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Multiple Solutions to a Periodic Boundary Value Problem for a Nonlinear Discrete Fourth Order Equation

John R. Graef, **Lingju Kong** and **Min Wang**
University of Tennessee at Chattanooga
Department of Mathematics
Chattanooga, TN 37403, USA
John-Graef@utc.edu, Lingju-Kong@utc.edu,
Min-Wang@utc.edu

Abstract

Sufficient conditions are given for the existence of multiple solutions to a periodic boundary value problem for a fourth order nonlinear difference equation. The analysis makes use of variational methods and critical point theory. Two examples of the main result are included.

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Dedicated to the memory of Professor Panayiotis D. Siafarikas

1 Introduction

We consider the periodic boundary value problem (PBVP) consisting of the fourth order nonlinear difference equation

$$\Delta^4 u(t-2) - \Delta\big(p(t-1)\Delta u(t-1)\big) + q(t)u(t) = f(t, u(t)), \quad t \in [1, N]_{\mathbb{Z}}, \tag{1.1}$$

and the periodic boundary condition (BC)

$$\Delta^i u(-1) = \Delta^i u(N-1), \quad i = 0, 1, 2, 3, \tag{1.2}$$

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where $N \geq 1$ is an integer, Δ is the forward difference operator given by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^0 u(t) = u(t)$, $\Delta^i u(t) = \Delta^{i-1}(\Delta u(t))$ for $i = 1, 2, 3, 4$, and the functions p, q, and f satisfy $p \in C([0, N]_{\mathbb{Z}}, \mathbb{R})$ with $p(0) = p(N)$, $q \in C([1, N]_{\mathbb{Z}}, \mathbb{R})$, and $f \in C([1, N]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$. Here we are using the notation that for any integers a and b with $a \leq b$, $[a, b]_{\mathbb{Z}}$ denotes the discrete interval $\{a, a+1, \ldots, b\}$. By a *solution* of PBVP (1.1)–(1.2), we mean a function $u : [-1, N+2]_{\mathbb{Z}} \to \mathbb{R}$ that satisfies (1.1) and (1.2).

The problem (1.1)–(1.2) can be viewed as a discrete analogue of the fourth order periodic boundary value problem for differential equations given by

$$u^{(4)}(t) - \big(p(t)u'(t)\big)' + q(t)u(t) = f(t, u(t)), \quad t \in (0, 1), \tag{1.3}$$

$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0, 1, 2, 3. \tag{1.4}$$

The PBVP (1.3)–(1.4) and special cases of it have been studied using a number of different methods, and as some recent examples in the literature, we refer the reader to the papers of Bai [2], Bereanu [3], Conti et al. [9], and Li et al. [13, 14] and the references contained therein. The study of difference equations has long been an important one due to the fact that they arise in numerical solutions of both ordinary and partial differential equations as well as in applications in such areas as population dynamics, electrical circuit analysis, economics, biology, and a variety of other fields. Recent works on fourth–order difference equations include the papers of Anderson and Minhós [1], Cai and Guo [6], He and Su [10], He and Yu [11], Ji and Yang [12], Ma and Xu [16], and Zhang et al. [19]. There are relatively few papers on fourth-order periodic boundary value problems for difference equations and we mention as recent contributions to their study the papers of Cabada and Dimitrov [4], Cabada and Ferreiro [5], Cai, Yu, and Guo [7], and Zhou and Li [20].

In [4], the authors considered the periodic BVP

$$\begin{cases} u(t+4) + Mu(t) = \lambda g(t) f(u(t)) + h(t), \ t \in [0, N-1]_{\mathbb{Z}}, \\ u(i) = u(N+i), \ i = 0, 1, 2, 3, \end{cases} \tag{1.5}$$

where f, g, h are continuous functions, $\lambda > 0$ is a parameter, and M is a parameter for which the associated linear problem

$$\begin{cases} u(t+4) + Mu(t) = 0, \ t \in [0, N-1]_{\mathbb{Z}}, \\ u(i) = u(N+i), \ i = 0, 1, 2, 3, \end{cases} \tag{1.6}$$

has only the trivial solution and the corresponding Green's function has constant sign. They used the properties of the Green's function and applied Krasnosel'skii's fixed point theorem to obtain the existence of positive solutions of problem (1.5).

In the last few years, variational methods and critical point theory have been utilized to examine the existence of solutions of boundary value problems. Here we use this approach to obtain sufficient conditions for the existence of multiple solutions to the PBVP (1.1)–(1.2).

2 Preliminary Lemmas

In this section, we present the necessary background needed to apply the variational methods used to prove our main result. We let X be a real Banach space. We say that the functional $J \in C^1(X, \mathbb{R})$ satisfies the Palais–Smale (PS) condition if every sequence $\{u_n\} \subset X$, such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. Here, the sequence $\{u_n\}$ is called a *PS sequence*. Our first lemma is known as Clark's critical point theorem and can be found in [8, 17]. It has been used, for example, in the study of periodic solutions of fourth order differential equations (see [15, 18]).

Lemma 2.1 (See [17, Theorem 9.1])**.** *Let X be a real Banach space and $J \in C^1(X, R)$ be even, bounded from below, and satisfy the PS condition. Suppose that $J(0) = 0$ and there is a set $K \subset X$ such that K is homeomorphic to S^{n-1} by an odd map and $\sup_{u \in K} J(u) < 0$, where S^{n-1} is the $n-1$ dimensional unit sphere. Then, J has at least n disjoint pairs of nontrivial critical points.*

We define the set X by

$$X = \left\{u : [-1, N+2]_{\mathbb{Z}} \to \mathbb{R} \ : \ \Delta^i u(-1) = \Delta^i u(N-1), \ i = 0, 1, 2, 3\right\},$$

and for any $u \in X$, define

$$||u|| = \left(\sum_{t=1}^{N} |u(t)|^2\right)^{1/2}$$

and

$$||u||_\infty = \max_{t \in [1,N]_{\mathbb{Z}}} |u(t)|.$$

It is not hard to see that, for any $u \in X$,

$$u(-1) = u(N-1), \ u(0) = u(N), \ u(1) = u(N+1), \text{ and } u(2) = u(N+2). \quad (2.1)$$

Equipped with $||\cdot||$, X is an N dimensional reflexive Banach space. In fact, X is isomorphic to $\mathbb{R}^N$. When we say that the vector $u = (u(1), \ldots, u(N)) \in \mathbb{R}^N$, we understand that u can be extended to a vector in X so that (2.1) holds, that is, u can be extended to the vector

$$\big(u(N-1), u(N), u(1), \ldots, u(N), u(1), u(2)\big) \in X,$$

and when we write $X = \mathbb{R}^N$, we mean the elements in $\mathbb{R}^N$ have been extended in the above sense.

For $u \in X$, define the functional J by

$$J(u) = \frac{1}{2}\sum_{t=1}^{N}\left[|\Delta^2 u(t-2)|^2 + p(t-1)|\Delta u(t-1)|^2 + q(t)|u(t)|^2\right] - \sum_{t=1}^{N} F(t, u(t)), \quad (2.2)$$

where $F(t,x) = \int_0^x f(t,s)ds$ for $x \in \mathbb{R}$. It is easy to see that J is continuously differentiable and its derivative $J'(u)$ at $u \in X$ is given by

$$\begin{aligned} J'(u)(v) = \sum_{t=1}^{N} \big[&\Delta^2 u(t-2)\Delta^2 v(t-2) + p(t-1)\Delta u(t-1)\Delta v(t-1) \\ &+ q(t)u(t)v(t) - f(t,u(t))v(t)\big] \end{aligned} \tag{2.3}$$

for any $v \in X$.

Lemma 2.2. *For any $u, v \in X$, we have*

$$\sum_{t=1}^{N} \Delta^2 u(t-2)\Delta^2 v(t-2) = \sum_{t=1}^{N} \Delta^4 u(t-2)v(t) \tag{2.4}$$

and

$$\sum_{t=1}^{N} p(t-1)\Delta u(t-1)\Delta v(t-1) = -\sum_{t=1}^{N} \Delta\big(p(t-1)\Delta u(t-1)\big)v(t). \tag{2.5}$$

Proof. We first prove (2.4). For any $u, v \in X$, from the summation by parts formula and (2.1), it follows that

$$\begin{aligned} \sum_{t=1}^{N} \Delta^2 u(t-2)\Delta^2 v(t-2) &= \Delta^2 u(N-1)\Delta v(N-1) - \Delta^2 u(-1)\Delta v(-1) \\ &\quad - \sum_{t=1}^{N} \Delta^3 u(t-2)\Delta v(t-1) \\ &= -\sum_{t=1}^{N} \Delta^3 u(t-2)\Delta v(t-1) \\ &= -\Delta^3 u(N-1)v(N) + \Delta^3 u(-1)v(0) \\ &\quad + \sum_{t=1}^{N} \Delta^4 u(t-2)v(t) \\ &= \sum_{t=1}^{N} \Delta^4 u(t-2)v(t), \end{aligned}$$

i.e., (2.4) holds.

Now, we show (2.5). Again, by the summation by parts formula, (2.1), and the fact

that $p(0) = p(N)$, we have

$$\begin{aligned}
&\sum_{t=1}^{N} p(t-1)\Delta u(t-1)\Delta v(t-1) \\
= \;& p(N)\Delta u(N)v(N) - p(0)\Delta u(0)v(0) - \sum_{t=1}^{N} \Delta\big(p(t-1)\Delta u(t-1)\big)v(t) \\
= \;& -\sum_{t=1}^{N} \Delta\big(p(t-1)\Delta u(t-1)\big)v(t),
\end{aligned}$$

i.e., (2.5) holds. This completes the proof of the lemma. □

The following lemma plays an important role in obtaining our main results.

Lemma 2.3. *A function $u \in X$ is a critical point of J if and only if $u(t)$ is a solution of the PBVP* (1.1)–(1.2).

Proof. From (2.3) and Lemma 2.2, we see that $u \in X$ is a critical point of J if and only if

$$\begin{aligned}
J'(u)(v) = \sum_{t=1}^{N} [\Delta^4 u(t-2) - \Delta\big(p(t-1)\Delta u(t-1)\big) \\
+ q(t)u(t) - f(t, u(t))]v(t) = 0
\end{aligned}$$

for any $v \in X$, which is equivalent to

$$\Delta^4 u(t-2) - \Delta\big(p(t-1)\Delta u(t-1)\big) + q(t)u(t) = f(t, u(t)) \quad \text{for } t \in [1, N]_{\mathbb{Z}}.$$

Clearly, (1.2) holds (since $u \in X$), and this proves the lemma. □

Next, we will give an equivalent form for the functional J. Let

$$u = (u(1), u(2), \ldots, u(N))^T,$$

$$A = \begin{pmatrix}
6 & -4 & 1 & 0 & 0 & \ldots & 0 & 0 & 1 & -4 \\
-4 & 6 & -4 & 1 & 0 & \ldots & 0 & 0 & 0 & 1 \\
1 & -4 & 6 & -4 & 1 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 6 & -4 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -4 & 6 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 6 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & -4 & 6 & -4 & 1 \\
1 & 0 & 0 & 0 & 0 & \ldots & 1 & -4 & 6 & -4 \\
-4 & 1 & 0 & 0 & 0 & \ldots & 0 & 1 & -4 & 6
\end{pmatrix}_{N \times N}, \tag{2.6}$$

$$B = \begin{pmatrix} p(0)+p(1) & -p(1) & 0 & \dots & -p(0) \\ -p(1) & p(1)+p(2) & -p(2) & \dots & 0 \\ 0 & -p(2) & p(2)+p(3) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -p(N-1) \\ -p(0) & 0 & 0 & \dots & p(N-1)+p(0) \end{pmatrix}_{N\times N}, \tag{2.7}$$

and

$$C = \begin{pmatrix} q(1) & 0 & 0 & \dots & 0 & 0 \\ 0 & q(2) & 0 & \dots & 0 & 0 \\ 0 & 0 & q(3) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q(N-1) & 0 \\ 0 & 0 & 0 & \dots & 0 & q(N) \end{pmatrix}_{N\times N}. \tag{2.8}$$

Then, A, B, and C are symmetric, and calculations show that for $u \in X$,

$$\sum_{t=1}^{N} |\Delta^2 u(t-2)|^2 = u^T A u,$$

$$\sum_{t=1}^{N} p(t-1)|\Delta u(t-1)|^2 = u^T B u$$

and

$$\sum_{t=1}^{N} q(t)|u(t)|^2 = u^T C u.$$

We can then rewrite the functional J as

$$J(u) = \frac{1}{2} u^T (A+B+C) u - \sum_{t=1}^{N} F(t, u(t)). \tag{2.9}$$

Remark 2.4. The matrices A, B, and C have the following properties:

(a) A is positive semidefinite. In fact, it is clear that 0 is an eigenvalue of A with an eigenvector $(1, 1, \dots, 1)^T$. Moreover, it can be shown that the $(N-1)\times(N-1)$

matrix

$$A_{N-1} = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ -4 & 6 & -4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -4 & 6 & -4 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -4 & 6 \end{pmatrix}$$

is positive definite. This implies that A is positive semidefinite.

(b) If $p(t) > 0$ for $t \in [0, N-1]_{\mathbb{Z}}$, then B is positive semidefinite. In fact, it is clear that 0 is an eigenvalue of B with an eigenvector $(1, 1, \dots, 1)^T$. Moreover, it can be shown that the $(N-1) \times (N-1)$ matrix

$$B_{N-1} = \begin{pmatrix} p(0)+p(1) & -p(1) & 0 & \dots & 0 \\ -p(1) & p(1)+p(2) & -p(2) & \dots & 0 \\ 0 & -p(2) & p(2)+p(3) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -p(N-2) \\ 0 & 0 & 0 & \dots & p(N-2)+p(N-1) \end{pmatrix}$$

is positive definite. This implies that B is positive semidefinite.

(c) If $q(t) > 0$ for $t \in [1, N]_{\mathbb{Z}}$, it is clear that C is positive definite.

3 Main Results

In this section we prove the main result in this paper and give an example to illustrate its applicability. Let λ_i, $i = 1, \dots, N$, be the eigenvalues of $A + B + C$ satisfying

$$\lambda_1 \le \lambda_2 \le \dots \le \lambda_N,$$

and let ξ_i be an eigenvector of $A + B + C$ associated with λ_i such that

$$(\xi_i, \xi_j) = \begin{cases} 0, & i \ne j, \\ 1, & i = j. \end{cases}$$

Then, for any $u = (u(1), u(2), \dots, u(N))^T \in \mathbb{R}^N$, we can easily see that

$$\lambda_1 ||u||^2 \le u^T (A + B + C) u \le \lambda_N ||u||^2.$$

Moreover, from Remark 2.4, if $p(t) > 0$ for $t \in [0, N-1]_{\mathbb{Z}}$ and $q(t) > 0$ for $t \in [1, N]_{\mathbb{Z}}$, then $A + B + C$ is positive definite, and so

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N.$$

We need to define the constants

$$f_0 = \liminf_{x \to 0} \min_{t \in [1,N]_{\mathbb{Z}}} \frac{f(t,x)}{x} \quad \text{and} \quad f^{\infty} = \limsup_{|x| \to \infty} \max_{t \in [1,N]_{\mathbb{Z}}} \frac{f(t,x)}{x}.$$

We will make use of the following conditions:

(H_1) $p(t) \geq 0$ and $q(t) \geq 0$ for $t \in [1, N]_{\mathbb{Z}}$ and there exists η with $\eta < \underline{q}$ such that $f^{\infty} \leq \eta$, where $\underline{q} = \min_{t \in [1,N]_{\mathbb{Z}}} q(t)$;

(H_2) $f(t, x)$ is odd in x, i.e., $f(t, -x) = -f(t, x)$ for $(t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$;

(H_3) there exists $m \in \{1, \ldots, N\}$ such that $f_0 > \lambda_m$.

To prove our main result, we will also need the following lemma.

Lemma 3.1. *If condition (H_1) holds, the functional J defined in* (2.2) *is bounded below and satisfies the PS condition.*

Proof. Since $f^{\infty} \leq \eta$, for $\epsilon > 0$ satisfying $\eta + \epsilon < \underline{q}$, there exists $C > 0$ such that

$$\frac{f(t,x)}{x} \leq \eta + \epsilon \quad \text{for } (t, |x|) \in [1, N]_{\mathbb{Z}} \times (C, \infty). \tag{3.1}$$

Let $t \in [1, N]_{\mathbb{Z}}$. If $x > C$, inequality (3.1) implies $f(t, x) \leq (\eta + \epsilon)x$, so

$$\begin{aligned} F(t,x) &= \int_0^x f(t,s)ds = \int_0^C f(t,s)ds + \int_C^x f(t,s)ds \\ &\leq \int_0^C f(t,s)ds + \int_C^x (\eta+\epsilon)s ds \\ &= \int_0^C f(t,s)ds + \frac{\eta+\epsilon}{2}(x^2 - C^2) \\ &\leq \frac{\eta+\epsilon}{2}x^2 + \max_{t \in [1,N]_{\mathbb{Z}}} \int_0^C f(t,s)ds. \end{aligned} \tag{3.2}$$

Now if $x < -C$, then (3.1) implies $f(t,x) \geq (\eta+\epsilon)x$, so

$$\begin{aligned} F(t,x) &= -\int_x^0 f(t,s)ds = -\int_x^{-C} f(t,s)ds - \int_{-C}^0 f(t,s)ds \\ &\leq -\int_x^{-C} (\eta+\epsilon)sds - \int_{-C}^0 f(t,s)ds \\ &= -\frac{\eta+\epsilon}{2}(C^2-x^2) - \int_{-C}^0 f(t,s)ds \\ &\leq \frac{\eta+\epsilon}{2}x^2 - \min_{t\in[1,N]_{\mathbb{Z}}} \int_{-C}^0 f(t,s)ds. \end{aligned} \tag{3.3}$$

Finally, if $|x| \leq C$, then

$$\begin{aligned} F(t,x) &= \frac{\eta+\epsilon}{2}x^2 + F(t,x) - \frac{\eta+\epsilon}{2}x^2 \\ &\leq \frac{\eta+\epsilon}{2}x^2 + \max_{\substack{t\in[1,N]_{\mathbb{Z}} \\ x\in[-C,C]}} \int_0^x f(t,s)ds. \end{aligned} \tag{3.4}$$

From (3.2)–(3.4), we see that

$$F(t,x) \leq \frac{\eta+\epsilon}{2}x^2 + C_1 \quad \text{for } (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R} \tag{3.5}$$

for some constant $C_1 \in \mathbb{R}$. Let J be defined by (2.2). For any $u \in X$, (3.5) implies

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{t=1}^N \left(|\Delta^2 u(t-2)|^2 + p(t-1)|\Delta u(t-1)|^2 + q(t)|u(t)|^2\right) - \sum_{t=1}^N F(t,u(t)) \\ &\geq \frac{1}{2}\underline{q}||u||^2 - \frac{\eta+\epsilon}{2}\sum_{t=1}^N |u(t)|^2 - C_1 N \\ &= \frac{1}{2}\big(\underline{q} - (\eta+\epsilon)\big)||u|| - C_1 N. \end{aligned}$$

Then, $J(u) \to \infty$ as $||u||_X \to \infty$, and J is bounded from below.

To see that the PS condition is satisfied, notice that any PS sequence $\{u_n\} \subset X$ must be bounded, and since the dimension of X is finite, $\{u_n\}$ has a convergent subsequence. This completes the proof of the lemma. □

Theorem 3.2. *If conditions (H_1)–(H_3) hold, then BVP* (1.1), (1.2) *has at least* $2m$ *non-trivial solutions.*

Proof. Clearly, $J(0) = 0$ and J is even by (H_2). By Lemma 3.1, J is bounded below and satisfies the PS condition. First suppose that $f_0 < \infty$ and let τ satisfy $0 < \tau < f_0 - \lambda_m$. Then, $f_0 > \lambda_m$ implies that there exists $\rho > 0$ such that

$$\frac{f(t,x)}{x} \geq f_0 - \tau \quad \text{for } (t,|x|) \in [1,T]_{\mathbb{Z}} \times [0,\rho].$$

Thus,

$$F(t,x) = \int_0^x f(t,s)ds \geq \frac{1}{2}(f_0 - \tau)x^2 \quad \text{for } (t,|x|) \in [1,T]_{\mathbb{Z}} \times [0,\rho]. \tag{3.6}$$

Define

$$K = \left\{ u = \sum_{i=1}^m c_i \xi_i \ : \ \sum_{i=1}^m c_i^2 = \rho^2 \right\}.$$

Let S^{m-1} be the unit sphere in $\mathbb{R}^m$, and define $T : K \to S^{m-1}$ by

$$T(u) = \frac{1}{\rho}\left(c_1, c_2, \ldots, c_m\right).$$

Then, T is an odd homeomorphism between K and S^{m-1}. For $u \in K$, from (2.9) and (3.6), we have

$$\begin{aligned} J(u) &= \frac{1}{2}u^T(A+B+C)u - \sum_{t=1}^N F(t,u(t)) \\ &\leq \frac{1}{2}\lambda_m ||u||^2 - \frac{1}{2}(f_0 - \tau)\sum_{t=1}^N |u(t)|^2 \\ &= \frac{1}{2}(\lambda_m - f^0 + \tau)\rho^2 < 0. \end{aligned}$$

Thus, $\sup_{u \in K} J(u) < 0$. When $f_0 = \infty$, we can show that this inequality also holds. Hence, all the conditions of Lemma 2.1 are satisfied, so J has at least m disjoint pairs of nontrivial critical points. An application of Lemma 2.3 completes the proof of the theorem. □

We end this paper with the following examples.

Example 3.3. In PBVP (1.1)–(1.2), let $N = 6$, $p(0) = 1 + \sin 6$, $p(t) = 1 + \sin t$, $q(t) = t^2 + 2$ for $t \in [1,6]_{\mathbb{Z}}$, and

$$f(t,x) = t^{-1/4} \begin{cases} x + 24, & x \in (1,\infty), \\ 25x, & x \in [-1,1], \\ x - 24, & x \in (-\infty,-1). \end{cases}$$

Then, we claim that PBVP (1.1)–(1.2) has at least 4 nontrivial solutions.

In fact, it is easy to see that $f_0 \approx 15.9736$, $f^\infty = 1$, and $f(t,x)$ is odd in x. Note that $\underline{q} = \min_{[1,6]_{\mathbb{Z}}} q(t) = 3$. Then, ($H_1$) and ($H_2$) hold.

With the above N, p, and q, let the matrices A, B, C be defined by (2.6)–(2.8). Then, using *MATLAB*, we find that the eigenvalues of $A+B+C$ are given by

$$\lambda_1 \approx 6.5981, \quad \lambda_2 \approx 13.7406, \quad \lambda_3 \approx 21.1240,$$
$$\lambda_4 \approx 27.7973, \quad \lambda_5 \approx 34.4156, \quad \lambda_6 \approx 47.1180.$$

Thus, $\lambda_2 < f_0 < \lambda_3$, i.e., ($H_3$) holds with $m = 2$. The claim now follows from Theorem 3.2.

Example 3.4. Consider the equation

$$\Delta^4 u(t-2) + t^4 u(t) = \frac{10+u^2}{1+2u^2}u, \quad t \in [1,2]_{\mathbb{Z}}, \tag{3.7}$$

and the periodic boundary condition

$$u(-1) = u(1),\ \Delta u(-1) = \Delta u(1),\ \Delta^2 u(-1) = \Delta^2 u(1),\ \Delta^3 u(-1) = \Delta^3 u(1). \tag{3.8}$$

Then,

$$A+B+C = \begin{pmatrix} 7 & -4 \\ -4 & 22 \end{pmatrix}$$

and the eigenvalues of $A+B+C$ are $\lambda_1 = 6$ and $\lambda_2 = 23$. Since $f_0 = 10$ and $f^\infty = 1/2$, we have $m = 1$. The hypotheses of Theorem 3.2 are satisfied and so the problem (3.7)–(3.8) has at least two nontrivial solutions.

Remark 3.5. In conclusion, we should point out that it is also possible to obtain similar results using the conditions

$$f^0 = \limsup_{x\to 0} \max_{t\in[1,N]_{\mathbb{Z}}} \frac{f(t,x)}{x} \quad \text{and} \quad f_\infty = \liminf_{|x|\to\infty} \min_{t\in[1,N]_{\mathbb{Z}}} \frac{f(t,x)}{x}.$$

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