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Fractional Integro-Differential Equations with State-Dependent Delay

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Abstract

In this article, we deal with the existence of mild solutions for a class of fractional integro-differential equations with state-dependent delay. Our results are based on the technique of measures of noncompactness and Darbo's fixed point theorem. An example is provided to illustrate the main result.

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1 Introduction

Fractional differential equations have become an important object of investigation in recent years stimulated by their numerous applications to problems arising in physics, mechanics and other fields (see [14, 15, 24, 30, 34–36]). The theory of differential equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to fractional differential equations, for example see [1, 2, 6, 10, 13, 27, 28, 37].

On the other hand, functional differential equations with state-dependent delay appears frequently in applications as models of equations. Investigations of these classes of delay equations essentially differ from once of equations with constant or time-dependent delay. For these reasons the theory of differential equations with state-dependent delay has drawn the attention of researchers in the recent years, see for

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instance [3, 4, 17–19, 21–23, 29, 32]. Recently, Carvalho dos Santos et al [8] studied the existence of solutions for fractional neutral functional integro-differential equations with state-dependent delay. Kavitha et al [26] established the existence of mild solutions for a class of neutral functional fractional differential equations with state-dependent delay. In [9, 11], the authors provide sufficient conditions for the existence of solutions of fractional functional differential equation with state-dependent delay. Very recently, Benchohra et al [7] investigated the existence of solutions on a compact interval for fractional integro-differential equations with state-dependent delay by using standard fixed point theorems.

Motivated by the previous literature, the purpose of this article is to establish the existence of mild solutions for fractional functional integro-differential equation with state-dependent delay of the form

$$\begin{aligned} D_t^q x(t) &= Ax(t) + \int_0^t a(t,s) f(s, x_{\rho(s,x_s)}, x(s)) ds, \qquad t \in J = [0,T], \\ x(t) &= \phi(t), \qquad\qquad\qquad\qquad\qquad\qquad t \in (-\infty, 0], \end{aligned} \tag{1.1}$$

where D_t^q is the Caputo fractional derivative of order $0 < q < 1$, A is a generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$ of uniformly bounded linear operators on X, $f : J \times \mathcal{B} \times X \longrightarrow X$ and $\rho : J \times \mathcal{B} \to (-\infty, T]$ are appropriated functions. $a : D \to \mathbb{R}$ $(D = \{(t,s) \in J \times J : t \geq s\})$, $\phi \in \mathcal{B}$ where $\mathcal{B}$ is called phase space to be defined in Section 2. For any continuous function x defined on $(-\infty, T]$ and any $t \in J$, we denote by x_t the element of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t + \theta), \qquad \theta \in (-\infty, 0].$$

Here x_t represents the history of the state up to the present time t.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space, $C = C(J, X)$ the space of all X-valued continuous functions on J, $L(X)$ the Banach space of all linear and bounded operators on X, $L^1(J, X)$ the space of X-valued Bochner integrable functions on J with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt,$$

and $L^\infty(J, \mathbb{R})$ the Banach space of essentially bounded functions, normed by

$$\|y\|_{L^\infty} = \inf\{d > 0 : |y(t)| \leq d, \text{ a.e. } t \in J\}.$$

We need some basic definitions of the fractional calculus theory which are used in this paper.

Definition 2.1. Let $\alpha > 0$ and $f : \mathbb{R}_+ \to X$ be in $L^1(\mathbb{R}_+, X)$. Then the Riemann–Liouville integral is given by:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

For more details on the Riemann–Liouville fractional derivative, we refer the reader to [12].

Definition 2.2 (See [33])**.** The Caputo derivative of order α for $f : [0, +\infty) \to X$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)(s)}}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \ n-1 \leq \alpha < n.$$

If $0 < \alpha \leq 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^{(1)}(s)}{(t-s)^\alpha} ds.$$

Obviously, the Caputo derivative of a constant is equal to zero.

Definition 2.3. A function $f : J \times \mathcal{B} \times X \longrightarrow X$ is said to be a Carathéodory function if it satisfies

(i) for each $t \in J$ the function $f(t, \cdot, \cdot) : \mathcal{B} \times X \longrightarrow X$ is continuous;

(ii) for each $(v, w) \in \mathcal{B} \times X$ the function $f(\cdot, v, w) : J \to X$ is measurable.

Next we give the concept of a measure of noncompactness [5].

Definition 2.4. Let B be a bounded subset of a seminormed linear space Y. Kuratowski's measure of noncompactness of B is defined as

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter} \leq d\}.$$

We note that this measure of noncompactness satisfies interesting regularity properties (for more information, we refer to [5]).

Lemma 2.5. *1. If $A \subseteq B$, then $\alpha(A) \leq \alpha(B)$,*

2. $\alpha(A) = \alpha(\overline{A})$, where $\overline{A}$ denotes the closure of A,

3. $\alpha(A) = 0 \Leftrightarrow \overline{A}$ is compact (A is relatively compact),

4. $\alpha(\lambda A) = |\lambda| A, \ with \ \lambda \in \mathbb{R}$,

5. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,

6. $\alpha(A+B) \leq \alpha(A)+\alpha(B)$, *where*

$$A+B=\{x+y : x \in A, y \in B\},$$

7. $\alpha(A+a)=\alpha(A)$ *for any* $a \in Y$,

8. $\alpha(\overline{\mathrm{conv}}A)=\alpha(A)$, *where* $\overline{\mathrm{conv}}A$ *is the closed convex hull of* A.

For $H \subset C(J,X)$, we define

$$\int_0^t H(s)ds = \left\{ \int_0^t u(s)ds : u \in H \right\} \text{ for } t \in J,$$

where $H(s)=\{u(s) \in X : u \in H\}$.

Lemma 2.6 (See [5]). *If* $H \subset C(J,X)$ *is a bounded, equicontinuous set, then*

$$\alpha(H)=\sup_{t \in J} \alpha(H(t)).$$

Lemma 2.7 (See [20]). *If* $\{u_n\}_{n=1}^{\infty} \subset L^1(J,X)$ *and there exists* $m \in L^1(J,\mathbb{R}^+)$ *such that* $\|u_n(t)\| \leq m(t)$, *a.e.* $t \in J$, *then* $\alpha(\{u_n(t)\}_{n=1}^{\infty})$ *is integrable and*

$$\alpha\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^{\infty}\right) \leq 2\int_0^t \alpha(\{u_n(s)\}_{n=1}^{\infty})ds.$$

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [16]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following axioms:

(A1) If $x : (-\infty,T\,] \longrightarrow X$ is continuous on J and $x_0 \in \mathcal{B}$, then $x_t \in \mathcal{B}$ and x_t is continuous in $t \in J$ and

$$\|x(t)\| \leq C\|x_t\|_{\mathcal{B}}, \tag{2.1}$$

where $C \geq 0$ is a constant.

(A2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{B}} \leq C_1(t) \sup_{s \in [0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}}, \tag{2.2}$$

for $t \in [0,T]$ and x as in $(A1)$.

(A3) The space $\mathcal{B}$ is complete.

Remark 2.8. Condition (2.1) in (A1) is equivalent to $\|\phi(0)\| \leq C\|\phi\|_{\mathcal{B}}$, for all $\phi \in \mathcal{B}$.

Example 2.9 (The phase space $C_r \times L^p(g, X)$)**.** Let $r \geq 0, 1 \leq p < \infty$, and let $g : (-\infty, -r) \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions $(g-5), (g-6)$ in the terminology of [25]. Briefly, this means that g is locally integrable and there exists a nonnegative, locally bounded function Λ on $(-\infty, 0]$, such that $g(\xi + \theta) \leq \Lambda(\xi) g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \backslash N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero.

The space $C_r \times L^p(g, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$, such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ on $(-\infty, -r)$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup_{\theta \in [-r,0]} \|\varphi(\theta)\| + \left(\int_{-\infty}^{-r} g(\theta) \|\varphi(\theta)\|^p d\theta \right)^{\frac{1}{p}}.$$

The space $\mathcal{B} = C_r \times L^p(g, X)$ satisfies axioms **(A1)**, **(A2)**, **(A3)**. Moreover, for $r = 0$ and $p = 2$, this space coincides with $C_0 \times L^2(g, X), H = 1, M(t) = \Lambda(-t)^{\frac{1}{2}}, K(t) = 1 + \left(\int_{-r}^{0} g(\tau) d\tau \right)^{\frac{1}{2}}$, for $t \geq 0$ (see [25, Theorem 1.3.8] for details).

A continuous map $N : D \subseteq Y \to Y$ is said to be a α-contraction if there exists a positive constant $\nu < 1$ such that $\alpha(N(C)) \leq \nu \alpha(C)$ for any bounded closed subset $C \subseteq D$.

Theorem 2.10 (Darbo–Sadovskii, see [5])**.** *If $D \subseteq Y$ is bounded closed and convex, the continuous map $N : D \to D$ is a α-contraction, then the map N has at least one fixed point in D.*

3 Existence of Mild Solutions

In this section, we study the existence of mild solutions for the system (1.1). We give first the definition of the mild solution of the our problem.

Definition 3.1. A function $x : (-\infty, T] \to X$ is said to be a mild solution of (1.1) if $x_0 = \phi, x_{\rho(\tau, x_\tau)} \in \mathcal{B}$ for every $\tau \in J$ and

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ -Q(t)\phi(0) + \int_0^t \int_0^s R(t-s) a(s, \tau) f(\tau, x_{\rho(\tau, x_\tau)}, x(\tau)) d\tau ds, & t \in J, \end{cases}$$

where

$$Q(t) = \int_0^\infty \xi_q(\sigma) S(t^q \sigma) d\sigma, \qquad R(t) = q \int_0^\infty \sigma t^{q-1} \xi_q(\sigma) S(t^q \sigma) d\sigma$$

and ξ_q is a probability density function defined on $(0,\infty)$ such that

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-(\frac{1}{q})}\varpi_q(\sigma^{-\frac{1}{q}}) \geq 0,$$

where

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\sigma^{-qk-1}\frac{\Gamma(kq+1)}{k!}\sin(k\pi q), \quad \sigma \in (0,\infty).$$

Remark 3.2. Note that $\{S(t)\}_{t\geq 0}$ is a uniformly bounded, i.e.,

there exists a constant $M > 0$ such that $\|S(t)\| \leq M$ for all $t \geq 0$.

Remark 3.3. According to [31], direct calculation gives that

$$\|R(t)\| \leq C_{q,M}t^{q-1}, \quad t > 0, \tag{3.1}$$

where $C_{q,M} = \dfrac{qM}{\Gamma(1+q)}$.

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to (-\infty, T]$ is continuous. Additionally, we introduce following hypothesis:

(H_φ) The function $t \to \varphi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.4. The condition (H_φ) is frequently verified by functions continuous and bounded. For more details, see for instance [25].

Remark 3.5. In the rest of this section, C_1^* and C_2^* are the constants

$$C_1^* = \sup_{s\in J} C_1(s) \text{ and } C_2^* = \sup_{s\in J} C_2(s).$$

Lemma 3.6 (See [23]). *If $x : (-\infty,T] \to X$ is a function such that $x_0 = \phi$, then*

$$\|x_s\|_{\mathcal{B}} \leq (C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* \sup\{|y(\theta)|; \theta \in [0, \max\{0,s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t\in\mathcal{R}(\rho^-)} L^\phi(t)$.

Let us introduce the following hypotheses:

(H1) The function $f : J \times \mathcal{B} \times X \longrightarrow X$ is Carathéodory.

(H2) There exists a function $\mu \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to (0, +\infty)$ such that

$$\|f(t, v, w)\| \leq \mu(t)\psi\left(\|v\|_{\mathcal{B}} + \|w\|_X\right), \quad (t, v, w) \in J \times \mathcal{B} \times X.$$

(H3) For each $t \in J$, $a(t, s)$ is measurable on $[0, t]$ and $a(t) = \operatorname{ess\,sup}\{|a(t, s)|, 0 \leq s \leq t\}$ is bounded on J. The map $t \to a_t$ is continuous from J to $L^\infty(J, \mathbb{R})$, here, $a_t(s) = a(t, s)$.

(H4) For any bounded sets $D_1 \subset \mathcal{B}, D_2 \subset X$, and $0 \leq s \leq t \leq T$, there exists an integrable positive function η such that

$$\alpha\left(R(t-s)f(\tau, D_1, D_2)\right) \leq \eta_t(s, \tau)\left(\sup_{0<\theta\leq\tau} \alpha(D_1(\theta)) + \alpha(D_2)\right),$$

where $\eta_t(s, \tau) = \eta(t, s, \tau)$ and $\sup\limits_{t\in J} \displaystyle\int_0^t \int_0^s \eta_t(s, \tau)d\tau ds = \eta^* < \infty$.

Theorem 3.7. *Suppose that the assumptions* (H_φ), $(H1)$–$(H4)$ *hold, and if*

$$4\, a\, \eta^*(1 + C_1^*) < 1, \tag{3.2}$$

then the problem (1.1) *has at least one mild solution on* $(-\infty, T]$.

Proof. Let $\overline{\phi} : (-\infty, T] \longrightarrow X$ be the extension of ϕ to $(-\infty, T]$ such that $\overline{\phi}(\theta) = \phi(0) = 0$ on J. Consider the space $Y = \{u \in C(J, E) : u(0) = \phi(0) = 0\}$ endowed with the uniform convergence topology and define the operator $\Phi : Y \to Y$ by

$$\Phi(x)(t) = -Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s, \tau)f(\tau, \overline{x}_{\rho(\tau, \overline{x}_\tau)}, \overline{x}(\tau))d\tau ds, \ t \in J.$$

Let the set

$$B_r = \{x \in Y : \|x\| \leq r\},$$

where r is any fixed finite real number which satisfies the inequality

$$r \geq M\|\phi\|_{\mathcal{B}} + aC_{q,M}\frac{T^q}{q}\psi\left((C_2^* + L^\phi)\|\phi\|_{\mathcal{B}} + (C_1^* + 1)r\right)\int_0^T \mu(\tau)d\tau. \tag{3.3}$$

Clearly, the subset B_r is closed, bounded, and convex. In order to apply Theorem 2.10, we give the proof in several steps.

Step 1: Φ is continuous. Let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence such that $x^k \to x$ in Y. Then for each $t \in J$, we have

$$\begin{aligned}\|\phi(x^k)(t)-\phi(x)(t)\| &\leq \int_0^t\int_0^s \|R(t-s)\|\|a(s,\tau)\|\|f(\tau,\overline{x}^k_{\rho(\tau,\overline{x}^k_\tau)},\overline{x}^k(\tau))\\ &- f(\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)},\overline{x}(\tau))\|d\tau ds\\ &\leq aC_{q,M}\int_0^t\int_0^s (t-s)^{q-1}\|f(\tau,\overline{x}^k_{\rho(\tau,\overline{x}^k_\tau)},\overline{x}^k(\tau))\\ &- f(\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)},\overline{x}(\tau))\|d\tau ds.\end{aligned}$$

Since f is of Carathéodory type, we have by the Lebesgue dominated convergence theorem that

$$\|\Phi(x^k)(t)-\Phi(x)(t)\| \to 0 \text{ as } k\to\infty.$$

Thus Φ is continuous.

Step 2: Φ maps B_r into itself. If $x \in B_r$, from Lemma 3.6 follows that

$$\|\overline{x}_{\rho(\tau,\overline{x}_\tau)}\|_{\mathcal{B}} \leq (C_2^*+L^\phi)\|\phi\|_{\mathcal{B}} + C_1^* r.$$

For each $x \in B_r$, by (H2) and (3.3), we have for each $t \in J$

$$\begin{aligned}\|\Phi(x)(t)\| &\leq \|-Q(t)\phi(0)\| + \int_0^t\int_0^s \|R(t-s)a(s,\tau)f(\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)},\overline{x}(\tau))\|d\tau ds\\ &\leq M\|\phi\|_{\mathcal{B}} + aC_{q,M}\int_0^t\int_0^s (t-s)^{q-1}\mu(\tau)\psi(\|\overline{x}_{\rho(\tau,\overline{x}_\tau)}\|_{\mathcal{B}}+\|\overline{x}\|)d\tau ds\\ &\leq M\|\phi\|_{\mathcal{B}} + aC_{q,M}\int_0^t\int_0^s (t-s)^{q-1}\mu(\tau)\\ &\times\ \psi\left((C_2^*+L^\phi)\|\phi\|_{\mathcal{B}}+C_1^*r+r\right)d\tau ds\\ &\leq M\|\phi\|_{\mathcal{B}} + aC_{q,M}\frac{T^q}{q}\psi\left((C_2^*+L^\phi)\|\phi\|_{\mathcal{B}}+(C_1^*+1)r\right)\int_0^T \mu(\tau)d\tau\\ &\leq r.\end{aligned}$$

Step 3: $\Phi(B_r)$ is bounded and equicontinuous. By Step 2, it is obvious that $\Phi(B_r) \subset B_r$ is bounded. For the equicontinuity of $\Phi(B_r)$. Set

$$G(\cdot,\overline{x}_{\rho(\cdot,\overline{x}_{(\cdot)})},\overline{x}(\cdot)) = \int_0^{\cdot} a(\cdot,\tau)f(\tau,\overline{x}_{\rho(\tau,\overline{x}_\tau)},\overline{x}(\tau))d\tau.$$

Let $\tau_1,\tau_2 \in J$ with $\tau_1 > \tau_2$, and let $x \in B_r$. Then

$$\|\Phi(x)(\tau_1)-\Phi(x)(\tau_2)\| \leq I_1+I_2+I_3,$$

where

$$
\begin{aligned}
I_1 &= \|Q(\tau_1) - Q(\tau_2)\|\|\phi(0)\| \\
I_2 &= \left\| \int_0^{\tau_2} [R(\tau_1 - s) - R(\tau_2 - s)] G(s, \overline{x}_{\rho(s,\overline{x}_s)}, \overline{x}(s)) ds \right\| \\
I_3 &= \int_{\tau_2}^{\tau_1} \|R(\tau_1 - s)\| \|G(s, \overline{x}_{\rho(s,\overline{x}_s)}, \overline{x}(s))\| ds.
\end{aligned}
$$

I_1 tends to zero as $\tau_2 \to \tau_1$, since $S(t)$ is uniformly continuous operator. For I_2, using (3.1) and (H2), we have

$$
\begin{aligned}
I_2 &\leq \left\| \int_0^{\tau_2} \left[q \int_0^{\infty} \sigma(\tau_1 - s)^{q-1} \xi_q(\sigma) S((\tau_1 - s)^q \sigma) d\sigma \right.\right. \\
&- \left.\left. q \int_0^{\infty} \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) S((\tau_2 - s)^q \sigma) d\sigma \right] G(s, \overline{x}_{\rho(s,\overline{x}_s)}, \overline{x}(s)) ds \right\| \\
&\leq q \int_0^{\tau_2} \int_0^{\infty} \sigma \| [(\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1}] \xi_q(\sigma) S((\tau_1 - s)^q \sigma) \\
&\quad \times G(s, \overline{x}_{\rho(s,\overline{x}_s)}, \overline{x}(s)) \| d\sigma ds \\
&+ q \int_0^{\tau_2} \int_0^{\infty} \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) \| S((\tau_1 - s)^q \sigma) - S((\tau_2 - s)^q \sigma) \| \\
&\quad \times \| G(s, \overline{x}_{\rho(s,\overline{x}_s)}, \overline{x}(s)) \| d\sigma ds \\
&\leq C_{q,M} \int_0^{\tau_2} \left| (\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1} \right| \| G(s, \overline{x}_{\rho(s,\overline{x}_s)}, \overline{x}(s)) \| ds \\
&+ q \int_0^{\tau_2} \int_0^{\infty} \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) \| S((\tau_1 - s)^q \sigma) - S((\tau_2 - s)^q \sigma) \| \\
&\quad \times \| G(s, \overline{x}_{\rho(s,\overline{x}_s)}, \overline{x}(s)) \| d\sigma ds \\
&\leq a\psi \left((C_2^* + L^{\phi}) \|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \right) \|\mu\|_{L^1} \\
&\times \left[C_{q,M} \int_0^{\tau_2} \left| (\tau_1 - s)^{q-1} - (\tau_2 - s)^{q-1} \right| ds \right. \\
&+ \left. q \int_0^{\tau_2} \int_0^{\infty} \sigma(\tau_2 - s)^{q-1} \xi_q(\sigma) \| S((\tau_1 - s)^q \sigma) - S((\tau_2 - s)^q \sigma) \| d\sigma ds \right].
\end{aligned}
$$

Clearly, the first term on the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. From the continuity of $S(t)$ in the uniform operator topology for $t > 0$, The second term on the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. In view of (H2), we have

$$
\begin{aligned}
I_3 &\leq C_{q,M} \int_{\tau_2}^{\tau_1} (\tau_1 - s)^{q-1} \| G(s, \overline{x}_{\rho(s,\overline{x}_s)}, \overline{x}(s)) \| ds \\
&\leq a\, C_{q,M} \psi \left((C_2^* + L^{\phi}) \|\phi\|_{\mathcal{B}} + (C_1^* + 1) r \right) \|\mu\|_{L^1} \int_{\tau_2}^{\tau_1} (\tau_1 - s)^{q-1} ds.
\end{aligned}
$$

As $\tau_2 \to \tau_1$, I_3 tends to zero. So $\Phi(B_r)$ is equicontinuous.

Now let V be a subset of B_r such that $V \subset \overline{\mathrm{conv}}(\Phi(V) \cup \{0\})$. Using Lemmas 2.5–2.7 and $(H4)$, we get

$$\begin{aligned}
\alpha(\Phi V) &= \sup_{t\in J} \alpha(\Phi V(t)) \\
&= \sup_{t\in J} \alpha\left(-Q(t)\phi(0) + \int_0^t \int_0^s R(t-s)a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))d\tau ds\right) \\
&\leq \sup_{t\in J} \alpha\left(-Q(t)\phi(0)\right) \\
&+ \sup_{t\in J} \alpha\left(\left\{\int_0^t \int_0^s R(t-s)a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))d\tau ds\right\}\right) \\
&\leq 2\sup_{t\in J} \int_0^t \alpha\left(\left\{R(t-s)\int_0^s a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))d\tau ds\right\}\right) \\
&\leq 4\sup_{t\in J} \int_0^t \int_0^s \alpha\left(\left\{R(t-s)a(s,\tau)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))d\tau ds\right\}\right) \\
&\leq 4\,a\sup_{t\in J} \int_0^t \int_0^s \alpha\left(\left\{R(t-s)f(\tau, \overline{x}_{\rho(\tau,\overline{x}_\tau)}, \overline{x}(\tau))d\tau ds\right\}\right) \\
&\leq 4\,a\sup_{t\in J} \int_0^t \int_0^s \eta_t(s,\tau)\left[C_1(t)\sup_{0<\mu\leq\tau}\alpha(V(\mu)) + \alpha(V(\tau))\right]d\tau ds \\
&\leq 4\,a(1+C_1^*)\alpha(V)\sup_{t\in J}\int_0^t \int_0^s \eta_t(s,\tau)d\tau ds \\
&\leq 4\,a\,\eta^*(1+C_1^*)\alpha(V).
\end{aligned}$$

By (3.2) it follows that Φ is a α-contraction. According to Theorem 2.10, the operator Φ has at least one fixed point x in B_r. □

4 An Example

In this section we give an example to illustrate the above results. Consider the following integrodifferential model:

$$\begin{aligned}
\frac{\partial^q}{\partial t^q}v(t,\zeta) &= \frac{\partial^2}{\partial\zeta^2}v(t,\zeta) + \int_0^t (t-s)\int_{-\infty}^s \gamma(\tau-s)v(\tau-\rho_1(s)\rho_2(|v(s,\zeta)|),\zeta)d\tau ds \\
&+ \int_0^t (t-s)\frac{s^2}{2}\cos|v(s,\zeta)|ds, \qquad t\in[0,T],\ \zeta\in[0,\pi], \\
v(t,0) &= v(t,\pi) = 0, \qquad t\in[0,T], \\
v(\theta,\zeta) &= \varphi(\theta,\zeta), \qquad \theta\in(-\infty,0],\ \zeta\in[0,\pi],
\end{aligned} \tag{4.1}$$

where $0 < q < 1, \rho_i : [0, +\infty) \to [0, +\infty), i = 1, 2, \gamma : \mathbb{R} \to \mathbb{R}$ are continuous functions, and $\partial^q / \partial t^q = D_t^\alpha$.

Set $X = L^2([0, \pi])$ and define A by

$$D(A) = \{u \in X : u'' \in X, u(0) = u(\pi) = 0\},$$

$$Au = u^{''}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $(S(t))_{t \geq 0}$ on X. For the phase space, we choose $\mathcal{B} = C_0 \times L^2(g, X)$, see Example 2.9 for details.

For $t \in [0, T]$ and $\zeta \in [0, \pi]$, we set

$$x(t)(\zeta) = v(t, \zeta),$$

$$a(t, s) = t - s,$$

$$f(t, \varphi, x(t))(\zeta) = \int_{-\infty}^{0} \gamma(\tau)\varphi(\tau, \zeta) d\tau + \frac{t^2}{2} \cos |x(t)(\zeta)|.$$

$$\rho(t, \varphi) = \rho_1(t)\rho_2(|\varphi(0)|).$$

Under the above conditions, we can represent the system (4.1) in the abstract form (1.1). The following result is a direct consequence of Theorem 3.7.

Proposition 4.1. *Let $\varphi \in \mathcal{B}$ be such that (H_φ) holds, and let $t \to \varphi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then there exists a mild solution of* (4.1).

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