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A Method for Solving Nonlinear Volterra Integral Equations

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Abstract

It is known that to construct the stable multistep method with the higher order of accuracy for solving integral equation is actual. For this aim here we suggest some ways for the construction of hybrid methods for solving nonlinear Volterra integral equations of the second kind. Thus, foundational this extends stable hybrid method with higher order of accuracy. Note that the hybrid methods which has been constructed here guarantee the minimal calculation of the kernel of the integral in the Volterra integral equation. Also the concrete methods with the degree $p = 4$, $p = 5$ and $p = 6$ for two mesh point has been suggested. As a consequence of the given algorithm the hybrid methods have some preference.

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1 Introduction

Many scientists for solving integral equations, used methods from the theory of numerical methods for solving ordinary differential equations. As it is known, there is a wide arsenal of numerical methods for solving ordinary differential equations, each of which has its own advantages and disadvantages. One of the classical methods for solving differential equations is the Runge–Kutta and Adams methods, which developed from Euler's method in different directions (constructions of one-step and multistep methods (see for example [11, 17])). Scientists in the middle of the XX century decided that giving an advantage to one of these methods is not correct, so they decided to construct

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methods with better properties of these methods, which are called hybrids. The first hybrid methods of Runge–Kutta type are constructed in [6] and Adams type in [2] and [4]. The advantage of hybrid methods is shown in many papers of different authors (see, for example [5, 7]). But the authors of [8] investigated the relation between one and multistep methods and gave a way to construct multistep methods by using one step and vice versa construction of one step method by using a multistep. Here, we take into consideration the advantage of hybrid methods, by using them as an application in solving Volterra integral equations. Also the comparison of the suggested methods with well-known ones has been considered. By this purpose we tried to explain these methods by chronological way.

Consider solving a nonlinear Volterra integral equation of the second kind, which has the following form:

$$y(x) = g(x) + \int_{x_0}^{x} K(x,\, s,\, y(s))ds, x_0 \le s \le x \le X. \tag{1.1}$$

Sometimes correlation (1.1) is called the equation of Volterra–Uryson. Under the assumption that equation (1.1) has a unique continuous solution, defined on the segment $[x_0, X]$ we consider the determination of its approximate values at the mesh points, defined as: $x_i = x_0 + ih$ $(i = 0, 1, ..., N)$. Here the quantity $h > 0$ is a step size dividing the segment $[x_0, X]$ to N equal parts.

V. Volterra thoroughly investigated equation (1.1) in the case when the kernel of integral is linear function on y, i.e., $K(x,\, z,\, y) \equiv b(x,\, z)y$. He also described a wide range of applications of integral equations with variable boundary, which is one of the most important factors in the development of the theory of integral equations. Naturally, V. Volterra constructed a method for the numerical solution of integral equations and for this purpose used the quadrature formula. Note that the method of quadrature has been successfully applied to the solution of equation (1.1), up to this day. The basic idea in the construction of quadrature methods, is to replace some of the integral with the integral sum, which in the simplest case is the following:

$$\vartheta(x_n) = \int_{x_0}^{x_n} K(x_n,\, s,\, y(s))ds \approx h \sum_{j=0}^{n} a_j K(x_n,\, x_j,\, y_j), \tag{1.2}$$

where the quantities a_j $(j = 0,\, 1,\, 2, ...,\, n)$ are the coefficients of the quadrature formula. It is easy to make the transition from the mesh point x_n to the next point x_{n+1} (to calculate $\vartheta(x_{n+1})$) the integral sum is computed again, because the value $K(x_n,\, x_j,\, y_j)$ is not replaced by the value $K(x_{n+1},\, x_j,\, y_j)$, consequently in the calculation of the quantity $\vartheta(x_{n+1})$ the value $\vartheta(x_n)$ is not used and this is the main lack of the method of quadratures. For the dispensation of this lack of quadrature methods, the author of [10] suggested a method that provides the constancy volume of computational operation at each mesh point. These methods are reminiscent of multistep methods with constant coefficients, but they have some properties which are not extrinsic to multistep methods. To construct methods with high accuracy in [16], the proposed multistep forward-jumping method with the second derivative was used. Multistep methods such as forward-jumping methods have some advantages and disadvantages. To solve this problem, the authors of [9] suggested a predictor-corrector scheme, which extends the region of stability of forward-jumping methods. To construct methods with improved properties, scientists investigated the numerical solution of equation (1) by using Runge–Kutta and collocation method, spline functions, etc. (see, for example [1, 12]). In [18], the application of different methods for the solution of equation (1.1) is examined, and in [13], by using some advantages of hybrid methods, their application to the solution of the Volterra integral equation was investigated. This hybrid method, constructed by Makroglou can be obtained from the following method as a special case:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i+\nu_i} \quad (|\nu_i| < 1,\ i = 0, 1, 2, \ldots, k). \tag{1.3}$$

But the authors of [15], considered the application of method (1.3) to the solution of equation (1.1). Here we investigate the application of the following method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h \sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i}, \tag{1.4}$$

to the solution of the integral equation (1.1). Now, consider the construction methods for the solution of equation (1.1) by using formula (1.4).

2 Construction of Hybrid Methods

As noted above, we attempt to apply to the solution of equation (1.1) some of the methods from the arsenal of numerical methods constructed for solving ordinary differential equations. To this end, consider the connection between the differential and integral equations.

Consider the following initial value problem for ordinary differential equations of first order:

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x_0 \leqslant x \leqslant X. \tag{2.1}$$

Integrating the differential equation on the segment $[x_0,\, x]$, we obtain an integral equation of type (1.1) in which $g(x) = y_0$, and the kernel of the integral is defined as:

$$K(x,\, s,\, y) \equiv f(s,\, y).$$

So we get that, if the kernel of the integral in equation (1.1) does not depend on the variation x, equation (1.1) and problem (2.1) are equivalent. Using this equivalence for the solution of equation (1.1), we modify some of the methods used to solve the problem (1.1). As it is known, the generalized method of rectangles can be written as:

$$\vartheta^{(1)}(x_n) = \int\limits_{x_0}^{x_n} K(x_n,\, s,\, y(s))ds = h\sum_{i=0}^{n-1} K(x_n,\, x_{i+1/2},\, y_{i+1/2}) + R_n^{(1)}, \qquad (2.2)$$

and a generalized method of trapezoids can be written as:

$$\vartheta^{(2)}(x_n) = \int\limits_{x_0}^{x_n} K(x_n,\, s,\, y(s))ds = h\sum_{i=1}^{n-1} K(x_n,\, x_i,\, y_i) +$$

$$+h\left(K(x_n,\, x_0,\, y_0) + K(x_n,\, x_n,\, y_n)\right)/2 + R_n^{(2)}. \qquad (2.3)$$

Here $R_n^{(1)}$ and $R_n^{(2)}$ are the remainder term of the methods. The accuracy of these methods is the same, but the coefficients of the remaining members of these methods have different signs. Therefore, the exact value of the integral lies between the values calculated by the method of (2.2) and (2.3). If we assume that

$$\frac{d^3}{ds^3}K(x_n,\, s,\, y(s)) \geqslant 0,$$

then we can write

$$\vartheta^{(1)}(x_n) \leqslant \vartheta(x_n) \leqslant \vartheta^{(2)}(x_n)$$

which is equivalent to the following:

$$\vartheta(x_n) \in \left[\vartheta^{(1)}(x_n),\, \vartheta^{(2)}(x_n)\right].$$

It is known that the equation of the segment $\left[\vartheta^{(1)}(x_n),\, \vartheta^{(2)}(x_n)\right]$ can be written as

$$t\vartheta^{(1)}(x_n) + (1-t)\vartheta^{(2)}(x_n) \quad (0 \leqslant t \leqslant 1).$$

There exists $t_0 \in [0,\, 1]$, for which the following holds:

$$\vartheta(x_n) = t_0\vartheta^{(1)}(x_n) + (1-t_0)\vartheta^{(2)}(x_n). \qquad (2.4)$$

But, the method for finding the exact value of t_0 is unknown. So scientists are developing different ways to determine a value of the quantity t_0 with high order. For example, for the methods (2.2) and (2.3) the approximate values t_0 are determined in the form $t_0 = 1/2$. Obviously, taking into account $t_0 = 1/3$, we obtain a new formula, which is more accurate than formulae (2.2) and (2.3). Then, generalizing the linear combination of (2.2) and (2.3) we can write:

$$\int\limits_{x_0}^{x_n} K(x_n,\, s,\, y(s))ds = h\sum_{i=0}^{n} a_i K(x_n,\, x_i,\, y_i) +$$

$$+h\sum_{i=0}^{n} b_i K(x_n,\, x_{i+\nu_i},\, y_{i+\nu_i}) + R_n^{(3)} \quad (|\nu_i| < 1;\ i = 0,1,2,\ldots,n)\,. \tag{2.5}$$

Now consider the construction methods using the values $y(x_{n-m})$ $(m = 1,2,\ldots,k)$ for the calculation of the value $y(x_n)$. Here k is a fixed quantity. For constructing the method, consider the following difference:

$$y(x_{n+1}) - y(x_n) = g(x_{n+1}) - g(x_n) +$$

$$+h\int_{x_0}^{x_n} K'_x(\xi_n,\, s,\, y(s))ds + \int_{x_n}^{x_{n+1}} K(x_{n+1},\, s,\, y(s))ds, \tag{2.6}$$

where $x_n < \xi_n < x_n + h$.

Assume that by any method found the solution of equation (1.1), after taking into account that, in (1.1) we obtain the identity. Then from the resulting equality, we have

$$y'(x) = g'(x) + K(x,\, x,\, y(x)) + \int_{x_0}^{x} K'_x(x,\, s,\, y(s))ds.$$

Here, if we put $x = \xi_n$, we obtain the following:

$$h\int_{x_0}^{x_n} K'_x(\xi_n,\, s,\, y(s))ds = h\left(y'(\xi_n) - g'(\xi_n) - K(\xi_n,\, \xi_n,\, y(\xi_n))\right) -$$

$$-h\int_{x_n}^{\xi_n} K'_x(\xi_n,\, s,\, y(s))ds. \tag{2.7}$$

By using equality (2.7) in equality (2.6) we obtain (see, for example [15]):

$$y(x_{n+1}) - y(x_n) = g(x_{n+1}) - g(x_n) + h\left(y'(\xi_n) - g'(\xi_n) - K(\xi_n,\, \xi_n,\, y(\xi_n))\right) +$$

$$+\int_{x_n}^{\xi_n} K(x_n,\, s,\, y(s))ds + \int_{\xi_n}^{x_{n+1}} K(x_{n+1},\, s,\, y(s))ds.$$

By using, in this equality, some transformations (see, for example [15]), and considering it in equality (2.5), after discarding the remainder terms, we obtain:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i g_{n+i} + h\sum_{j=0}^{k}\sum_{i=0}^{k} \beta_i^{(j)} K(x_{n+j},\, x_{n+i},\, y_{n+i}) +$$

$$+h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)}K(x_{n+j},\,x_{n+i+\nu_i},\,y_{n+i+\nu_i})\ \ (|\nu_i|<1,\ i=0,1,2,\ldots,k). \tag{2.8}$$

Consider the case where $K(x,\,s,\,y)\equiv f(s,\,y)$ and $g(x)\equiv y_0$. Then from (2.8) we have

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}\beta_i f_{n+i}+h\sum_{i=0}^{k}\gamma_i f_{n+i+\nu_i}, \tag{2.9}$$

where the coefficients $\beta_i,\,\gamma_i\ (i=0,1,2,\ldots,k)$ satisfy the following conditions:

$$\sum_{j=0}^{k}\beta_i^{(j)}=\beta_i;\ \sum_{j=0}^{k}\gamma_i^{(j)}=\gamma_i\ (i=0,1,2,\ldots,k). \tag{2.10}$$

Method (2.9) coincides with method (1.4). So we get that if we know the coefficients of method (1.4), the coefficients of method (2.8) $\beta_i^{(j)},\,\gamma_i^{(j)}\ (i,j=0,1,2,\ldots,k)$ can be determined from system (2.10). Thus, the construction method of type (2.8) is reduced to the construction method of type (1.4), which was studied in [14].

It is easy to verify that method (2.5))is more accurate than methods (2.2) and (2.3). Since in the construction of method (2.9) we used formula (2.5), we can expect that method (2.9))will be more accurate than the multistep method with constant coefficients, which is obtained from (2.9) for $\gamma_i=0\ (i=0,1,\ldots,k)$. By Dahlquist's theorem we know that if method (2.9) is stable for $\gamma_i=0\ (i=0,1,\ldots,k)$ and has the degree p, then $p\leqslant 2\,[k/2]+2$ (see [3]). But if method (2.9) is stable and has the degree p, then $p\leqslant 3k+1$. Note that in some cases the multistep method (2.9) is called the finite-difference and in this case the quantity k is called the order of the method, therefore, to determine the order of accuracy of the method we use the notion of the degree of the method, which is defined in the following form.

Definition 2.1. For a sufficiently smooth function $z(x)$, the integer quantity $p>0$ is called the degree of method (1.4), if the following holds:

$$\sum_{i=0}^{k}(\alpha_i z(x+ih)-h\beta_i z'(x+ih)-h\gamma_i z'(x+(i+\nu_i)h))=$$

$$=O(h^{p+1}),\ \ h\to 0. \tag{2.11}$$

It is known that the stability of the multistep methods is determined by the linear part of the considered method, because stability of them is understood in the classical sense (see, for example [3]).

As noted above, the methods of type (2.8) are based on the coefficients of method (2.9). However, these methods can have different properties. For example, the method of type (2.8) with the maximum degree is not unique, but the method of type (2.9) with the maximum degree can be unique. Indeed, the method of type (2.9) for $k=1$ with a degree $p = 4$ is unique, but the method of type (2.8) with a degree $p = 4$ is not unique. For confirmation of this fact it is enough to recall system (2.10), by which we determine the coefficients of the methods of type (2.8), that always has a more than one solution. Therefore, the methods of type (2.8) with the maximum degree are not unique. Note that the solution of (2.10) is also not unique. Usually, the coefficients of method (1.4) are determined as the solution of the homogeneous system of nonlinear algebraic equations (see for example [14]):

$$\begin{aligned}&\sum_{i=0}^{k}\alpha_i = 0,\ \sum_{i=0}^{k} i\alpha_i - \sum_{i=0}^{k}(\beta_i+\gamma_i) = 0,\\ &\sum_{i=0}^{k}\left(\frac{i^l}{l!}\alpha_i - \frac{i^{l-1}}{(l-1)!}\beta_i - \frac{(i+\nu_i)^{l-1}}{(l-1)!}\gamma_i\right) = 0 \ \ (l = 2, 3, \ldots, p).\end{aligned} \tag{2.12}$$

The homogeneous system (2.12) consists of $p+1$ homogeneous nonlinear equations, and $4k+4$ unknowns. It is known that the homogeneous system always has the trivial (zero) solution. But to construct a method, one must find a nontrivial (nonzero) solution. For the existence of nontrivial solutions the inequality $p + 1 < 4k + 4$, between the number of unknowns and equations, must hold. This implies that $p \leqslant 4k + 2$.

We can prove that the condition that the coefficients of method (1.4) satisfy system (2.12) is necessary and sufficient for method (1.4) to have degree p. Thus, we find that for determining the degree of method (1.4), one can use the homogeneous system (2.12). Usually for the investigation of method (1.4), we impose the following assumptions on the coefficients:

A: The coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i$ $(i = 0, 1, 2, \ldots, k)$ are some real numbers. Moreover, $\alpha_k \neq 0$.

B: The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i\lambda^i, \quad \sigma(\lambda) \equiv \sum_{i=0}^{k}\beta_i\lambda^i; \quad \gamma(\lambda) \equiv \sum_{i=0}^{k}\gamma_i\lambda^{i+\nu_i}$$

have no common multipliers different from the constant.

C: $\sigma(1) + \gamma(1) \neq 0$ and $p \geqslant 1$.

Now consider the construction of specific methods of type (2.8) for $k = 1$. Then from system (2.12) we have:

$$\begin{aligned}&\beta_1 + \beta_0 + \gamma_1 + \gamma_0 = \alpha_1,\\ &\beta_1 + l_1\gamma_1 + l_0\gamma_0 = \alpha_1/2,\\ &\beta_1 + l_1^3\gamma_1 + l_0^3\gamma_0 = \alpha_1/4,\\ &\beta_1 + l_1^4\gamma_1 + l_0^4\gamma_0 = \alpha_1/5,\\ &\beta_1 + l_1^5\gamma_1 + l_0^5\gamma_0 = \alpha_1/6.\end{aligned} \tag{2.13}$$

Solving system (2.13) for $\alpha_1 = -\alpha_0 = 1$, we obtain:

$$\beta_0 = \beta_1 = 1/12, \ \ \gamma_0 = \gamma_1 = 5/12,$$

$$l_0 = 1/2 - \sqrt{5}/10, \ \ l_1 = 1/2 + \sqrt{5}/10.$$

The method with degree $p = 6$ has the following form:

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/12+$$

$$+5h(f_{n+1/2-\sqrt{5/10}} + f_{n+1/2+\sqrt{5/10}})/12. \quad (2.14)$$

For applying hybrid method to the solution of some problems, we should know some values of the quantities $y_{n+1/2-\sqrt{5/10}}$ and $y_{n+1/2+\sqrt{5/10}}$ and the accuracy of these values should have at least $O(h^6)$ order. Note that hybrid method (2.14) is implicit and while applying it to the solution of initial problem (1.1) the predictor-corrector scheme that contains even one explicit method is used. Therefore, we consider construction of an explicit method that in one variant has the following form:

$$y_{n+1} = y_n + hf_n/9 + h((16 + \sqrt{6})f_{n+(6-\sqrt{6})/10}+$$

$$+(16 - \sqrt{6})f_{n+(6+\sqrt{6})/10})/36. \quad (2.15)$$

This method is explicit and has degree $p = 5$. Note that in the case $\beta_0 = \beta_1 = 0$ for $k = 1$, after solving system (2.13), we obtain the following steady hybrid method with the highest accuracy $p = 4$:

$$y_{n+1} = y_n + h(f_{n+1/2-\alpha} + f_{n+1/2+\alpha})/2 \quad (\alpha = \sqrt{3}/6). \quad (2.16)$$

Using the coefficients of method (2.14), the next method is used for solving equation (1.1):

$$y_{n+1} = y_n + g_{n+1} - g_n + h(2K(x_{n+1}, x_{n+1}, y_{n+1}) + K(x_{n+1}, x_n, y_n)+$$

$$+K(x_n, x_n, y_n))/24 + 5h(K(x_{n+1}, x_{n+1/2-\sqrt{5}/10}, y_{n+1/2-\sqrt{5}/10})+$$

$$+K(x_{n+1/2-\sqrt{5}/10}, x_{n+1/2-\sqrt{5}/10}, y_{n+1/2-\sqrt{5}/10})+$$

$$+K(x_{n+1}, x_{n+1/2-\sqrt{5}/10}, y_{n+1/2+\sqrt{5}/10})+$$

$$+K(x_{n+1/2+\sqrt{5}/10}, x_{n+1/2+\sqrt{5}/10}, y_{n+1/2+\sqrt{5}/10}))/24.$$

Consequently from here, it is not difficult to construct methods for solving equation (1.1) based on method (2.15). Therefore we recommend here the next algorithm.

Algorithm 2.2. To approximate the solution of the initial-value problem (2.1)

$$y' = f(x, y), \;\; x_0 \leqslant x \leqslant X, \quad y(x_0) = y_0,$$

at $(N + 1)$ equally spaced numbers in the interval $[x_0, X]$:
INPUT endpoints x_0, X; integer N;
Initial values $y_0, \;\; y_{1/2}$.
OUTPUT approximating y_i to $y(x_i)$ at the $(N + 1)$ values of x.

Step 1. Set $h = (x - x_0)/N$;

Step 2. For $i = 1, 2, \ldots, N$ do Steps 3–6.

Step 3.

$$\hat{y}_{i+1} = y_i + hf_{i+1/2}; \quad y_{i+1} = y_i + h(\hat{f}_{i+1} + 4f_{i+1/2} + f_i)/6;$$

$$y_{i+3/2} = y_{i+1/2} + h(7\hat{f}_{i+1} - 2f_{i+1/2} + f_i)/6.$$

Step 4. For $\alpha = (6 - \sqrt{6})/10, \; (6 + \sqrt{6})/10$ do

$$y_{i+\alpha} = y_i + \alpha h y_i' + \alpha^2 h((\alpha^2 - 12\alpha + 6)f_{i+3/2} - (3\alpha^2 - 48\alpha + 27)f_{i+1} +$$

$$+ (3\alpha^2 - 60\alpha + 54)f_{i+1/2} - (\alpha^2 - 24\alpha + 33)f_i)/18.$$

Step 5.

$$y_{i+1} = y_i + hf_i/9 + h((16 + \sqrt{6})f_{i+(6-\sqrt{6})/10} +$$

$$+ (16 - \sqrt{6})f_{i+(6+\sqrt{6})/10})/36.$$

Step 6. OUTPUT (x_{i+1}, y_{i+1}).

Step 7. STOP.

Numerical results are presented for four examples, all the examples are considered in [18], and can be written as follows:

1. $y(x) = 1 + x^2/2 + \int\limits_0^x y(s)ds, \quad 0 \leqslant s \leqslant x \leqslant 1, \quad h = 0.1$ and $h = 0.02$, exact solution is $y(x) = 2\exp(x) - x - 1$.

2. $y(x) = x + \int\limits_0^x \sin(x - s)y(s)ds$, $0 \leqslant s \leqslant x \leqslant 1, \quad h = 0.1$ and $h = 0.02$, exact solution is $y(x) = x + x^3/6$.

3. $y(x) = e^{-x} + \int\limits_0^x e^{-(x-s)} y(s)ds$, $0 \leqslant s \leqslant x \leqslant 0.1$, $\quad h = 0.02$, exact solution is $y(x) = 1$.

4. $y(x) = e^{-x} + \int\limits_0^x e^{-(x-s)} y^2(s)ds$, $0 \leqslant s \leqslant x \leqslant 0.1$, $\quad h = 0.02$, exact solution is $y(x) = 1$.

The results obtained here are compared with known ones in table 2.1. Note that in [18] a trapezoid was used, which has degree $p = 2a$. Method (2.16) has degree $p = 4$. Therefore, the solution obtained by method (2.16) is more accurate. In [18] using the trapezoidal method, with increasing values of the number of calls to the kernel of the integral increases with size, but in the method (2.16) the number of calls to the kernel of the integral does not depend on the value of and is constant at each step.

Number example	X	Maximal error for the method from [18]	Maximal error for method (2.16) h=0.02	Maximal error for method (2.16) h=0.1
I	0.1	7.9 E-02	2.6 E-11	1.6 E-08
	0.5	7.0 E-04	1.4 E-10	9.1 E-08
	1.0	7.5 E-02	3.4 E-10	2.1 E-07
II	0.1	1.0 E-03	2.2 E-06	7.9 E-05
	0.4	8.0 E-03	2.5 E-04	9.8 E-05
	0.6	5.0 E-03	1.0 E-02	5.6 E-04
III	0.02	1.0 E-06	3.2 E-07	
	0.06	2.0 E-06	9.4 E-07	
	0.1	9.0 E-06	1.5 E-06	3.7 E-05
	0.5		5.1 E-06	1.3 E-04
	1.0		6.6 E-06	1.7 E-04
IV	0.02	1.0 E-02	6.5 E-09	
	0.06	1.0 E-02	5.7 E-08	
	0.1	1.1 E-02	1.5 E-07	3.7 E-06
	0.5		2.4 E-06	7.1 E-05
	1.0		6.6 E-06	2.0 E-04

Table 2.1: Comparative results.

3 Conclusions

We constructed a multistep hybrid method with constant coefficients and some concrete hybrid method with degree $4 \leqslant p \leqslant 6$ for $k = 1$. It is known that for $k = 1$ k–step method with constant coefficients has maximal degree $p_{\max} = 2$ that is a trapezoidal method. But the hybrid method constructed here has maximal degree $p_{\max} = 6$. On the base of this, a method has been constructed which can be applied to the solution of equation (1.1). Taking into account some equivalences between equation (1.1) and problem (2.1) we suggested an algorithm for solving problem (2.1) by method (2.15). Note that for $k = 2$ we constructed stable methods with degree $p = 8$ and $p = 9$.

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