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On Some Inequalities for the Gamma Function

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In Memory of Panos Siafarikas

Abstract

We present some elementary proofs of well-known inequalities for the gamma function and for the ratio of two gamma functions. The paper is purely expository and it is based on the talk that the first author gave during the memorial conference in Patras, 2012.

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1 Introduction and Background

The Euler gamma function is defined, for $\Re x > 0$, by

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt. \tag{1.1}$$

The incomplete gamma functions arise by decomposing the integral in (1.1) into the sum of an integral from 0 to x, and another from x to ∞. Namely

$$\gamma(a,x) = \int_0^x e^{-t}\, t^{a-1} dt, \qquad \Re a > 0$$
$$\Gamma(a,x) = \int_x^\infty e^{-t}\, t^{a-1} dt, \qquad |\arg a| < \pi.$$

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This decomposition was studied, for the first time, by Prym [16] and therefore these functions are also referred to as Prym's functions.

Gautschi [1] proved an interesting inequality for the incomplete gamma function $\Gamma(a,x)$:

$$\frac{1}{2a}\left[(x+2)^a - x^a\right] < e^x\Gamma\left(a;x\right) \leq \frac{c_a}{a}\left[(x+c_a^{-1})^a - x^a\right], \tag{1.2}$$

where $0 \leq x < \infty$, $0 < a < 1$ and

$$c_a = \left[\Gamma\left(1+a\right)\right]^{1/(1-a)}.$$

Inequalities (1.2) are very general. For example for $a = 1/2$, the lower bound reduces to an inequality of Komatu [9] for $e^{x^2}\int_x^\infty e^{-t^2}dt$, while the upper bound reduces to a result of Pollak [15]. When $a \to 1$, both bounds tend to 1 and $e^x\Gamma(1,x) = 1$, since $\Gamma(1,x) = e^{-x}$. Moreover as $a \to 0$, we find

$$\frac{1}{2}\log\left(1+\frac{2}{x}\right) \leq e^x E_1(x) \leq \log\left(1+\frac{1}{x}\right),$$

where $0 \leq x < \infty$, and

$$E_1(x) = \int_1^\infty e^{-xt}t^{-1}dt$$

is the exponential integral.

We shall consider also the logarithmic derivative of the Γ function

$$\psi(x) := \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

called the *psi* or *digamma function*.

2 Inequalities for Gamma Function

An interesting special case of (1.2) is obtained by setting $x = 0$ and using the property $\Gamma(1+a) \leq 1$, for $0 \leq a \leq 1$. We get

$$2^{a-1} \leq \Gamma\left(1+a\right) \leq 1, \qquad 0 \leq a \leq 1. \tag{2.1}$$

Inequality (2.1) can be improved by using a very simple method [12] that we now describe.

Theorem 2.1. *Let Γ denote the gamma function and $\gamma = 0.577\ldots$ the Euler–Mascheroni constant. Then, for $0 \leq x \leq 1$*

$$e^{(x-1)(1-\gamma)} \leq \Gamma\left(1+x\right) \leq 1, \tag{2.2}$$

where in the lower bound equality occurs if and only if $x = 1$.

Proof. The upper bound is trivial. It is well known that $\Gamma(1+x) \le 1$ for $0 \le x \le 1$ ($\Gamma(1) = \Gamma(2) = 1$ and $\Gamma(x)$ is convex on $[1,2]$). The lower bound is also trivial for $x = 0$.

For $0 < x < 1$ and $x < t < 1$, let us consider the function $\log\Gamma(1+t)$ and apply the mean value theorem of differential calculus. We get

$$-\log\Gamma(1+x) = (1-x)\psi(1+c), \qquad x < c < 1.$$

Hence, recalling that $\psi(x)$ increases for $x > 0$, we find

$$(x-1)\psi(2) < \log\Gamma(1+x)$$

and finally, since $\psi(2) = 1-\gamma$,

$$e^{(x-1)(1-\gamma)} < \Gamma(1+x).$$

This completes the proof. □

We can compare (2.1) with (2.2). Our lower bound in (2.2) will be sharper if

$$e^{(x-1)(1-\gamma)} > 2^{x-1}; \qquad \text{or} \qquad 1-\gamma < \log 2$$

and this is the case, because $1-\gamma = 1 - 0.577\ldots < \log 2 = 0.69\ldots$.

A similar analysis shows that for $x > 0$

$$e^{-\gamma x} < \Gamma(1+x) < e^{x\psi(x+1)}.$$

However, this result is not new. The inequalities have been proved by Feng Qi [17].

3 Inequalities for the Ratio of Two Gamma Functions

In 1959, W. Gautschi [1] established, for $x > 0$ and $0 < \lambda < 1$, the inequalities

$$x^{1-\lambda} \le \frac{\Gamma(x+1)}{\Gamma(x+\lambda)} \le e^{(1-\lambda)\psi(x+1)}, \tag{3.1}$$

$$x^{1-\lambda} \le \frac{\Gamma(x+1)}{\Gamma(x+\lambda)} \le (x+1)^{1-\lambda}. \tag{3.2}$$

Actually, the results have been proved only for x an integer, $n = 1, 2, \ldots$, but the proof is valid for arbitrary $x > 0$. (See the review of [1]: MR0103289 (21 #2067)). Inequalities (3.1) have been the subject of several improvements, extensions and generalizations [4, 7, 8, 10, 13, 14]; (see [3] for further references).

We take the occasion to point out that, already in 1948, in order to establish the asymptotic relation

$$\lim_{x\to\infty} \frac{\Gamma(x+\lambda)}{x^{\lambda}\Gamma(x)} = 1,$$

for real λ and x, Wendel [19] proved the inequalities

$$\left(\frac{x}{x+a}\right)^{1-a} \le \frac{\Gamma(x+a)}{x^a\Gamma(x)} \le 1, \tag{3.3}$$

for $0 < a < 1$ and $x > 0$. These are equivalent to inequalities (3.2). It is our opinion that the Wendel paper should be better known. The importance of this paper, also in order to fix the priority of the results, has already been pointed out by Feng Qi [17]. The Wendel proof is based on the classical Hölder inequality

$$\int_0^\infty f(t)g(t)dt \le \left(\int_0^\infty [f(t)]^p\,dt\right)^{1/p}\left(\int_0^\infty [g(t)]^q\,dt\right)^{1/q},$$

which holds for $p > 0, q > 0, 1/p + 1/q = 1$. Choosing $x > 0,\ p = 1/a,\ q = 1/(1-a),\ f(t) = e^{-at}t^{ax},\ g(t) = e^{-(1-a)t}t^{(1-a)x+a-1}$ gives the right-hand inequality in (3.3) while successively replacing a by $1-a$ and x by $x+a$ gives the left-hand one.

It is possible to find a lower bound for $\dfrac{\Gamma(x+1)}{\Gamma(x+\lambda)}$ of the same form as the upper bound (3.1), i.e., involving the exponential and the *psi* function. Moreover it is also possible to extend the validity of the inequalities to every $\lambda > 0$. More precisely we have the following result [12]:

Theorem 3.1. *For $x > 0$ and $\lambda \ge 0$, the inequalities*

$$e^{(1-\lambda)\psi(x+\lambda)} \le \frac{\Gamma(x+1)}{\Gamma(x+\lambda)} \le e^{(1-\lambda)\psi(x+1)}, \tag{3.4}$$

hold true. In particular, equalities occur if and only if $\lambda = 1$.

Proof. The case $\lambda = 1$ is trivial. Consider the case $0 \le \lambda < 1$. For fixed $x > 0$, the classical mean value theorem, applied to the function $f(t) = \log\Gamma(t)$ on the interval $[x+\lambda, x+1]$, gives

$$\log\Gamma(x+1) - \log\Gamma(x+\lambda) = (1-\lambda)\psi(x+c),$$

or

$$\log\frac{\Gamma(x+1)}{\Gamma(x+\lambda)} = (1-\lambda)\psi(x+c), \qquad \lambda < c < 1.$$

From the increasing character of *psi* function it follows that

$$(1-\lambda)\psi(x+\lambda) < \log\frac{\Gamma(x+1)}{\Gamma(x+\lambda)} < (1-\lambda)\psi(1+x),$$

which gives (3.4).

The proof of Theorem 3.1 in the case $\lambda > 1$ is similar. $\square$

4 On Some Mean Value Inequalities

W. Gautschi [2] proved the following conjecture of V.R. Uppuluri [18]

$$\frac{1}{2}\left[\Gamma(x)+\Gamma\left(\frac{1}{x}\right)\right] \le \Gamma(x)\Gamma\left(\frac{1}{x}\right) \tag{4.1}$$

which, because of the arithmetic-geometric mean inequality, implies that

$$\Gamma(x)\Gamma\left(\frac{1}{x}\right) \ge 1. \tag{4.2}$$

An alternative proof of (4.2) was given by Kairies [6]. The lower bound in (4.2) is very pessimistic. For example we have $\Gamma(5)\Gamma(0.2) = 110.179\ldots$. Motivated by this observation, Giordano and Laforgia [4] proved a result for a product of gamma functions which provides more accurate inequalities than (4.2).

Theorem 4.1. *For $x_1, x_2 > 0$ and $x_1 x_2 = 1$, we get*

$$\frac{1}{2}\Gamma(1+x_1+x_2) \le \Gamma(x_1)\Gamma(x_2) \le \Gamma(1+x_1+x_2). \tag{4.3}$$

At the lower bound, equality occurs if and only if $x_1 = x_2 = 1$.

Proof. Putting $x_1 = x$ and $x_2 = 1/x$, from the consequence

$$\Gamma(1+z) = \prod_{n=1}^{\infty} \frac{(1+1/n)^z}{1+z/n},$$

of the Euler formula

$$\Gamma(1+z) = \lim_{n\to\infty} \frac{n!\, n^z}{(z+1)\ldots(z+n)},$$

we have

$$\frac{\Gamma(1+x)\Gamma(1+1/x)}{\Gamma(1+x+1/x)} = \prod_{n=1}^{\infty} \frac{1+(1/n)(x+1/x)}{(1+x/n)(1+1/(nx))}.$$

Let $x + 1/x = 2y$; we clearly have

$$\frac{1+2y/n}{1+2y/n+1/n^2} < 1 \qquad \text{with } y \ge 1.$$

As a consequence, we obtain the upper bound in (4.3). If we prove that, for $y \ge 1$,

$$\frac{1+2y/n}{1+2y/n+1/n^2} \ge \frac{1+2/n}{(1+1/n)^2},$$

we can conclude that

$$\frac{\Gamma(1+x)\Gamma(1+1/x)}{\Gamma(1+x+1/x)} \geq \prod_{n=1}^{\infty} \frac{1+2/n}{(1+1/n)^2} = \frac{1}{\Gamma(3)} = \frac{1}{2},$$

which gives the lower bound in (4.3). But this is clearly true because it is equivalent to

$$\frac{2}{n^2}(y-1) \geq 0 \qquad \text{with } \ y \geq 1.$$

Equality occurs, of course, when $y = 1$, i.e., $x = 1$. □

Recently, G. J. O. Jameson and T. P. Jameson [5] proved the following inequality

$$\Gamma(x) + \Gamma\left(\frac{1}{x}\right) \leq \Gamma\left(1 + x + \frac{1}{x}\right) \tag{4.4}$$

conjectured by D. Kershaw. From (4.4) it follows immediately that the lower bound in (4.3) is sharper than the Uppuluri one (4.1). An alternative proof of (4.4) is also given in [11].

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