}^{T}(t_0), t_0)e_p(\delta_{\pm}^{T}(t_0), t_0) \\
&= (e_p(\delta_{\pm}^{T}(t_0), t_0))^2.
\end{aligned}$$

The proof can be finished by induction. □

Lemma 3.4. *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_{\pm}$ with the period P. Suppose that the shifts $\delta_{\pm}^{T}$ are Δ-differentiable on $t \in \mathbb{T}^*$ where $T \in [P, \infty)_{\mathbb{T}^*}$ and $p \in \mathcal{R}$ is Δ-periodic in shifts $\delta_{\pm}$ with the period T. Then*

$$e_p(\delta_{\pm}^{T}(t), \sigma(\delta_{\pm}^{T}(s))) = e_p(t, \sigma(s)) = \frac{e_p(t, s)}{1 + \mu(t)p(t)} \ for\ t, s \in \mathbb{T}^*.$$

Proof. From Corollary 2.10, we know $\sigma(\delta_\pm^T(s)) = \delta_\pm^T(\sigma(s))$. By Lemma 3.1 and Lemma 2.2 we obtain

$$e_p(\delta_\pm^T(t), \sigma(\delta_\pm^T(s))) = e_p(\delta_\pm^T(t), \delta_\pm^T(\sigma(s))) = e_p(t, \sigma(s)) = \frac{e_p(t,s)}{1+\mu(t)p(t)}.$$

The proof is complete. □

4 Main Result

In this section, we consider the linear initial value problem (1.2)–(1.3). Firstly, we give a definition of a bounded solution. Hereafter, we denote the solution of (1.2) by $x(t, t_0, x_0)$.

Definition 4.1. A solution of (1.2) is bounded if there exists a real number $M > 0$ such that $|x(t, t_0, x_0)| < M$ for $t \in \mathbb{T}$.

Theorem 4.2. *The linear initial value problem* (1.2)–(1.3) *has a periodic solution in shifts $\delta_\pm$ with period T if and only if it has a bounded solution in $\mathbb{T}$.*

Proof. Let $x(t)$ be a periodic solution in shifts $\delta_\pm$ with period T. Since, $x(t) = x(\delta_+^T(t))$ for all $t \in \mathbb{T}^*$ and $x(t)$ is continuous, the necessity is obvious.

Now we show the sufficiency using Brouwer's fixed point theorem. Let $\overline{x}(t)$ be a bounded solution of (1.2). Then there exists a constant $M > 0$ such that $|\overline{x}(t)| \le M$ for $t \in \mathbb{T}$. We take $\overline{x}_0 := \overline{x}(t_0) \in \mathbb{R}$ and define the set $D \subset \mathbb{R}$ by

$$D := \{x_0 \in \mathbb{R} : |x_0| \le M,\ |x(t, t_0, x_0)| \le M,\ t \in \mathbb{T}\} \subset \mathbb{R}$$

where $x(t, t_0, x_0)$ is the unique solution of (1.2) through (t_0, x_0).

Since $x_0 \in D$, D is nonempty. We show that D is a compact convex set in $\mathbb{R}$. It is easy to see from the definition of D that D is closed and bounded. Thus, D is compact. For any $x_1, x_2 \in D$ and $\alpha \in [0, 1]$, we have

$$|\alpha x_1 + (1-\alpha)x_2| \le \alpha|x_1| + (1-\alpha)|x_2| \le M.$$

Moreover, by Theorem 2.3 we get

$$\begin{aligned}
\alpha x(t, t_0, x_1) \quad &+ \quad (1-\alpha)x(t, t_0, x_2) = \alpha e_p(t, t_0)x_1 + \alpha \int_{t_0}^t e_p(t, \sigma(s))q(s)\Delta s \\
&+ \quad (1-\alpha)e_p(t, t_0)x_2 + (1-\alpha)\int_{t_0}^t e_p(t, \sigma(s))q(s)\Delta s \\
&= \quad x(t, t_0, \alpha x_1 + (1-\alpha)x_2).
\end{aligned}$$

So we have for $t \ge t_0$

$$|x(t, t_0, \alpha x_1 + (1-\alpha)x_2)| \le \alpha|x(t, t_0, x_1)| + (1-\alpha)|x(t, t_0, x_2)| \le M.$$

We define a mapping $F : D \to \mathbb{R}$ by

$$Fx_0 = x(\delta_+^T(t_0), t_0, x_0) = e_p(\delta_+^T(t_0), t_0)x_0 + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s.$$

Because of $\delta_+^T(t_0) \in \mathbb{T}$, for any $x_0 \in D$, we get $|x(\delta_+^T(t_0), t_0, x_0)| \leq M$. By Lemma 3.1, Lemma 3.2, Lemma 3.3 and Theorem 2.3, we have

$$\begin{aligned}
F\ \ (Fx_0) &= x(\delta_+^T(t_0), t_0, x(\delta_+^T(t_0), t_0, x_0)) \\
&= e_p(\delta_+^T(t_0), t_0)x(\delta_+^T(t_0), t_0, x_0) + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s \\
&= e_p(\delta_+^T(t_0), t_0)\left(e_p(\delta_+^T(t_0), t_0)x_0 + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s\right) \\
&\quad + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s \\
&= (e_p(\delta_+^T(t_0), t_0))^2 x_0 + (e_p(\delta_+^T(t_0), t_0) + 1)\int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s \\
&= e_p(\delta_+^T(\delta_+^T(t_0)), t_0)x_0 + (e_p(\delta_+^T(t_0), t_0) + 1)\int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s.
\end{aligned}$$

Substituting $\upsilon(s) = \delta_+^T(s)$ and $g(s) = e_p(\delta_+^T(\delta_+^T(t_0)), \sigma(\delta_+^T(s)))q(\delta_+^T(s))$ in (2.7) and taking q is Δ-periodic in shifts $\delta_\pm$ into account we have

$$\begin{aligned}
x\ \ (\delta_+^T(\delta_+^T(t_0)), t_0, x_0) &= e_p(\delta_+^T(\delta_+^T(t_0)), t_0)x_0 + \int_{t_0}^{\delta_+^T(\delta_+^T(t_0))} e_p(\delta_+^T(\delta_+^T(t_0)), \sigma(s))q(s)\Delta s \\
&= e_p(\delta_+^T(\delta_+^T(t_0)), t_0)x_0 + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(\delta_+^T(t_0)), \sigma(s))q(s)\Delta s \\
&\quad + \int_{\delta_+^T(t_0)}^{\delta_+^T(\delta_+^T(t_0))} e_p(\delta_+^T(\delta_+^T(t_0)), \sigma(s))q(s)\Delta s \\
&= e_p(\delta_+^T(\delta_+^T(t_0)), t_0)x_0 + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(\delta_+^T(t_0)), \delta_+^T(t_0))e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s \\
&\quad + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(\delta_+^T(t_0)), \sigma(\delta_+^T(s)))q(\delta_+^T(s))\delta_+^{T\Delta}(s)\Delta s \\
&= e_p(\delta_+^T(\delta_+^T(t_0)), t_0)x_0 + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), t_0)e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s \\
&\quad + \int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s \\
&= e_p(\delta_+^T(\delta_+^T(t_0)), t_0)x_0 + (e_p(\delta_+^T(t_0), t_0) + 1)\int_{t_0}^{\delta_+^T(t_0)} e_p(\delta_+^T(t_0), \sigma(s))q(s)\Delta s.
\end{aligned}$$

Therefore, we have $F(Fx_0) = x(\delta_+^T(\delta_+^T(t_0)), t_0, x_0)$. Also, since $\delta_+^T(\delta_+^T(t_0)) \in \mathbb{T}$ and $|x(\delta_+^T(\delta_+^T(t_0)), t_0, x_0)| \leq M$ we obtain $Fx_0 \in D$ which means that $FD \subset D$ and F is compact. By the continuous dependence of solutions of dynamic systems on time scales with respect to initial values (see [10]), F is continuous. Thus F has a fixed point in D by Brouwer's fixed point theorem. That is $x(\delta_+^T(t), t_0, x_0) = x_0$. So

$$x(\delta_+^T(t), t_0, x_0) = x(t, t_0, x_0), \ \ t \in \mathbb{T}$$

is a periodic solution in shifts $\delta_\pm$ with period T of (1.2). The proof is complete. $\square$

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