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Using Hybrid Functions to solve a Coupled System of Fredholm Integro-Differential Equations of the Second Kind

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Abstract

This article introduces a numerical method that uses hybrid functions for approximating solutions of systems of Fredholm integro-differential equations of the second kind. This method reduces a system of Fredholm integro-differential equations to a system of algebraic equations and is illustrated by some numerical examples.

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1 Introduction

Many physical phenomena may be modeled by a system of integro-differential equations. Lots of work has been done on nonlinear integro-differential equations using pulse functions and Legendre polynomials, see [5, 7, 9, 10, 12], as well as a recent work using this technique to solve higher dimensional problem, see [1]. Also different methods were used to approximate its solutions such as Chebyshev wavelets method, Galerkin method or the modified decomposition method, see [4, 8, 13, 14].

In this article, we develop a method using hybrid functions

$$b_{km}(t) = b_k(t)p_m\left(\frac{2t - t_{k-1} - t_k}{t_k - t_{k-1}}\right)$$

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on the interval $[0,T)$ defined in terms of a pulse function $b_k(t)_{k=1}^q$ and Legendre polynomials to approximate the solution of a system of Fredholm integro-differential equations of the form

$$\begin{cases} u'(t) + v(t) + \int_0^1 k_1(t,s)u(s)ds = x(t), \\ v'(t) + u(t) + \int_0^1 k_2(t,s)v(s)ds = y(t), \\ u(0) = u_0, \quad v(0) = v_0, \quad t \in [0,1], \end{cases}$$

where $k_1(t,s), k_2(t,s) \in L_2([0,1] \times [0,1])$ and $x(t), y(t) \in L_2([0,1])$ are known functions and $u(t), v(t)$ are unknown functions.

The objective of using hybrid functions in this article is to show that there are important orthogonal basis functions other than those mentioned above that also yield good approximating solutions for systems of integro-differential equations by converting them into systems of linear algebraic equations. Also, new proofs for some properties of hybrid functions will be given.

2 Preliminaries

In this section, we define block pulse and hybrid functions, and recall function approximations in $L_2[-1,1]$.

Definition 2.1. Let $\{b_k(t)\}_{k=1}^q$ be a finite set of block pulse functions [8, 15] on the interval [0, T) defined by

$$b_k(t) = \begin{cases} 1 & \text{if } t_{k-1} \le t < t_k \\ 0 & \text{elsewhere,} \end{cases}$$

where $t_0 = 0$, $t_q = T$ and $[t_{k-1}, t_k) \subset [0,T)$ for $k = 1, 2, \dots, q$.

It follows that for $t \in [0,1)$, $t_{k-1} = (k-1)/q$, $t_k = k/q$ and $T = 1$, we have

$$b_k(t) = \begin{cases} 1 & \text{if } \dfrac{k-1}{q} \le t < \dfrac{k}{q} \\ 0 & \text{elsewhere.} \end{cases}$$

This set of block pulse functions is orthogonal [14], since

$$b_i(t)b_j(t) = \begin{cases} 0 & \text{if } i \ne j, \quad i = 1, 2, \dots, q \\ b_i(t) & \text{if } i = j, \quad j = 1, 2, \dots, q \end{cases}$$

and

$$\langle b_i(t), b_j(t) \rangle = \begin{cases} 0 & \text{if } i \ne j \\ \dfrac{1}{q} & \text{if } i = j. \end{cases}$$

Definition 2.2. We define the hybrid Legendre block pulse functions (or simply hybrid functions) on the interval $[0, T)$ by $b_{km}(t) = b_k(t) p_m\left(\dfrac{2t - t_{k-1} - t_k}{t_k - t_{k-1}}\right)$ or equivalently by

$$b_{km}(t) = \begin{cases} p_m\left(\dfrac{2t - t_{k-1} - t_k}{t_k - t_{k-1}}\right) & \text{if } t_{k-1} \le t < t_k \\ 0 & \text{elsewhere,} \end{cases}$$

where $1 \le k \le q$, $0 \le m \le r - 1$ and $r, q \in \mathbb{N}$, see [8].

If we let $t_k = k/q$, $t_{k-1} = (k-1)/q$ and $T = 1$, then

$$\begin{aligned} p_m\left(\frac{2t - t_{k-1} - t_k}{t_k - t_{k-1}}\right) &= p_m\left(\frac{2t - \frac{k-1}{q} - \frac{k}{q}}{\frac{k}{q} - \frac{k-1}{q}}\right) = p_m\left(\frac{\frac{2tq-(k-1)-k}{q}}{\frac{k-(k-1)}{q}}\right) \\ &= p_m\left(\frac{\frac{2tq-k+1-k}{q}}{\frac{1}{q}}\right) = p_m(2qt - 2k + 1), \end{aligned}$$

and hence we have the hybrid function, $t \in [0, 1)$,

$$b_{km}(t) = \begin{cases} p_m(2qt - 2k + 1) & \text{if } \dfrac{k-1}{q} \le t < \dfrac{k}{q} \\ 0 & \text{elsewhere.} \end{cases}$$

Function Approximation in $L_2[-1, 1]$

Consider the orthogonal set of hybrid functions

$$M = \{b_{km}(t) : \ 0 \le m \le r - 1, \quad 1 \le k \le q\}$$

in the Hilbert space $L_2[-1, 1]$. Then any function $f \in L_2[-1, 1]$ may be approximated arbitrary close by a (finite or possibly infinite) linear combination of elements of M (i.e., f may be expanded to a hybrid function). Thus, $f(t) \simeq \sum_{k=1}^{+\infty} \sum_{m=0}^{\infty} f_{km} b_{km}(t)$, where

$$f_{km} = \frac{\langle f(t), b_{km}(t)\rangle}{\langle b_{km}(t), b_{km}(t)\rangle}.$$

Let f be an arbitrary function in $L_2[-1, 1]$ and

$$\begin{aligned} Y = \operatorname{span}\{&b_{10}(t), b_{11}(t), \ldots, b_{1(r-1)}(t), b_{20}(t), b_{21}(t), \ldots, b_{2(r-1)}(t), \\ &\ldots, b_{q0}(t), \ldots, b_{q(r-1)}(t)\} \end{aligned}$$

be a finite dimensional vector space [6]. Then f has the unique best approximation f_0 in Y in the sense that for any $y \in Y$, we have $\|f - f_0\|_2 \le \|f - y\|_2$. Since

$f_0 \in Y$, there exist real numbers (coefficients) $c_{10}, c_{20}, \ldots, c_{q(r-1)}$ such that f can be uniquely approximated by $f \simeq f_0 = \sum_{m=0}^{r-1} \sum_{k=1}^{q} c_{km} b_{km}(t)$, where the coefficients $c_{km} = \dfrac{\langle f, b_{km}(t)\rangle}{\langle b_{km}(t), b_{km}(t)\rangle}$ are determined by

$$
\begin{aligned}
\langle f, b_{km}(t)\rangle &= \left\langle \sum_{i=0}^{r-1} \sum_{j=1}^{q} c_{ij} b_{ij}(t), b_{km}(t) \right\rangle \\
&= \sum_{i=0}^{r-1} \sum_{j=1}^{q} c_{ij} \langle b_{ij}(t), b_{km}(t)\rangle \\
&= c_{km} \langle b_{km}(t), b_{km}(t)\rangle .
\end{aligned}
$$

3 New Proofs

Now, we generalize the concept of hybrid functions [10, 11] and prove that they can be extended to properly handle coupled systems of Fredholm integro-differential equations of the second kind. These properties are fundamental for establishing the main results of this paper. This will be done by using some properties of Legendre polynomials. Let $B(t) = \left(B_1^T(t), \ldots, B_q^T(t)\right)^T$ be a vector function of hybrid functions on $[0, 1)$, where

$$
B_i(t) = \left(b_{i0}(t), \ldots, b_{i(r-1)}(t)\right)^T, \quad i = 1, 2, \ldots, q.
$$

Proposition 3.1 (Operational matrix of integration)**.** *The integration of the vector function $B(t)$ may be approximated by $\int_0^t B(s)ds \simeq PB(t)$, where P is an $rq \times rq$ matrix [8, 11], known as the operation matrix for hybrid functions and given by*

$$
P = \begin{pmatrix}
E & H & H & \cdots & H \\
0 & E & H & \cdots & H \\
0 & 0 & E & \ddots & \vdots \\
 & \vdots & & \ddots & H \\
0 & 0 & 0 & \cdots & E
\end{pmatrix},
$$

where H and E are $r \times r$ matrices defined by

$$
H = \frac{1}{q} \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
 & \vdots & & \ddots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

and

$$E = \frac{1}{2q}\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2r-3} & 0 & \frac{1}{2r-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2r-1} & 0 \end{pmatrix}.$$

Proof. Since $(2m+1)p_m(t) = p'_{m+1}(t) - p'_{m-1}(t)$, see [3], we get

$$\begin{aligned} \int_0^t b_{km}(s)ds &= \underbrace{\int_0^{\frac{k-1}{q}} b_{km}(s)ds}_{0} + \int_{\frac{k-1}{q}}^t b_{km}(s)ds \\ &= \int_{\frac{k-1}{q}}^t b_{km}(s)ds = \int_{\frac{k-1}{q}}^t p_m(2qs-2k+1)ds. \end{aligned}$$

We now discuss the different cases according to the values of t: First, we discuss the case that $(k-1)/q \le t < k/q$. Two situations will be considered: If $m \neq 0$, then

$$\begin{aligned} \int_0^t b_{km}(s)ds &= \int_{\frac{k-1}{q}}^t b_{km}(s)ds \\ &= \int_{\frac{k-1}{q}}^t p_m(2qs-2k+1)ds \\ &= \int_{\frac{k-1}{q}}^t \frac{1}{2q\,(2m+1)}\left(p'_{m+1}(2qs-2k+1) - p'_{m-1}(2qs-2k+1)\right) ds \\ &= \frac{1}{2q\,(2m+1)}\left(p_{m+1}(2qt-2k+1) - p_{m-1}(2qt-2k+1)\right) \\ &= \frac{1}{2q\,(2m+1)}\left(b_{k(m+1)}(t) - b_{k(m-1)}(t)\right). \end{aligned}$$

If $m = 0$, then

$$\int_0^t b_{k0}(s)ds = \int_{\frac{k-1}{q}}^t b_{k0}(s)ds = \int_{\frac{k-1}{q}}^t p_0(2qs-2k+1)ds.$$

According to [3], we have $p_0(t) = p'_1(t) + p'_0(t)$. Then

$$p_0(2qs-2k+1) = \frac{1}{2q}\left(p'_1(2qs-2k+1) + p'_0(2qs-2k+1)\right)$$

and hence

$$\begin{aligned}\int_0^t b_{k0}(s)ds &= \int_{\frac{k-1}{q}}^t \frac{1}{2q}\left(p_1'(2qs-2k+1)+p_0'(2qs-2k+1)\right)\\ &= \frac{1}{2q}\left(p_1(2qt-2k+1)+p_0(2qt-2k+1)\right)\\ &= \frac{1}{2q}\left(b_{k1}(t)+b_{k0}(t)\right).\end{aligned}$$

Second, we discuss the case that $t \geq k/q$. Again two situations will be considered: If $m \neq 0$, then we similarly get

$$\int_0^t b_{km}(s)ds = \underbrace{\int_0^{\frac{k-1}{q}} b_{km}(s)ds}_{0} + \int_{\frac{k-1}{q}}^{\frac{k}{q}} b_{km}(s)ds + \underbrace{\int_{\frac{k}{q}}^t b_{km}(s)\,ds}_{0}$$

$$\begin{aligned}&= \int_{\frac{k-1}{q}}^{\frac{k}{q}} p_m(2qs-2k+1)ds\\ &= \frac{1}{2q\,(2m+1)}\int_{\frac{k-1}{q}}^{\frac{k}{q}} \left(p_{m+1}'(2qs-2k+1)-p_{m-1}'(2qs-2k+1)\right)ds.\end{aligned}$$

Now, let $u = 2qs-2k+1$. Then

$$\begin{aligned}\int_0^t b_{km}(s)ds &= \frac{1}{(2m+1)(2q)^2}\int_{-1}^1 \left(p_{m+1}'(u)-p_{m-1}'(u)\right)du\\ &= \frac{1}{(2m+1)(2q)^2}(p_{m+1}(u)|_{-1}^1 - p_{m-1}(u)|_{-1}^1)\\ &= \frac{1}{(2m+1)(2q)^2}[1-(-1)^{m+1}-(1-(-1)^{m-1})]\\ &= 0.\end{aligned}$$

If $m=0$, then

$$\int_0^t b_{k0}(s)ds = \int_{\frac{k-1}{q}}^{\frac{k}{q}} b_{k0}(s)ds = \int_{\frac{k-1}{q}}^{\frac{k}{q}} 1ds = \frac{1}{q}.$$

Thus, in summary, we have

$$\int_0^t b_{km}ds = \begin{cases} 0 & \text{if } m\in\mathbb{N}\cup\{0\},\ 0\leq t<\dfrac{k-1}{q}\\ \dfrac{b_{k1}(t)+b_{k0}(t)}{2q} & \text{if } m=0,\ \dfrac{k-1}{q}\leq t<\dfrac{k}{q}\\ \dfrac{b_{k(m+1)}(t)-b_{k(m-1)}(t)}{2q(2m+1)} & \text{if } m\neq 0\\ \dfrac{1}{q} & \text{if } m=0,\ \dfrac{k}{q}\leq t<1\\ 0 & \text{if } m\neq 0.\end{cases}$$

Now, consider $B_k(s) = \big(b_{k0}(s), b_{k1}(s), \ldots, b_{k(r-1)}(s)\big)^T$. Then

$$\begin{aligned}
\int_0^t B_k(s)ds &= \left(\int_0^t b_{k0}(s)ds, \int_0^t b_{k1}(s)ds, \ldots, \int_0^1 b_{k(r-1)}(s)ds\right)^T \\
&= \begin{cases}
(0,0,\ldots,0)^T \text{ if } 0 \le t < \dfrac{k-1}{q} \\
\left(\dfrac{b_{k1}(t)+b_{k0}(t))}{2q}, \dfrac{b_{k2}(t)-b_{k0}(t)}{2q\cdot 3}, \ldots, \dfrac{-b_{k(r-2)}(t)}{2q(2r-1)}\right)^T \\
\qquad \text{if } \dfrac{k-1}{q} \le t < \dfrac{k}{q} \\
\left(\dfrac{1}{q}b_{l0}(t), 0, \ldots, 0\right)^T \text{ if } \dfrac{l-1}{q} \le t < \dfrac{l}{q},\ l = k+1, \ldots, q
\end{cases} \\
&= \begin{cases}
0_{r\times r}B_k(t) & \text{if } 0 \le t < \dfrac{k-1}{q} \\
EB_k(t) & \text{if } \dfrac{k-1}{q} \le t < \dfrac{k}{q} \\
HB_l(t) & \text{if } \dfrac{l-1}{q} \le t < \dfrac{l}{q},\ l = k+1, \ldots, q
\end{cases} \\
&= EB_k(t) + \sum_{l=k+1}^{q} HB_l(t) \text{ for all } 0 \le t < 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^t B(s)ds &= \begin{pmatrix} \int_0^t B_1(s)ds \\ \int_0^t B_2(s)ds \\ \vdots \\ \int_0^t B_q(s)ds \end{pmatrix} \\
&= \begin{pmatrix} EB_1(t) + HB_2(t) + HB_3(t) + HB_3(t) + \ldots + HB_q(t) \\ EB_2(t) + HB_3(t) + \ldots + HB_q(t) \\ \vdots \\ EB_q(t) \end{pmatrix} \\
&= \begin{pmatrix} E & H & H & \cdots & H \\ 0 & E & H & \cdots & H \\ 0 & 0 & E & \ddots & H \\ \vdots & \vdots & \ddots & \ddots & H \\ 0 & 0 & 0 & 0 & E \end{pmatrix} B(t).
\end{aligned}$$

This concludes the proof. □

Proposition 3.2 (The integration of two hybrid functions)**.** *The integration of two hybrid Legendre block pulse function vectors, see [7], is* $L = \int_0^1 B(t)B^T(t)dt$*, where* $L = \operatorname{diag}(D, D, \ldots, D)$ *is an* $rq \times rq$ *diagonal matrix and* $D = \frac{1}{q}\left(1, \frac{1}{3}, \ldots, \frac{1}{2r-1}\right)$ *is an* $r \times r$ *diagonal matrix for* $r \in \mathbb{N}$.

Proof. Consider

$$B(t) = (B_1^T, B_2^T, B_3^T, \ldots, B_q^T)^T, \text{ where } B_i(t) = \left(b_{i0}(t), b_{i1}(t), \ldots, b_{i(r-1)}(t)\right)^T.$$

Then

$$B(t)B^T(t) = \begin{pmatrix} b_{10}b_{10} & b_{10}b_{11} & \cdots & b_{10}b_{1(r-1)} & b_{10}b_{20} & \cdots & b_{10}b_{q(r-1)} \\ b_{11}b_{10} & b_{11}b_{11} & \cdots & b_{11}b_{1(r-1)} & b_{11}b_{20} & \cdots & b_{11}b_{q(r-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{1(r-1)}b_{10} & b_{1(r-1)}b_{11} & \cdots & b_{1(r-1)}b_{1(r-1)} & b_{1(r-1)}b_{20} & \cdots & b_{1(r-1)}b_{q(r-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{q(r-1)}b_{10} & b_{q(r-1)}b_{11} & \cdots & b_{q(r-1)}b_{1(r-1)} & b_{q(r-1)}b_{20} & \cdots & b_{q(r-1)}b_{q(r-1)} \end{pmatrix}$$

and hence

$$\int_0^1 B(t)B^T(t)dt = \begin{pmatrix} \int_0^1 b_{10}(t)b_{10}(t)dt & \cdots & \int_0^1 b_{10}(t)b_{q(r-1)}(t)dt \\ \int_0^1 b_{11}(t)b_{10}(t)dt & \cdots & \int_0^1 b_{11}(t)b_{q(r-1)}(t)dt \\ \vdots & \vdots & \vdots \\ \int_0^1 b_{1(r-1)}(t)b_{10}(t)dt & \cdots & \int_0^1 b_{1(r-1)}(t)b_{q(r-1)}(t)dt \\ \vdots & \vdots & \vdots \\ \int_0^1 b_{q(r-1)}(t)b_{10}(t)dt & \cdots & \int_0^1 b_{q(r-1)}(t)b_{q(r-1)}(t)dt \end{pmatrix}.$$

Thus,

$$L = \int_0^1 B(t)B^T(t)dt$$

$$= \begin{pmatrix} \langle b_{10}, b_{10}\rangle & \cdots & \langle b_{10}, b_{1(r-1)}\rangle & \cdots & \langle b_{10}, b_{q(r-1)}\rangle \\ \langle b_{11}, b_{10}\rangle & \cdots & \langle b_{11}, b_{1(r-1)}\rangle & \cdots & \langle b_{11}, b_{q(r-1)}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle b_{1(r-1)}, b_{10}\rangle & \cdots & \langle b_{1(r-1)}, b_{1(r-1)}\rangle & \cdots & \langle b_{1(r-1)}, b_{q(r-1)}\rangle \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \langle b_{q(r-1)}, b_{10}\rangle & \cdots & \langle b_{q(r-1)}, b_{1(r-1)}\rangle & \cdots & \langle b_{q(r-1)}, b_{q(r-1)}\rangle \end{pmatrix}.$$

Consequently, $L = \left(d_{km}^{ij}\right)$, where $d_{km}^{ij} = \langle b_{km}, b_{ij}\rangle$ with $k, i = 1, 2, \ldots, q$ and $j = 0, 1, 2, \ldots, r-1$. Hence, there are two cases to be considered: First, if $k \neq i$, then $d_{km}^{ij} = 0$. Second, if $k = i$, then

$$\begin{aligned} d_{km}^{ij} &= \langle b_{km}, b_{ij}\rangle \\ &= \int_0^1 b_{km}(t) b_{ij}(t)dt = \int_0^1 b_k(t) p_m(2qt-2k+1) p_j(2qt-2k+1)dt \\ &= \int_{\frac{k-1}{q}}^{\frac{k}{q}} p_m(2qt-2k+1) p_j(2qt-2k+1)dt \\ &= \frac{1}{2q}\int_{-1}^1 p_m(s)p_j(s) = \frac{1}{2q}\left(\frac{2}{2m+1}\right)\delta_{mj}, \text{ where } m, j = 0, 1, \ldots, r-1. \end{aligned}$$

Therefore,

$$d_{km}^{ij} = \begin{cases} 0 & \text{if } k \neq i \text{ or } m \neq j \\ \dfrac{1}{q}\left(\dfrac{1}{2m+1}\right) & \text{if } k = i \text{ and } m = j \end{cases}$$

and hence $L = \text{diag}(D, D, \ldots, D)$ is an $rq \times rq$ matrix, where

$$\begin{aligned} D &= \begin{pmatrix} \langle b_{i0}, b_{i0}\rangle & \langle b_{i0}, b_{i1}\rangle & \cdots & \langle b_{i0}, b_{i(r-1)}\rangle \\ \langle b_{i1}, b_{i0}\rangle & \langle b_{i1}, b_{i1}\rangle & \cdots & \langle b_{i1}, b_{i(r-1)}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle b_{i(r-1)}, b_{i0}\rangle & \langle b_{i(r-1)}, b_{i1}\rangle & \cdots & \langle b_{i(r-1)}, b_{i(r-1)}\rangle \end{pmatrix} \\ &= \frac{1}{q}\text{diag}\left(1, \frac{1}{3}, \ldots, \frac{1}{2r-1}\right) \end{aligned}$$

is an $r \times r$ matrix for all $i = 1, 2, \ldots, q$. This concludes the proof. □

Now, we will clearly present the approximation of $k(t, s)$ in $L_2([0,1] \times [0,1])$ and provide a new proof for it. Let $B_{(i)}(t)$ denote the ith component of $B(t)$ and $B_{(j)}(s)$ denote the j^{th} component of $B(s)$.

Approximation of $k(t, s)$ in $L_2([0,1] \times [0,1])$

Now we will approximate the function $k(t, s)$ in $L_2([0,1] \times [0,1])$ as

$$k(t, s) \simeq B^T(t) G B(s),$$

[11], where G is an $rq \times rq$ matrix such that

$$g_{ij} = \frac{\langle B_{(i)}(t), \langle k(t,s), B_{(j)}(s)\rangle\rangle}{\langle B_{(i)}(t), B_{(i)}(t)\rangle\langle B_{(j)}(s), B_{(j)}(s)\rangle},\ i,j = 1,2,\ldots,rq.$$

Proof. Consider

$$k(t,s) \simeq \sum_{l=1}^{q}\sum_{d=1}^{q}\sum_{k=0}^{r-1}\sum_{m=0}^{r-1} b_{lk}(t)b_{dm}(s)c_{ld}^{km},$$

where c_{ld}^{km} is a real coefficient to be determined for all $l,d = 1,2,\ldots,q$ and $k,m = 0,1,\ldots,r-1$. Let $B_{(i)}(t) = b_{au}(t)$ be the ith component of $B(t)$ and $B_{(j)}(s) = b_{vp}(s)$ be the jth component of $B(s)$, $a,v = 1,2,\ldots,q$ and $u,p = 0,1,\ldots,r-1$. Then

$$\begin{aligned}\langle k(t,s), B_{(j)}(s)\rangle &= \left\langle \sum_{l=1}^{q}\sum_{d=1}^{q}\sum_{k=0}^{r-1}\sum_{m=0}^{r-1} b_{lk}(t)b_{dm}(s)c_{ld}^{km}, B_{(j)}(s)\right\rangle \\ &= \sum_{l=1}^{q}\sum_{k=0}^{r-1} c_{lv}^{kp} b_{lk}(t)\langle B_{(j)}(s), B_{(j)}(s)\rangle.\end{aligned}$$

Now,

$$\begin{aligned}\langle B_{(i)}(t), \langle k(t,s), B_{(j)}(s)\rangle\rangle &= \left\langle B_{(i)}(t), \sum_{l=1}^{q}\sum_{k=0}^{r-1} c_{lv}^{kp} b_{lk}(t)\langle B_{(j)}(s), B_{(j)}(s)\rangle\right\rangle \\ &= \left\langle B_{(i)}(t), \sum_{l=1}^{q}\sum_{k=0}^{r-1} c_{lv}^{kp} b_{lk}(t)\right\rangle \langle B_{(j)}(s), B_{(j)}(s)\rangle \\ &= c_{av}^{up}\langle B_{(i)}(t), B_{(i)}(t)\rangle\langle B_{(j)}(s), B_{(j)}(s)\rangle.\end{aligned}$$

Thus, we conclude that the real coefficient c_{av}^{up} in $\sum_{l=1}^{q}\sum_{d=1}^{q}\sum_{k=0}^{r-1}\sum_{m=0}^{r-1} b_{lk}(t)b_{dm}(s)c_{ld}^{km}$ that is multiplied by $b_{au}(t)$ (the ith component $B_{(i)}(t)$ of $B(t)$) and $b_{vp}(s)$ (the jth component $B_{(j)}(s)$ of $B(s)$) has the form

$$c_{av}^{up} = \frac{\langle B_{(i)}(t), \langle k(t,s), B_{(j)}(s)\rangle\rangle}{\langle B_{(i)}(t), B_{(i)}(t)\rangle\langle B_{(j)}(s), B_{(j)}(s)\rangle},\ i,j = 1,2,\ldots,rq.$$

Thus,

$$\sum_{l=1}^{q}\sum_{d=1}^{q}\sum_{k=0}^{r-1}\sum_{m=0}^{r-1} b_{lk}(t)b_{dm}(s)c_{ld}^{km} = \sum_{i=1}^{rq}\sum_{j=1}^{rq} g_{ij}B_{(i)}(t)B_{(j)}(s),$$

where

$$g_{ij} = \frac{\langle B_{(i)}(t), \langle k(t,s), B_{(j)}(s)\rangle\rangle}{\langle B_{(i)}(t), B_{(i)}(t)\rangle\langle B_{(j)}(s), B_{(j)}(s)\rangle},\ i,j = 1,2,\ldots,rq.$$

Moreover,

$$\sum_{i=1}^{rq}\sum_{j=1}^{rq} g_{ij}B_{(i)}(t)B_{(j)}(s) = B^T(t)GB(s),$$

where $G = (g_{ij})$, $i, j = 1, 2, \ldots, rq$ is an $rq \times rq$ matrix. Thus

$$k(t,s) \simeq \sum_{l=1}^{q}\sum_{d=1}^{q}\sum_{k=0}^{r-1}\sum_{m=0}^{r-1} b_{lk}(t)b_{dm}(s)c_{ld}^{km} = B^T(t)GB(s).$$

This concludes the proof. □

4 Main Results

In [13], a system of integro-differential equations was approximated using the modified decomposition method, and in [2], a similar system was approximated using the approximation method. We now consider a system of Fredholm integro-differential equations of the form

$$\begin{cases} u'(t) + v(t) + \displaystyle\int_0^1 k_1(t,s)u(s)ds = x(t), \\ v'(t) + u(t) + \displaystyle\int_0^1 k_2(t,s)v(s)ds = y(t), \\ u(0) = u_0, \quad v(0) = v_0, \quad t \in [0,1], \end{cases}$$

where $k_1(t,s), k_2(t,s) \in L_2([0,1]\times[0,1])$ and $x(t), y(t) \in L_2([0,1])$ are known functions while $u(t), v(t)$ are unknown functions, and we propose solving it by using the following approximations

$$\begin{aligned} u(t) &\simeq (U)^T B(t) = B^T(t)U, \\ u'(t) &\simeq (U')^T B(t) = B^T(t)U', \\ v(t) &\simeq (V)^T B(t) = B^T(t)V, \\ v'(t) &\simeq (V')^T B(t) = B^T(t)V', \\ k_1(t,s) &\simeq B^T(t)G_1B(s), \\ k_2(t,s) &\simeq B^T(t)G_2B(s). \end{aligned}$$

Also, by the fundamental theorem of calculus, [7], we have

$$u(t) = \int_0^t u'(s)ds + u(0).$$

Substituting $u(t)$, $u'(t)$ and $u_0(t)$ in the above equation, we get

$$U^T B(t) \simeq \int_0^t U'^T B(s)ds + U_0^T B(t)$$

$$\begin{aligned} &\simeq U'^T \int_0^t B(s)ds + U_0^T(t)B(t) \\ &\simeq U'^T PB(t) + U_0^T B(t) \simeq (U'^T P + U_0^T)B(t). \end{aligned}$$

Thus,

$$U^T = U'^T P + U_0^T \text{ and so } U' = (P^T)^{-1}(U - U_0).$$

Substituting the approximated functions in the above system, we get

$$\begin{cases} (U')^T B(t) + (V)^T B(t) + \int_0^1 B^T(t) G_1 B(s) B^T(s) U ds = (X)^T B(t), \\ (V')^T B(t) + (U)^T B(t) + \int_0^1 B^T(t) G_2 B(s) B^T(s) V ds = (Y)^T B(t). \end{cases}$$

Hence,

$$\begin{cases} (U')^T + (V)^T + (G_1 LU)^T = (X)^T, \\ (V')^T + (U)^T + (G_2 LU)^T = (Y)^T. \end{cases}$$

Therefore

$$\begin{cases} U' + V + G_1 LU = X, \\ V' + U + G_2 LU = Y. \end{cases}$$

Thus

$$\begin{cases} U - U_0 + P^T V + p^T G_1 LU = P^T X, \\ V - V_0 + P^T U + P^T G_2 LU = P^T Y, \end{cases}$$

and hence

$$\begin{cases} (I + P^T G_1 L)U = P^T X - P^T V + U_0, \\ V - V_0 + P^T U + P^T G_2 LU = P^T Y. \end{cases}$$

After some calculations, we get

$$\begin{aligned} V = &\left(I - P^T(I + P^T G_1 L)^{-1} P^T + P^T G_2 L\right)^{-1} \\ &\times \left(P^T Y - P^T(I + P^T G_1 L)^{-1} U_0 - P^T(I + P^T G_1 L)^{-1} P^T X\right) \end{aligned}$$

and

$$U = (I + P^T G_1 L)^{-1}(P^T X - P^T Y + U_0).$$

Using $u(t) \simeq U^T B(t)$ and $v(t) \simeq V^T B(t)$, we get the approximated solution.

5 Numerical Examples

In this section, the method introduced above will be numerically applied to solve two systems.

Example 5.1. Consider the system of Fredholm integro-differential equations

$$\begin{cases} u'(t) + v(t) + \displaystyle\int_0^1 (s+1)\sin(t)u(s)ds = -\frac{1}{(1+t)^2} + \mathrm{e}^{t-1} + \sin(t), \\ v'(t) + u(t) + \displaystyle\int_0^1 \mathrm{e}^{t-s+1} v(s)ds = \mathrm{e}^{t-1} + \mathrm{e}^{t} + \frac{1}{1+t}, \end{cases}$$

$t \in [0,1]$, whose exact solution is given by $u(t) = \dfrac{1}{1+t}$ and $v(t) = \mathrm{e}^{t-1}$. For $q = 1$ and $r = 2$, we have the following approximations:

$$\begin{aligned} u(t) &\approx \frac{17209962558249181}{18014398509481984} - \frac{3697206315417981}{9007199254740992}t, \\ v(t) &\approx \frac{1567108479986707}{9007199254740992} + \frac{784906517646295}{1125899906842624}t. \end{aligned}$$

A comparison of approximate solutions versus exact solutions with an L_2-norm of the error is given in Table 5.1.

Table 5.1: Approximate versus exact solutions in L_2-norm error for Example 5.1

t	Approximate sol. of $u(t)$ $q=1,\ r=2$	Exact sol. of $u(t)$	L_2-norm error $u(t)$	Approximate sol. of $v(t)$ $q=1,\ r=2$	Exact sol. of $v(t)$	L_2-norm error $v(t)$
0.1	0.9143	0.9091	0.0889	0.2437	0.4060	0.1852
0.2	0.8733	0.8333	0.0843	0.3134	0.4490	0.1871
0.3	0.8322	0.7692	0.0854	0.3831	0.4960	0.1873
0.4	0.7912	0.7143	0.0906	0.4528	0.5480	0.1863
0.5	0.7501	0.6667	0.0980	0.5226	0.6060	0.1841
0.6	0.7091	0.6250	0.1063	0.5923	0.6700	0.1808
0.7	0.6680	0.5882	0.1149	0.6620	0.7400	0.1764
0.8	0.6270	0.5556	0.1233	0.7317	0.8180	0.1712
0.9	0.5859	0.5263	0.1315	0.8014	0.9040	0.1652
1	0.5449	0.5000	0.1389	0.8711	1.0000	0.1587

Example 5.2. Consider the system of Fredholm integro-differential equations

$$\begin{cases} u'(t) + v(t) + \displaystyle\int_0^1 (s+1)u(s)ds = t + \frac{10}{3}, \\ v'(t) + u(t) + \displaystyle\int_0^1 (s+1)tv(s)ds = 2 + \frac{11}{6}t, \end{cases}$$

$t \in [0,1]$, whose exact solutions are $u(t) = t + 1$ and $v(t) = t$. For $q = 1$ and $r = 2$, we have the following approximations:

$$u(t) \approx \frac{35757158052806621}{36028797018963968} + \frac{34995271409874301}{18014398509481984}t - \frac{20013428180405163}{18014398509481984}t^2,$$

$$v(t) \approx \frac{710517630996031}{72057594037927936} + \frac{36164191621359827}{36028797018963968}t - \frac{4899263276533599}{36028797018963968}t^2.$$

A comparison of approximate solutions versus exact solutions with an L_2-norm error is given in Table 5.2.

Table 5.2: Approximate versus exact solutions with L_2-norm error for Example 5.2

t	Approximate sol. of $u(t)$ $q=1,\ r=2$	Exact sol. of $u(t)$	L_2-norm error $u(t)$	Approximate sol. of $v(t)$ $q=1,\ r=2$	Exact sol. of $v(t)$	L_2-norm error $v(t)$
0.1000	1.2685	1.1000	0.0137	0.1100	0.1000	0.0030
0.2000	1.3500	1.2000	0.0346	0.1986	0.2000	0.0023
0.3000	1.4315	1.3000	0.0500	0.2872	0.3000	0.0009
0.4000	1.5130	1.4000	0.0586	0.3758	0.4000	0.0020
0.5000	1.5945	1.5000	0.0603	0.4644	0.5000	0.0052
0.6000	1.6760	1.6000	0.0551	0.5529	0.6000	0.0094
0.7000	1.7576	1.7000	0.0429	0.6415	0.7000	0.0144
0.8000	1.8391	1.8000	0.0242	0.7301	0.8000	0.0203
0.9000	1.9206	1.9000	0.0092	0.8187	0.9000	0.0271
1.0000	2.0021	2.0000	0.0381	0.9073	1.0000	0.0347

6 Conclusion

This paper shows that hybrid functions may be also used to effectively approximate solutions of systems of Fredholm integro-differential equations.

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