Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 8, Number 2, pp. 233–245 (2013)
http://campus.mst.edu/adsa

Existence and Multiplicity of Positive Solutions for a System of Higher-Order Multi-Point Boundary Value Problems

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Abstract

We study the existence and multiplicity of positive solutions of a system of nonlinear higher-order ordinary differential equations subject to multi-point boundary conditions.

AMS Subject Classifications: 34B10, 34B18.
Keywords: Higher-order differential system, multi-point boundary conditions, positive solutions.

1 Introduction

We consider the system of nonlinear higher-order ordinary differential equations

$$\begin{cases} u^{(n)}(t) + f(t, v(t)) = 0, & t \in (0, T), \\ v^{(m)}(t) + g(t, u(t)) = 0, & t \in (0, T), \end{cases} \tag{1.1}$$

Received October 30, 2012; Accepted January 10, 2013
Communicated by Lance Littlejohn

with the multi-point boundary conditions

$$\begin{cases} u(0)=\sum_{i=1}^{p} a_i u(\xi_i),\ \ u'(0)=\ldots=u^{(n-2)}(0)=0,\ \ u(T)=\sum_{i=1}^{q} b_i u(\eta_i), \\ v(0)=\sum_{i=1}^{r} c_i v(\zeta_i),\ \ v'(0)=\ldots=v^{(m-2)}(0)=0,\ \ v(T)=\sum_{i=1}^{l} d_i v(\rho_i), \end{cases} \tag{1.2}$$

where $n,\, m \in \mathbb{N}$, $n,\, m \geq 2$, $p,\, q,\, r,\, l \in \mathbb{N}$. In the case $n = 2$ or $m = 2$ the above conditions are of the form $u(0) = \sum_{i=1}^{p} a_i u(\xi_i)$, $u(T) = \sum_{i=1}^{q} b_i u(\eta_i)$, or $v(0) = \sum_{i=1}^{r} c_i v(\zeta_i)$, $v(T) = \sum_{i=1}^{l} d_i v(\rho_i)$, respectively, that is, without conditions on the derivatives of u and v in the point 0.

Under sufficient conditions on f and g, we prove the existence and multiplicity of positive solutions of the above problem, by applying the fixed point index theory. By a positive solution of (1.1)–(1.2), we understand a pair of functions $(u, v) \in C^n([0, T]) \times C^m([0, T])$ satisfying (1.1) and (1.2) with $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0, T]$, and $\sup_{t\in[0,T]} u(t) > 0$, $\sup_{t\in[0,T]} v(t) > 0$. This problem is a generalization of the problem studied in [5], where in the boundary conditions we have $a_i = 0$, $i = 1, \ldots, p$, and $c_i = 0$, $i = 1, \ldots, r$ (denoted by $\widetilde{(1.2)}$). If $n = m = 2$, $a_i = 0$, $i = 1, \ldots, p$, and $c_i = 0$, $i = 1, \ldots, r$, the above problem was investigated in [2]. In [11], the authors studied the existence and multiplicity of positive solutions for system (1.1) with $n = m = 2$ and $T = 1$ with the boundary conditions $u(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = 0$, $v(1) = \alpha v(\eta)$, $\eta \in (0, 1)$, $0 < \alpha\eta < 1$. We also mention the paper [10], where the authors used the fixed point index theory to prove the existence of positive solutions for the system (1.1) with $f(t, v(t))$ and $g(t, u(t))$ replaced by $c(t)\widetilde{f}(u(t), v(t))$ and $d(t)\widetilde{g}(u(t), v(t))$, respectively, and $\widetilde{(1.2)}$, where $1/2 \leq \eta_1 < \eta_2 < \cdots < \eta_q < 1$, $1/2 \leq \rho_1 < \rho_2 < \cdots < \rho_l < 1$, $(T = 1)$.

Some multi-point boundary value problems for systems of ordinary differential equations, which involve positive eigenvalues, were studied in recent years by Henderson, Luca, Ntouyas and Purnaras, by using the Guo–Krasnosel'skii fixed point theorem (see [3,4,6,7]). In the last decades, the multi-point boundary value problems for second-order or higher-order differential or difference equations/systems have been investigated by many authors, by using different methods such as fixed point theorems in cones, the Leray–Schauder continuation theorem and its nonlinear alternatives and the coincidence degree theory.

In Section 2, we shall present some auxiliary results which investigate two boundary value problems for higher-order equations (the problems (2.1)–(2.2) and (2.6)–(2.7) below). In Section 3, we shall prove some existence and multiplicity results for positive solutions with respect to a cone for our problem (1.1)–(1.2) which are based on three fixed point index theorems of Amann [1] and Zhou and Xu [11, Lemma 3].

2 Auxiliary Results

In this section, we shall present some auxiliary results from [3] and [8] related to the nth-order differential equation with multi-point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \;\; t \in (0, T), \tag{2.1}$$

$$u(0) = \sum_{i=1}^{p} a_i u(\xi_i), \;\; u'(0) = \ldots = u^{(n-2)}(0) = 0, \;\; u(T) = \sum_{i=1}^{q} b_i u(\eta_i), \tag{2.2}$$

where $n \in \mathbb{N}$, $n \geq 2$, p, $q \in \mathbb{N}$. If $n = 2$, the condition (2.2) has the form $u(0) = \sum_{i=1}^{p} a_i u(\xi_i)$, $u(T) = \sum_{i=1}^{q} b_i u(\eta_i)$.

Lemma 2.1 (See [3]). *If*

$$\Delta_1 = \left(1 - \sum_{i=1}^{q} b_i\right) \sum_{i=1}^{p} a_i \xi_i^{n-1} + \left(1 - \sum_{i=1}^{p} a_i\right)\left(T^{n-1} - \sum_{i=1}^{q} b_i \eta_i^{n-1}\right) \neq 0,$$

$0 < \xi_1 < \ldots < \xi_p < T$, $0 < \eta_1 < \ldots < \eta_q < T$ *and* $y \in C([0, T])$, *then the solution of* (2.1)–(2.2) *is given by*

$$\begin{aligned} u(t) = &- \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\, ds + \frac{t^{n-1}}{\Delta_1} \left\{ \left(1 - \sum_{i=1}^{q} b_i\right) \sum_{i=1}^{p} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)\, ds \right. \\ &\left. + \left(1 - \sum_{i=1}^{p} a_i\right) \frac{1}{(n-1)!} \left[\int_0^T (T-s)^{n-1} y(s)\, ds - \sum_{i=1}^{q} b_i \int_0^{\eta_i} (\eta_i - s)^{n-1} y(s)\, ds \right] \right\} \\ &+ \frac{1}{\Delta_1} \left\{ \left(\sum_{i=1}^{p} a_i \xi_i^{n-1} \right) \frac{1}{(n-1)!} \left[\int_0^T (T-s)^{n-1} y(s)\, ds - \sum_{i=1}^{q} b_i \int_0^{\eta_i} (\eta_i - s)^{n-1} y(s)\, ds \right] \right. \\ &\left. - \left(T^{n-1} - \sum_{i=1}^{q} b_i \eta_i^{n-1} \right) \sum_{i=1}^{p} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s) ds \right\}. \end{aligned} \tag{2.3}$$

Lemma 2.2 (See [3]). *Under the assumptions of Lemma 2.1, the Green function for the boundary value problem* (2.1)–(2.2) *is given by*

$$\begin{aligned} &G_1(t,s) = g_1(t,s) \\ &+ \frac{1}{\Delta_1} \left[\left(T^{n-1} - t^{n-1}\right) \left(1 - \sum_{i=1}^{q} b_i\right) + \sum_{i=1}^{q} b_i \left(T^{n-1} - \eta_i^{n-1}\right) \right] \sum_{i=1}^{p} a_i g_1(\xi_i, s) \\ &+ \frac{1}{\Delta_1} \left[t^{n-1} \left(1 - \sum_{i=1}^{p} a_i\right) + \sum_{i=1}^{p} a_i \xi_i^{n-1} \right] \sum_{i=1}^{q} b_i g_1(\eta_i, s), \;\; (t,s) \in [0,T] \times [0,T], \end{aligned} \tag{2.4}$$

where

$$g_1(t,s) = \frac{1}{(n-1)!T^{n-1}} \begin{cases} t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}, 0 \le s \le t \le T, \\ t^{n-1}(T-s)^{n-1}, 0 \le t \le s \le T. \end{cases} \tag{2.5}$$

By using G_1, the solution u of problem (2.1)–(2.2) given by (2.3) can be written as $u(t) = \int_0^T G_1(t,s)y(s)\,ds$.

Lemma 2.3 (See [8]; also [5]). *The function g_1 given by* (2.5) *has the properties*

a) $g_1 : [0,T] \times [0,T] \to \mathbb{R}_+$ *is a continuous function and* $g_1(t,s) \ge 0$ *for all* $(t,s) \in [0,T] \times [0,T]$.

b) $g_1(t,s) \le g_1(\theta_1(s),s)$, *for all* $(t,s) \in [0,T] \times [0,T]$.

c) For any $c \in \left(0, \frac{T}{2}\right)$,

$$\min_{t\in[c,T-c]} g_1(t,s) \ge \frac{c^{n-1}}{T^{n-1}} g_1(\theta_1(s),s), \text{ for all } s \in [0,T],$$

where $\theta_1(s) = s$ *if* $n = 2$ *and* $\theta_1(s) = \begin{cases} \dfrac{s}{1-\left(1-\frac{s}{T}\right)^{\frac{n-1}{n-2}}}, & s \in (0,T], \\ \dfrac{T(n-2)}{n-1}, & s = 0, \end{cases}$ *if* $n \ge 3$.

Lemma 2.4 (See [3]). *If* $a_i \ge 0$ *for all* $i = 1,\dots,p$, $\sum_{i=1}^{p} a_i < 1$, *and* $b_i \ge 0$ *for all* $i = 1,\dots,q$, $\sum_{i=1}^{q} b_i < 1$, $0 < \xi_1 < \dots < \xi_p < T$, $0 < \eta_1 < \dots < \eta_q < T$, *then the Green function* G_1 *of problem* (2.1)–(2.2) *(given by* (2.4)*) is continuous on* $[0,T] \times [0,T]$ *and satisfies* $G_1(t,s) \ge 0$ *for all* $(t,s) \in [0,T] \times [0,T]$. *Moreover, if* $y \in C([0,T])$ *satisfies* $y(t) \ge 0$ *for all* $t \in [0,T]$, *then the unique solution* u *of problem* (2.1)-(2.2) *satisfies* $u(t) \ge 0$ *for all* $t \in [0,T]$.

Lemma 2.5 (See [3]). *Assume that* $a_i \ge 0$ *for all* $i = 1,\dots,p$, $\sum_{i=1}^{p} a_i < 1$, *and* $b_i \ge 0$ *for all* $i = 1,\dots,q$, $\sum_{i=1}^{q} b_i < 1$, $0 < \xi_1 < \dots < \xi_p < T$, $0 < \eta_1 < \dots < \eta_q < T$. *Then the Green function* G_1 *of the problem* (2.1)–(2.2) *satisfies the inequalities*

a) $G_1(t,s) \le J_1(s), \ \ \forall\,(t,s) \in [0,T]\times[0,T]$, *where*

$$J_1(s) = g_1(\theta_1(s),s) + \frac{1}{\Delta_1}\left[T^{n-1}\left(1-\sum_{j=1}^{q} b_j\right) + \sum_{i=1}^{q} b_i\left(T^{n-1}-\eta_i^{n-1}\right)\right]\times$$
$$\times\sum_{i=1}^{p} a_i g_1(\xi_i,s) + \frac{1}{\Delta_1}\left[T^{n-1}\left(1-\sum_{j=1}^{p} a_j\right) + \sum_{i=1}^{p} a_i\xi_i^{n-1}\right]\sum_{i=1}^{q} b_i g_1(\eta_i,s).$$

b) For every $c \in (0,T/2)$, *we have*

$$\min_{t\in[c,T-c]} G_1(t,s) \ge \gamma_1 J_1(s) \ge \gamma_1 G_1(t',s), \ \ \forall\, t',\, s \in [0,T],$$

where

$$\gamma_1 = \min\left\{\frac{c^{n-1}}{T^{n-1}}, \frac{\left(T^{n-1}-(T-c)^{n-1}\right)\left(1-\sum_{j=1}^{q} b_j\right) + \sum_{i=1}^{q} b_i\left(T^{n-1}-\eta_i^{n-1}\right)}{T^{n-1}\left(1-\sum_{j=1}^{q} b_j\right) + \sum_{i=1}^{q} b_i\left(T^{n-1}-\eta_i^{n-1}\right)},\right.$$
$$\left.\frac{c^{n-1}\left(1-\sum_{j=1}^{p} a_j\right) + \sum_{i=1}^{p} a_i\xi_i^{n-1}}{T^{n-1}\left(1-\sum_{j=1}^{p} a_j\right) + \sum_{i=1}^{p} a_i\xi_i^{n-1}}\right\} > 0.$$

Lemma 2.6 (See [3]). *Assume that* $a_i \ge 0$ *for all* $i = 1,\ldots,p$, $\sum_{i=1}^{p} a_i < 1$, $b_i \ge 0$ *for all* $i = 1,\ldots,q$, $\sum_{i=1}^{q} b_i < 1$, $0 < \xi_1 < \ldots < \xi_p < T$, $0 < \eta_1 < \ldots < \eta_q < T$, $c \in (0,T/2)$ *and* $y \in C([0,T])$, $y(t) \ge 0$ *for all* $t \in [0,T]$. *Then the solution* $u(t)$, $t \in [0,T]$ *of problem* (2.1)–(2.2) *satisfies the inequality* $\min_{t\in[c,T-c]} u(t) \ge \gamma_1 \max_{t'\in[0,T]} u(t')$.

We can also formulate similar results as Lemmas 2.1–2.6 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \ \ t \in (0,T), \tag{2.6}$$

$$v(0) = \sum_{i=1}^{r} c_i v(\zeta_i), \ \ v'(0) = \ldots = v^{(m-2)}(0) = 0, \ \ v(T) = \sum_{i=1}^{l} d_i v(\rho_i), \tag{2.7}$$

where $0 < \zeta_1 < \ldots < \zeta_r < T$, $c_i \ge 0$ for $i = 1,\ldots,r$, $0 < \rho_1 < \ldots < \rho_l < T$, $d_i \ge 0$ for $i = 1,\ldots,l$ and $h \in C([0,T])$. We denote by Δ_2, γ_2, g_2, θ_2, G_2 and J_2 the corresponding constants and functions for the problem (2.6)–(2.7) defined in a similar manner as Δ_1, γ_1, g_1, θ_1, G_1 and J_1, respectively.

3 Main Results

In this section, we shall investigate the existence and multiplicity of positive solutions for our problem (1.1)–(1.2), under various assumptions on f and g.

We present the assumptions that we shall use in the sequel.

(H1) $0 < \xi_1 < \ldots < \xi_p < T$, $a_i \geq 0$ for all $i = 1, \ldots, p$, $\sum_{i=1}^{p} a_i < 1$, $0 < \eta_1 < \ldots < \eta_q < T$, $b_i \geq 0$ for all $i = 1, \ldots, q$, $\sum_{i=1}^{q} b_i < 1$, $0 < \zeta_1 < \ldots < \zeta_r < T$, $c_i \geq 0$ for all $i = 1, \ldots, r$, $\sum_{i=1}^{r} c_i < 1$, $0 < \rho_1 < \ldots < \rho_l < T$, and $d_i \geq 0$ for all $i = 1, \ldots, l$, $\sum_{i=1}^{l} d_i < 1$.

(H2) The functions $f,\ g \in C([0,T] \times [0,\infty), [0,\infty))$ and $f(t,0) = 0,\ g(t,0) = 0$ for all $t \in [0,T]$.

(H3) There exists a positive constant $p_1 \in (0,1]$ such that

$$i)\ \ f_\infty^i = \liminf_{u\to\infty} \inf_{t\in[0,T]} \frac{f(t,u)}{u^{p_1}} \in (0,\infty]; \quad ii)\ \ g_\infty^i = \liminf_{u\to\infty} \inf_{t\in[0,T]} \frac{g(t,u)}{u^{1/p_1}} = \infty.$$

(H4) There exists a positive constant $q_1 \in (0,\infty)$ such that

$$i)\ \ f_0^s = \limsup_{u\to 0^+} \sup_{t\in[0,T]} \frac{f(t,u)}{u^{q_1}} \in [0,\infty); \quad ii)\ \ g_0^s = \limsup_{u\to 0^+} \sup_{t\in[0,T]} \frac{g(t,u)}{u^{1/q_1}} = 0.$$

(H5) There exists a positive constant $r \in (0,\infty)$ such that

$$i)\ \ f_\infty^s = \limsup_{u\to\infty} \sup_{t\in[0,T]} \frac{f(t,u)}{u^{r}} \in [0,\infty); \quad ii)\ \ g_\infty^s = \limsup_{u\to\infty} \sup_{t\in[0,T]} \frac{g(t,u)}{u^{1/r}} = 0.$$

(H6) The following conditions are satisfied

$$f_0^i = \liminf_{u\to 0^+} \inf_{t\in[0,T]} \frac{f(t,u)}{u} \in (0,\infty]; \quad ii)\ \ g_0^i = \liminf_{u\to 0^+} \inf_{t\in[0,T]} \frac{g(t,u)}{u} = \infty.$$

(H7) For each $t \in [0,T]$, $f(t,u)$ and $g(t,u)$ are nondecreasing with respect to u, and there exists a constant $N > 0$ such that

$$f\left(t, m_0 \int_0^T g(s,N)\,ds\right) < \frac{N}{m_0}, \quad \forall\, t \in [0,T],$$

where $m_0 = \max\{K_1 T, K_2\}$, $K_1 = \max_{s\in[0,T]} J_1(s)$, $K_2 = \max_{s\in[0,T]} J_2(s)$ and $J_1,\ J_2$ are defined in Section 2.

The pair of functions $(u, v) \in C^n([0,T]) \times C^m([0,T])$ is a solution for our problem (1.1)–(1.2) if and only if $(u, v) \in C([0,T]) \times C([0,T])$ is a solution for the nonlinear integral system

$$\begin{cases} u(t) = \displaystyle\int_0^T G_1(t,s) f\left(s, \int_0^T G_2(s,\tau) g(\tau, u(\tau))\, d\tau\right) ds, \ \ t \in [0,T], \\ v(t) = \displaystyle\int_0^T G_2(t,s) g(s, u(s))\, ds, \ \ t \in [0,T]. \end{cases}$$

We consider the Banach space $X = C([0,T])$ with supremum norm $\|\cdot\|$ and define the cone $P \subset X$ by $P = \{u \in X, \ \ u(t) \geq 0, \ \ \forall\, t \in [0,T]\}$.

We also define the operators $\mathcal{A} : P \to X$ by

$$(\mathcal{A}u)(t) = \int_0^T G_1(t,s) f\left(s, \int_0^T G_2(s,\tau) g(\tau, u(\tau))\, d\tau\right) ds, \ \ t \in [0,T],$$

and $\mathcal{B} : P \to X, \mathcal{C} : P \to X$ by

$$(\mathcal{B}u)(t) = \int_0^T G_1(t,s) u(s)\, ds, \ \ (\mathcal{C}u)(t) = \int_0^T G_2(t,s) u(s)\, ds, \ \ t \in [0,T].$$

Under the assumptions $(H1)$ and $(H2)$, using also Lemma 2.4, it is easy to see that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are completely continuous from P to P. Thus the existence and multiplicity of positive solutions of the system (1.1)–(1.2) are equivalent to the existence and multiplicity of fixed points of the operator $\mathcal{A}$.

Theorem 3.1. *Assume that (H1)–(H4) hold. Then the problem* (1.1)–(1.2) *has at least one positive solution* $(u(t), v(t)), \ t \in [0,T]$.

Proof. From assumption i) of $(H3)$, we conclude that there exist $C_1, \ C_2 > 0$ such that

$$f(t,u) \geq C_1 u^{p_1} - C_2, \ \ \forall\, (t,u) \in [0,T] \times [0,\infty). \tag{3.1}$$

Then, for $u \in P$, by using (3.1) and Lemma 2.5, we obtain after some computations (see also [5])

$$(\mathcal{A}u)(t) \geq \widetilde{C}_1 \int_0^T G_1(t,s) \left(\int_0^T (G_2(s,\tau))^{p_1} (g(\tau, u(\tau)))^{p_1}\, d\tau\right) ds - C_3, \ \ \forall\, t \in [0,T], \tag{3.2}$$

where $\widetilde{C}_1 = C_1 T^{p_1/q_0}$ for $p_1 \in (0,1)$ and $q_0 = p_1/(p_1 - 1)$, $\widetilde{C}_1 = C_1$ for $p_1 = 1$ and $C_3 = C_2 \displaystyle\int_0^T J_1(s)\, ds$.

For $c \in (0, T/2)$, we define the cone $P_0 = \{u \in P; \ \inf\limits_{t \in [c, T-c]} u(t) \geq \gamma \|u\|\}$, where $\gamma = \min\{\gamma_1, \gamma_2\}$.

From our assumptions and Lemma 2.6, it can be shown that for any $y \in P$ the functions $u(t) = (\mathcal{B}y)(t)$ and $v(t) = (\mathcal{C}y)(t)$ satisfy the inequalities

$$\inf_{t\in[c,T-c]} u(t) \geq \gamma_1\|u\| \geq \gamma\|u\|, \quad \inf_{t\in[c,T-c]} v(t) \geq \gamma_2\|v\| \geq \gamma\|v\|.$$

So, $u = \mathcal{B}y \in P_0,\ v = \mathcal{C}y \in P_0$. Therefore, we deduce that $\mathcal{B}(P) \subset P_0, \mathcal{C}(P) \subset P_0$.

Now we consider the function $u_0(t),\ t \in [0,T]$, the solution of problem (2.1)–(2.2) with $y = y_0$, where $y_0(t) = 1$ for all $t \in [0,T]$. Then $u_0(t) = \displaystyle\int_0^T G_1(t,s)\,ds = (\mathcal{B}y_0)(t)$, $t \in [0,T]$. Obviously, we have $u_0(t) \geq 0$ for all $t \in [0,T]$. We also consider the set

$$M = \{u \in P;\ \text{there exists } \lambda \geq 0 \text{ such that } u = \mathcal{A}u + \lambda u_0\}.$$

We will show that $M \subset P_0$ and M is a bounded subset of X. If $u \in M$, then there exists $\lambda \geq 0$ such that $u(t) = (\mathcal{A}u)(t) + \lambda u_0(t),\ t \in [0,T]$. From the definition of u_0, we have

$$u(t) = (\mathcal{A}u)(t) + \lambda(\mathcal{B}y_0)(t) = \mathcal{B}(Fu(t)) + \lambda(\mathcal{B}y_0)(t) = \mathcal{B}(Fu(t) + \lambda y_0(t)) \in P_0,$$

where $F : P \to P$ is defined by $(Fu)(t) = f\left(t, \displaystyle\int_0^T G_2(t,s)g(s,u(s))\,ds\right)$. Therefore, $M \subset P_0$, and from the definition of P_0, we have

$$\|u\| \leq \frac{1}{\gamma} \inf_{t\in[c,T-c]} u(t), \quad \forall\, u \in M. \tag{3.3}$$

From ii) of assumption $(H3)$, we conclude that for $\varepsilon_0 = \left(2/(\widetilde{C}_1 m_1 m_2 \gamma_1 \gamma_2^{p_1})\right)^{1/p_1} > 0$ there exists $C_4 > 0$ such that

$$(g(t,u))^{p_1} \geq \varepsilon_0^{p_1} u - C_4, \quad \forall\,(t,u) \in [0,T] \times [0,\infty), \tag{3.4}$$

where $m_1 = \displaystyle\int_c^{T-c} J_1(\tau)\,d\tau > 0$, $m_2 = \displaystyle\int_c^{T-c} (J_2(\tau))^{p_1}\,d\tau > 0$.

For $u \in M$ and $t \in [c, T-c]$, by using Lemma 2.5 and the relations (3.2), (3.4), it follows that

$$\begin{aligned}
&u(t) = (\mathcal{A}u)(t) + \lambda u_0(t) \geq (\mathcal{A}u)(t) \\
&\geq \widetilde{C}_1 \int_c^{T-c} G_1(t,s) \left[\int_c^{T-c} (G_2(s,\tau))^{p_1} (g(\tau,u(\tau)))^{p_1}\,d\tau\right] ds - C_3 \\
&\geq \widetilde{C}_1 \gamma_1 \gamma_2^{p_1} \left(\int_c^{T-c} J_1(s)\,ds\right) \left(\int_c^{T-c} (J_2(\tau))^{p_1} \left(\varepsilon_0^{p_1} u(\tau) - C_4\right)\,d\tau\right) - C_3 \\
&\geq \widetilde{C}_1 \varepsilon_0^{p_1} \gamma_1 \gamma_2^{p_1} \left(\int_c^{T-c} J_1(s)\,ds\right) \left(\int_c^{T-c} (J_2(\tau))^{p_1}\,d\tau\right) \inf_{\tau\in[c,T-c]} u(\tau) - C_5 \\
&= 2 \inf_{\tau\in[c,T-c]} u(\tau) - C_5,
\end{aligned}$$

where $C_5 = C_3 + C_4\widetilde{C}_1 m_1 m_2 \gamma_1 \gamma_2^{p_1} > 0$.

Hence, $\inf\limits_{t\in[c,T-c]} u(t) \geq 2 \inf\limits_{t\in[c,T-c]} u(t) - C_5$, and so

$$\inf_{t\in[c,T-c]} u(t) \leq C_5, \ \ \forall\, u \in M. \tag{3.5}$$

Now from relations (3.3) and (3.5), it can be shown that $\|u\| \leq \dfrac{1}{\gamma} \inf\limits_{t\in[c,T-c]} u(t) \leq \dfrac{C_5}{\gamma}$, for all $u \in M$, that is, M is a bounded subset of X.

Besides, there exists a sufficiently large $L > 0$ such that

$$u \neq \mathcal{A}u + \lambda u_0, \ \ \forall\, u \in \partial B_L \cap P, \ \ \forall\, \lambda \geq 0.$$

From [1] (or [11, Lemma 2]), we deduce that the index

$$i(\mathcal{A}, B_L \cap P, P) = 0. \tag{3.6}$$

Next, from assumption $(H4)$, we conclude that there exist $M_0 > 0$ and $\delta_1 \in (0,1)$ such that

$$f(t,u) \leq M_0 u^q, \ \forall\,(t,u) \in [0,T]\times[0,1]; \ \ g(t,u) \leq \varepsilon_1 u^{1/q_1}, \ \forall\,(t,u) \in [0,T]\times[0,\delta_1], \tag{3.7}$$

where $\varepsilon_1 = \min\left\{1/M_2, (1/(2M_0M_1M_2^{q_1}))^{1/q_1}\right\} > 0$, $M_1 = \displaystyle\int_0^T J_1(s)\,ds > 0$, $M_2 = \displaystyle\int_0^T J_2(s)\,ds > 0$. Hence, for any $u \in \bar{B}_{\delta_1} \cap P$ and $t \in [0,T]$, we obtain

$$\int_0^T G_2(t,s)g(s,u(s))\,ds \leq \varepsilon_1 \int_0^T J_2(s)(u(s))^{1/q_1}\,ds \leq \varepsilon_1 M_2 \|u\|^{1/q_1} \leq 1. \tag{3.8}$$

Therefore, by (3.7) and (3.8), we deduce that for any $u \in \bar{B}_{\delta_1} \cap P$ and $t \in [0,T]$

$$\begin{aligned}(\mathcal{A}u)(t) &\leq M_0 \int_0^T G_1(t,s)\left(\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)^{q_1} ds \\ &\leq M_0 \varepsilon_1^{q_1} M_2^{q_1} \|u\| \int_0^T J_1(s)\,ds = M_0 \varepsilon_1^{q_1} M_1 M_2^{q_1} \|u\| \leq \frac{1}{2}\|u\|.\end{aligned}$$

This implies that $\|\mathcal{A}u\| \leq \|u\|/2, \ \ \forall\, u \in \partial B_{\delta_1} \cap P$. From [1] (or [11, Lemma 1]), we conclude that the index

$$i(\mathcal{A}, B_{\delta_1} \cap P, P) = 1. \tag{3.9}$$

Combining (3.6) and (3.9), we obtain

$$i(\mathcal{A}, (B_L \setminus \bar{B}_{\delta_1}) \cap P, P) = i(\mathcal{A}, B_L \cap P, P) - i(\mathcal{A}, B_{\delta_1} \cap P, P) = -1.$$

We conclude that $\mathcal{A}$ has at least one fixed point $u_1 \in (B_L \setminus \bar{B}_{\delta_1}) \cap P$, that is $\delta_1 < \|u_1\| < L$.

Let $v_1(t) = \displaystyle\int_0^T G_2(t,s)g(s,u_1(s))\,ds$. Then $(u_1, v_1) \in P \times P$ is a solution of (1.1)–(1.2). In addition $\|v_1\| > 0$. Indeed, if we suppose that $v_1(t) = 0$, for all $t \in [0,T]$, then by using $(H2)$ we have $f(s, v_1(s)) = f(s,0) = 0$, for all $s \in [0,T]$. This implies $u_1(t) = \displaystyle\int_0^T G_1(t,s)f(s,v_1(s))\,ds = 0$, for all $t \in [0,T]$, which contradicts $\|u_1\| > 0$. By using Theorem 1.1 from [9], we obtain $u_1(t) > 0$ and $v_1(t) > 0$ for all $t \in (0, T-c]$. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Assume that* $(H1)$, $(H2)$, $(H5)$ *and* $(H6)$ *hold. Then the problem* (1.1)–(1.2) *has at least one positive solution* $(u(t), v(t))$, $t \in [0,T]$.

Proof. From assumption $(H5)$, we deduce that there exist $C_6,\ C_7,\ C_8 > 0$ such that

$$f(t,u) \le C_6 u^r + C_7, \quad g(t,u) \le \varepsilon_2 u^{1/r} + C_8, \quad \forall\,(t,u) \in [0,T] \times [0,\infty), \tag{3.10}$$

where $\varepsilon_2 = (1/(2C_6M_1M_2^r))^{1/r}$.

Hence, for $u \in P$, by using (3.10), we obtain

$$\begin{aligned}
(\mathcal{A}u)(t) &\le \int_0^T G_1(t,s)\left[C_6\left(\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)^r + C_7\right]ds \\
&\le C_6\int_0^T G_1(t,s)\left[\int_0^T G_2(s,\tau)\left(\varepsilon_2\|u\|^{1/r} + C_8\right)d\tau\right]^r ds + M_1C_7 \\
&= C_6\left(\varepsilon_2\|u\|^{1/r} + C_8\right)^r\left(\int_0^T J_1(s)\,ds\right)\left(\int_0^T J_2(\tau)\,d\tau\right)^r + M_1C_7, \quad \forall\,t \in [0,T].
\end{aligned}$$

Therefore, we have

$$(\mathcal{A}u)(t) \le C_6M_1M_2^r\left(\varepsilon_2\|u\|^{1/r} + C_8\right)^r + M_1C_7, \quad \forall\,t \in [0,T]. \tag{3.11}$$

After some computations, we can show that there exists a sufficiently large $R > 0$ such that

$$C_6M_1M_2^r\left(\varepsilon_2\|u\|^{1/r} + C_8\right)^r + M_1C_7 \le \frac{3}{4}\|u\|, \quad \forall\,u \in P, \quad \|u\| \ge R. \tag{3.12}$$

Hence, from (3.11) and (3.12), we obtain $\|\mathcal{A}u\| \le \dfrac{3}{4}\|u\| < \|u\|$, for all $u \in \partial B_R \cap P$. Therefore, from [1] (or [11, Lemma 1]), we have

$$i(\mathcal{A}, B_R \cap P, P) = 1. \tag{3.13}$$

On the other hand, from assumption $(H6)$ we deduce that there exist positive constants $C_9 > 0$ and $u_1 > 0$ such that

$$f(t,u) \ge C_9 u, \quad g(t,u) \ge \frac{C_0}{C_9}u, \quad \forall\,(t,u) \in [0,T] \times [0,u_1], \tag{3.14}$$

where $C_0 = (\gamma_1\gamma_2 m_1 m_3)^{-1}$ and $m_3 = \displaystyle\int_c^{T-c} J_2(\tau)\,d\tau > 0$.

Because $g(t,0) = 0$ for all $t \in [0,T]$, and g is continuous, it can be shown that there exists a sufficiently small $\delta_2 \in (0,u_1)$ such that $g(t,u) \le \dfrac{u_1}{M_2}$ for all $(t,u) \in [0,T]\times[0,\delta_2]$. Hence,

$$\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau \le \int_0^T J_2(\tau)g(\tau,u(\tau))\,d\tau \le u_1,\ \ \forall\, u \in \bar{B}_{\delta_2}\cap P,\ \ s\in[0,T]. \tag{3.15}$$

From (3.14), (3.15) and Lemma 2.5, we deduce that for any $u \in \bar{B}_{\delta_2}\cap P$, we have

$$\begin{aligned}(\mathcal{A}u)(t) &\ge C_9\int_0^T G_1(t,s)\left(\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right)ds\\ &\ge C_0\gamma_2\int_c^{T-c} G_1(t,s)\left(\int_0^T J_2(\tau)u(\tau)\,d\tau\right)ds = (\mathcal{L}u)(t),\ \ \forall\, t\in[0,T].\end{aligned}$$

Hence, by using the above linear operator $\mathcal{L}: P\to P$, we obtain

$$\mathcal{A}u \ge \mathcal{L}u,\ \ \forall\, u\in\partial B_{\delta_2}\cap P. \tag{3.16}$$

For $w_0(t) = \displaystyle\int_c^{T-c} G_1(t,s)\,ds,\ t\in[0,T]$, we have $w_0 \in P\setminus\{\theta_0\}$ and

$$\begin{aligned}(\mathcal{L}w_0)(t) &\ge C_0\gamma_1\gamma_2\left(\int_c^{T-c} J_2(\tau)\,d\tau\right)\left(\int_c^{T-c} J_1(\tau)\,d\tau\right)\left(\int_c^{T-c} G_1(t,s)\,ds\right)\\ &= C_0\gamma_1\gamma_2 m_1 m_3\int_c^{T-c} G_1(t,s)\,ds = \int_c^{T-c} G_1(t,s)\,ds = w_0(t),\ \ t\in[0,T].\end{aligned}$$

Therefore,

$$\mathcal{L}w_0 \ge w_0. \tag{3.17}$$

We may suppose that $\mathcal{A}$ has no fixed point on $\partial B_{\delta_2}\cap P$ (otherwise the proof is finished). From (3.16), (3.17) and [11, Lemma 3] (with $u_0 = w_0$), we conclude that

$$i(\mathcal{A}, B_{\delta_2}\cap P, P) = 0. \tag{3.18}$$

Therefore, from (3.13) and (3.18), we have

$$i(\mathcal{A},(B_R\setminus\bar{B}_{\delta_2})\cap P,P) = i(\mathcal{A},B_R\cap P,P) - i(\mathcal{A},B_{\delta_2}\cap P,P) = 1.$$

Then $\mathcal{A}$ has at least one fixed point in $(B_R\setminus\bar{B}_{\delta_2})\cap P$. Thus, the problem (1.1)–(1.2) has at least one positive solution $(u,v)\in P\times P$ with $u(t)\ge 0,\ v(t)\ge 0$ for all $t\in[0,T]$ and $\|u\|>0,\ \|v\|>0$. This completes the proof of Theorem 3.2. □

Using similar arguments as those used in the proof of [5, Theorem 3.3], we obtain the following theorem.

Theorem 3.3. *Assume that* $(H1)-(H3)$, $(H6)$ *and* $(H7)$ *hold. Then the problem* (1.1)–(1.2) *has at least two positive solutions* $(u_1(t),v_1(t)),\ \ (u_2(t),v_2(t)),\ t\in[0,T]$.

Acknowledgements

The work of the second author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS–UEFISCDI, project number PN–II–ID–PCE–2011–3–0557.

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