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# European Call Option Pricing using the Adomian Decomposition Method

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**Abstract**

This article explores the Adomian decomposition method applied to the pricing of European call options in a risk-neutral world with an asset that pays and one that does not pay dividends. A brief introduction to existing methods of pricing, a numerical solution of the Black–Scholes equation, a construction of a payoff function consistent with the method, and finally some numerical results for a hypothetical experiment are given.

**AMS Subject Classifications:** 91G20, 91G30.
**Keywords:** Adomian decomposition method, Black–Scholes equation, European call option pricing.

## 1 Introduction

An option is a security that gives its owner the right to trade in a fixed number of shares of a specified common stock at a fixed price at any time on or before a given date. The act of making this transaction is referred to as exercising the option. The fixed price is

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termed the strike price, and the given date is termed the expiration date. A call option gives the right to buy the share while a put option gives the right to sell the share.

The option pricing problem has been treated using various numerical methods, e.g., Monte Carlo simulation, binomial trees, and finite difference methods. [7] presents two direct methods, a pathwise method and a likelihood ratio method, for estimating derivatives of security prices using simulation. It is important to remember that simulation has proved to be a valuable tool for estimating those prices for which simple closed form solutions do not exist. The main advantage of direct methods over simulation is increased computational speed.

[9] presents a simple discrete-time model for valuing options, using the Black–Scholes model. By its very construction, it gives rise to a simple and efficient numerical procedure for valuing options for which premature exercise is optimal. It should now be clear that whenever stock price movements conform to a discrete binomial process or to a limiting form of such a process, options can be priced solely on the basis of arbitrage considerations. Indeed, we could have significantly complicated the simple binomial process while still retaining this property. [11] presents a new method for pricing American options along with an efficient implementation procedure. The proposed method is efficient and accurate in computing both option values and various option hedge parameters. It also suggests how the method can be applied to the case of any American option for which a closed-form solution exists for the corresponding European option. The method presented has some attractive features. First, since its implementation is based on using an analytical formula for both option values and hedge parameters, the latter are computed directly rather than by perturbation of the option pricing formula. Second, as a result, the computation is both efficient and accurate, since the analytical formula involves only univariate integrals.

Numerical approximations of contingent claim partial differential equations (PDEs) are quickly becoming one of the most accepted techniques used in derivative security valuation. The most common methodology is the finite difference method (FDM). This procedure can be used as long as a well-posed PDE can be derived and therefore lends itself to contingent claims. The FDM requires prescribed conditions at the boundary. These boundary conditions are not readily available (at all boundaries) for most contingent claim PDEs. [8] presents an accurate and computationally inexpensive method for providing these boundary conditions. These absorbing and adjusting boundary conditions when applied to the contingent claim PDEs presented in this study increased the accuracy of the FDM solution at relatively little cost.

The Adomian decomposition method has been used to solve various equations. It is important first to know that in [2], George Adomian presented stochastic systems which define linear and nonlinear equations and different solution methods. In [4], nonlinear transformation of series are used together with the Adomian decomposition method. In [13], the efficiency and power of the technique is shown for wide classes of equations of mathematical physics. The analytical solutions for linear, one-dimensional, time-dependent partial differential equations subject to initial or lateral boundary conditions are reviewed and obtained in the form of convergent Adomian decomposition power series with easily computable components. [3] defines the decomposition method, which can be an effective procedure for the analytical solution of a wide class of dynamical systems without linearization or weak nonlinearity assumptions, closure approximations, perturbation theory, or restrictive assumptions on stochasticity. Finally [10] explains how the Adomian decomposition method is used to give explicit and numerical solutions of three types of diffusion-convection-reaction (DCRE) equations. The calculations are carried out for three different types of the DCRE such as the Black–Scholes equation used in financial market option pricing and the Fokker–Planck equation from plasma physics. The behaviour of the approximate solutions of the distribution functions is shown graphically and compared with that obtained by other theories such as the variational iteration method.

## 2 Adomian Decomposition Method for Black–Scholes

The classical Black–Scholes model of option pricing [5] assumes that the underlying dynamic behavior is associated with a linear homogeneous stochastic differential equation given by

$$dx_t = \mu x_t + \sigma x_t dB_t,$$

where $t \in [0, T]$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $\{B_t\}_{t \geq 0}$ is a one-dimensional standard Brownian motion. This model states that without making assumptions about the preferences of investors, one can obtain an expression for the value of options not directly dependent on the expected performance of the underlying stock or the option. The assumptions on which it is based form an ideal setting in which continuous negotiation is possible in a perfect market in which the interest rate is constant and risk free [1, 12].

An alternative method to numerically solve the Black–Scholes equation is the Adomian decomposition method [6], which is based on representing the solution as a series of functions, where each term is obtained by a polynomial expansion. In addition, this technique provides solutions for nonlinear partial differential equation such as partial differential equation of Black–Scholes type, which can be written [14] as

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 U}{\partial x^2} + (r - \delta)x\frac{\partial U}{\partial x} - ru = 0, \tag{2.1}$$

where $r$ is the risk free rate, $\sigma$ is the volatility, and $\delta$ is the dividend. The boundary condition for this equation is given by

$$u(x, T) = g(x) = \max\{x - K, 0\}, \tag{2.2}$$

where $K$ is the strike price and $T$ is the expiration date. We now describe the Adomian decomposition method [3] for the general problem

$$\begin{cases} \displaystyle\sum_{n=0}^{N} \alpha_n(x,t) G_n u(x,t) = \sum_{m=1}^{M} \beta_m(x,t) F_m u(x,t) + f(x,t), \\ G_n u(x,0) = g_n(x), \quad 0 \le n \le N-1, \end{cases}$$

where $G_n = \dfrac{\partial^n}{\partial t^n}$ for $0 \le n \le N$ and $F_m = \dfrac{\partial^m}{\partial x^m}$ for $0 \le m \le M$. The solution can be represented by

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t),$$

where

$$u_0(x,t) = G_N^{-1}\left(\frac{f(x,t)}{\alpha_n(x,t)}\right) + \sum_{l=0}^{N-1} \frac{t^l}{l!} g_l(x),$$

and for $k \in \mathbb{N}_0$,

$$u_{k+1}(x,t) = \left[\sum_{m=1}^{M} G_N^{-1}\left(\frac{\beta_m(x,t)}{\alpha_N(x,t)} F_m\right) - \sum_{n=0}^{N-1} G_N^{-1}\left(\frac{\alpha_n(x,t)}{\alpha_N(x,t)} G_n\right)\right] u_k(x,t).$$

Now considering the equation (2.1), we use the values $N = 1$, $M = 2$, $f = 0$, $\alpha_0 = r$, $\alpha_1 = -1$, $\beta_0 = 0$, $\beta_1 = (r-\delta)x$, $\beta_2 = \frac{1}{2}\sigma^2 x^2$ (see [6]). Applying them to equations (2.1) and (2.2), we obtain

$$u_0(x,t) = g(x), \tag{2.3}$$

and for $k \in \mathbb{N}_0$,

$$\begin{aligned} u_{k+1}(x,t) &= \left[\sum_{m=1}^{2} G_1^{-1}\left(\frac{\beta_m(x,t)}{\alpha_1(x,t)} F_m\right) - \sum_{n=0}^{0} G_1^{-1}\left(\frac{\alpha_n(x,t)}{\alpha_1(x,t)} G_n\right)\right] u_k(x,t) \\ &= G_1^{-1}\left[-(r-\delta)xF_1 - \frac{1}{2}\sigma^2 x^2 F_2 + rG_0\right] u_k(x,t) \\ &= \int_T^t \left[\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u_k(x,\tau)}{\partial x^2} + (r-\delta)x \frac{\partial u_k(x,\tau)}{\partial x} - r u_k(x,\tau)\right] d\tau. \end{aligned} \tag{2.4}$$

The sequence defined recursively in (2.3) and (2.4) can be represented explicitly [6] as

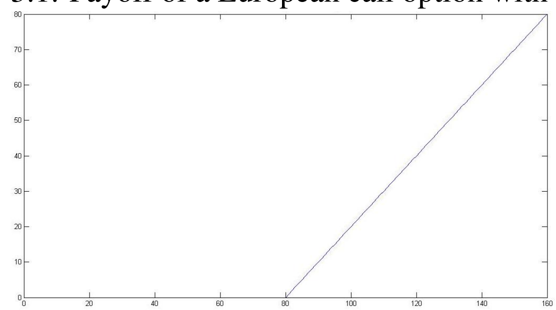
$$u_k(x,t) = \left[\sum_{m=0}^{2} k \left\{\sum_{v=0}^{m} \frac{(-1)^{m-v}}{v!(m-v)!} \rho_v^k\right\} x^m g^m(x)\right] \frac{(T-t)^k}{k!}$$

for all $k \in \mathbb{N}_0$, where $\rho_m = \left(\frac{1}{2}\sigma^2 m + r\right)(m-1) - \delta$ for all $m \in \mathbb{N}_0$, and $g$ is a function that has derivatives of all orders.

# 3 Construction of an Approximate Payoff Function

To use the Adomian decomposition, we will construct a function that meets two conditions: It has derivatives of all orders and is "consistent" with the payoff of a European call option, $\max\{S_T - K, 0\}$, where $S_T$ is the stock price at time $T$ and $K$ is the strike price. Figure 3.1 shows the graph of the payoff of a European call option. As this function is not differentiable at $K$, it cannot be used for the Adomian decomposition method of the Black–Scholes equation [6]. To find another function $g$, we perform a rotation and translation of axes of a hyperbola, taking into account the transformations

Figure 3.1: Payoff of a European call option with $K = 80$




$$\begin{cases} \tilde{x} = (x-K)\cos\theta + y\sin\theta, \\ \tilde{y} = -(x-K)\sin\theta + y\cos\theta. \end{cases} \tag{3.1}$$

The equation of a hyperbola with center $(0, 0)$ is given by

$$\frac{\tilde{y}^2}{b^2} - \frac{\tilde{x}^2}{a^2} = 1. \tag{3.2}$$

Substituting (3.1) in (3.2) and solving for $y$, we obtain

$$y = \frac{(x-K)B + \sqrt{(x-K)^2(B^2 - 4cd) + 4da^2b^2}}{2d},$$

where

$$B = \sin(2\theta)(a^2 + b^2),\ c = a^2\sin^2\theta - b^2\cos^2\theta,\ d = a^2\cos^2\theta - b^2\sin^2\theta.$$

For the special case when $b = a\tan\theta$, with $\theta = \dfrac{\pi}{8}$, this function can be rewritten as

$$g(x) = \frac{1}{2}(x-K) + \frac{1}{2}\sqrt{(x-K)^2 + 2a^2(\sqrt{2}-1)}.$$

**Proposition 3.1** (Convergence and analyticity)**.** *Let* $x_0 \in \mathbb{R}^+$ *and consider*

$$g_n(x_0) = \frac{1}{2}(x_0 - K) + \frac{1}{2}\sqrt{(x_0 - K)^2 + 2a_n^2(\sqrt{2} - 1)},\ a_n = \frac{1}{n},\ n \in \mathbb{N}.$$

*Then* $g_n$ *is analytic and*

$$\lim_{n\to\infty} g_n(x_0) = \max\{x_0 - K, 0\}.$$

*Proof.* We have

$$\begin{aligned} \lim_{n\to\infty} g_n(x_0) &= \lim_{n\to\infty}\left(\frac{1}{2}(x_0 - K) + \frac{1}{2}\sqrt{(x_0 - K)^2 + \frac{2(\sqrt{2}-1)}{n^2}}\right) \\ &= \frac{1}{2}(x_0 - K) + \frac{1}{2}\sqrt{(x_0 - K)^2} \\ &= \frac{1}{2}(x_0 - K) + \frac{1}{2}|x_0 - K| \\ &= \begin{cases} x_0 - K & \text{if} \quad x_0 > K \\ 0 & \text{if} \quad x_0 \le K \end{cases} \\ &= \max\{x_0 - K, 0\}. \end{aligned}$$

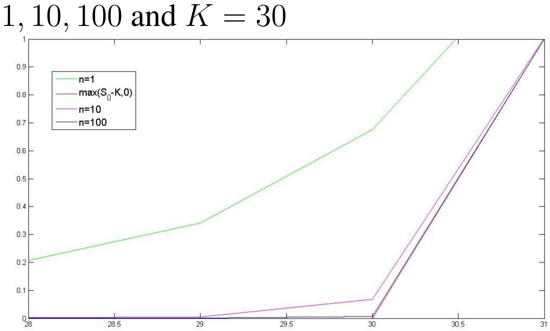
Consider now the functions $g_1$ and $g_2$ defined by

$$g_1(x) = \sqrt{x} \quad \text{and} \quad g_2(x) = (x - K)^2 + 2a_n^2(\sqrt{2} - 1).$$

Note that both $g_1$ and $g_2$ are analytic functions, and hence so is $g_1 \circ g_2 = g_n$. $\square$

Figure 3.2 shows the sequence of $g_n$ converging to the payoff of a European call option.

Figure 3.2: Graph of the convergence of the sequence $g_n$ to the payoff of a European call option with $n = 1, 10, 100$ and $K = 30$

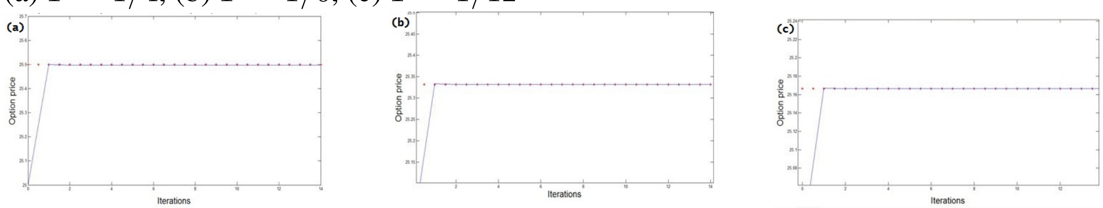


# 4 Experimental Numerical Results

## 4.1 Risk-Neutral Pricing of Nondivident Paying Assets

We give some experimental numerical results for the valuation of a European call option on a nondivident paying asset in a risk-neutral world. We use the Adomian decomposition method with different expiration times and compare it with the results obtained using the Black–Scholes formula. Table 4.1 presents a comparison of the valuation of

Figure 4.1: Pricing of a European call option with the Adomian decomposition method in a risk-neutral world with $S_0 = 65$, $K = 40$, $\sigma = 0.324366$, $r = 0.05$, $\delta = 0$, $k = 14$, (a) $T = 1/4$, (b) $T = 1/6$, (c) $T = 1/12$



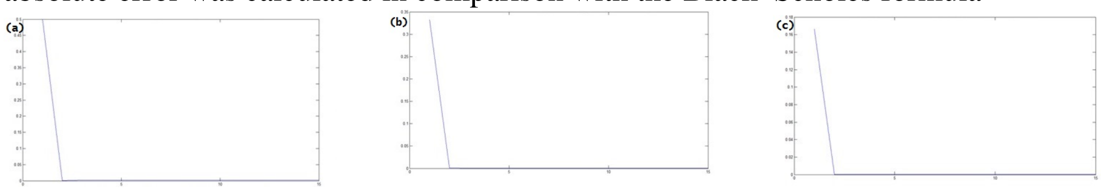

a European call option on a nondivident paying asset in a risk-neutral world using the Adomian decomposition method (12 iterations) with other numerical methods such as Monte Carlo Simulation (4000 trajectories), the binomial tree method (200 steps), explicit finite differences, and the Black–Scholes formula. In Figure 4.1 we can see that

Table 4.1: Comparison of the results obtained for the valuation of a European call option on a nondivident paying asset in a risk-neutral world with different methods and the Black–Scholes equation ($S_0 = 65$, $K = 40$, $\sigma = 0.324366$, $r = 0.05$, $\delta = 0$ for $T = 1/4$, $T = 1/6$, $T = 1/12$). The last column shows the relative error of the price obtained by the Adomian decomposition method

| Time | Adomian Price | Monte Carlo Simulation Price | Binomial Trees Price | Explicit Finite Difference Price | Black–Scholes Price | relative error (in %) |
|---|---|---|---|---|---|---|
| 1/4 | 25.4965 | 25.3034 | 25.3248 | 25.3535 | 25.4993 | 0.0106 |
| 1/6 | 25.3319 | 25.2173 | 25.2658 | 25.0037 | 25.3321 | 0.000611 |
| 1/12 | 25.1663 | 25.1289 | 25.1330 | 24.9485 | 25.1663 | 0.000000288 |

as the time step is smaller, ADM produces better results, which is noteworthy since the relative errors are becoming smaller. Figure 4.2 shows the behavior of the absolute error compared to the Black–Scholes formula. In the three graphs it can be seen that the error tends to zero, which refers to the closeness between the actual price and the price found by the Adomian decomposition method.

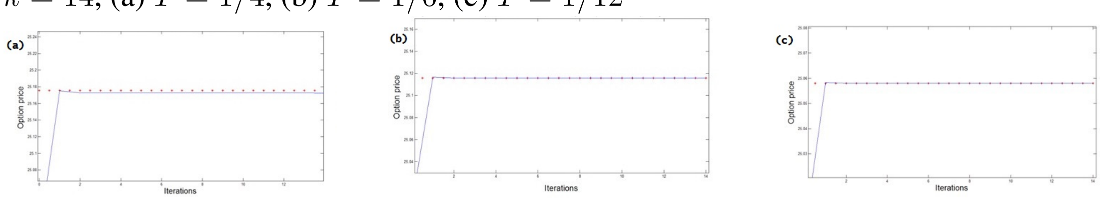
Figure 4.2: Convergence of Adomian decomposition method for a European call option on a nondivident paying asset in a risk-neutral world with $S_0 = 65$, $K = 40$, $\sigma = 0.324366$, $r = 0.05$, $\delta = 0$, $k = 14$, (a) $T = 1/4$, (b) $T = 1/6$, (c) $T = 1/12$. The absolute error was calculated in comparison with the Black–Scholes formula




## 4.2 Risk-Neutral Pricing of Divident Paying Assets

We give some numerical results for the valuation of a European call option on a dividend paying asset in a risk-neutral world. We use the Adomian decomposition method with different expiration times and compare it with the results obtained using the Black–Scholes formula. Table 4.2 presents a comparison of the valuation of a European call

Figure 4.3: Pricing of a European call option with the Adomian decomposition method in a risk-neutral world with $S_0 = 65$, $K = 40$, $\sigma = 0.324366$, $r = 0.05$, $\delta = 0.02$, $k = 14$, (a) $T = 1/4$, (b) $T = 1/6$, (c) $T = 1/12$

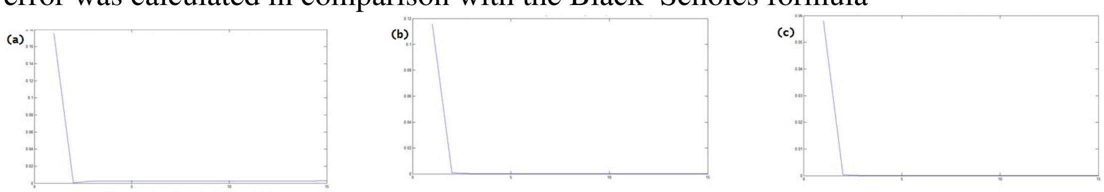



option on a divident paying asset in a risk-neutral world using the Adomian decomposition method (12 iterations) with other numerical methods such as Monte Carlo Simulation (4000 trajectories), the binomial tree method (200 steps), explicit finite differences, and the Black–Scholes formula. We see that the results are close to each other. Figure 4.3 shows these results, illustrating that the relative errors are small, indicating that the Adomian decomposition method yields good results. Figure 4.4 shows the behavior of the absolute error compared to the Black–Scholes formula. In the three graphs it can be seen that the error tends to zero, which refers to the closeness between the actual price and the price found by the Adomian decomposition method.

Table 4.2: Comparison of the results obtained for the valuation of a European call option on a divident paying asset in a risk-neutral world with different methods and the Black–Scholes equation ($S_0 = 65$, $K = 40$, $\sigma = 0.324366$, $r = 0.05$, $\delta = 0.02$ for $T = 1/4$, $T = 1/6$, $T = 1/12$). The last column shows the relative error of the price obtained by the Adomian decomposition method

| Time | Adomian Price | Monte Carlo Simulation Price | Binomial Trees Price | Explicit Finite Difference Price | Black–Scholes Price | relative error (in %) |
|---|---|---|---|---|---|---|
| 1/4 | 25.1724 | 25.1754 | 25.2948 | 26.3007 | 25.1753 | 0.0119 |
| 1/6 | 25.1156 | 25.1158 | 25.1956 | 26.1996 | 25.1158 | 0.000683 |
| 1/12 | 25.0580 | 25.0580 | 25.0979 | 26.0999 | 25.0580 | 0.000000319 |

Figure 4.4: Convergence of Adomian decomposition method for a European call option on a divident paying asset in a risk-neutral world with $S_0 = 65$, $K = 40$, $\sigma = 0.324366$, $r = 0.05$, $\delta = 0.02$, $k = 14$, (a) $T = 1/4$, (b) $T = 1/6$, (c) $T = 1/12$. The absolute error was calculated in comparison with the Black–Scholes formula




# 5 Conclusions

We have constructed a payoff function which is suitable for the implementation of the Adomian decomposition method because it meets the necessary conditions: It is nonnegative, infinitely often differentiable, and consistent with the payoff of a European call option in the sense that it converges pointwise to that payoff function. When considering the rotation and translation of axes, it is natural that the exercise price must be less than the initial price. On the other hand, this analysis can be extended to a European put option considering a corresponding nonpositive payoff function.

The results obtained by the Adomian decomposition method are pretty close to those obtained by other numerical methods such as Monte Carlo simulation, the binomial tree method, and explicit finite differences, suggesting that the Adomian decomposition method is a powerful method that can be used in the valuation of options of any kind. However, the method implementation can generate a large computational cost because it requires the calculation of a significant number of derivatives, some positive and some negative, to be offset as the iterations increase. The method converges rapidly as the numerical experiments support the conclusion that a small number of iterations is enough to find a favorable outcome. Also when the time of expiration of the option is small, the method gives better results.

For future work one could extend this analysis by performing a valuation of European options, both call and put, in a real world or physical world on assets whose dynamic behavior can be modeled by processes of mean reversion, which include interest rates, oil, aluminum, and natural gas, among others.

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