Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 8, Number 2, pp. 193–201 (2013)
http://campus.mst.edu/adsa

Symplectic Difference Systems: Natural Dependence on a Parameter

Ondřej Došlý
Masaryk University
Department of Mathematics and Statistics
Kotlářská 2, CZ 611 37 Brno, Czech Republic
dosly@math.muni.cz

Dedicated to the memory of our friend Professor Panayiotis D. Siararikas

Abstract

We investigate symplectic difference systems depending on a parameter. Using the time scale approach, we suggest a "natural" dependence on a parameter which preserves symplecticity of the investigated systems. In this case we present a result concerning the existence of a central stability zone when the considered symplectic system is periodic.

AMS Subject Classifications: 39A10.
Keywords: Symplectic difference system, dependence on a parameter, unifying time scale approach, periodic system, central stability zone.

1 Introduction

In this paper we consider the first order recurrence system

$$z_{k+1} = \mathcal{S}_k(\lambda) z_k, \quad k \in \mathbb{Z}, \tag{1.1}$$

with $\mathcal{S}_k : \mathbb{R} \to \mathbb{R}^{2n\times 2n}$ and $z \in \mathbb{R}^{2n}$. Recall first that a *symplectic difference system* is the first order system

$$z_{k+1} = \mathcal{S}_k z_k, \quad z_k \in \mathbb{R}^{2n}, \tag{1.2}$$

with $\mathcal{S}_k \in \mathbb{R}^{2n\times 2n}$ being *symplectic*, i.e.,

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.3}$$

Received October 2, 2012; Accepted October 31, 2012
Communicated by Eugenia N. Petropoulou

Symplectic difference systems attracted attention in several recent papers, see, e.g., [2,8, 12,14], partially because they are discrete counterparts of linear Hamiltonian *differential* systems

$$z' = \mathcal{J}\mathcal{H}(t)z, \tag{1.4}$$

$z \in \mathbb{R}^{2n}$, $\mathcal{H} \in R^{2n\times 2n}$, $\mathcal{H}^T(t) = \mathcal{H}(t)$, and qualitative theory of (1.2) is very similar to that of (1.4).

Note that symplectic matrices form a group with respect to the matrix multiplication, so the fundamental matrix of (1.2)

$$Z_k = \mathcal{S}_{k-1}\dots\mathcal{S}_1\mathcal{S}_0 Z_0, \quad Z_0 = I,$$

is a symplectic matrix at any index k. This is just the similarity between (1.2) and (1.4) since the fundamental matrix of (1.4) is also symplectic. Qualitative theory of (1.4) is deeply developed, see, e.g., [5, 13, 18], and the above mentioned papers on (1.2) show that many results of this theory can be "discretized", i.e., extended to (1.2).

If the matrix $\mathcal{H}$ in (1.4) is T-periodic, i.e., $\mathcal{H}(t+T) = \mathcal{H}(t)$, and we introduce a multiplicative parameter λ in this system, i.e., we consider the system

$$z' = \lambda\mathcal{J}\mathcal{H}(t)z, \tag{1.5}$$

there exists the deeply developed stability theory of periodic linear Hamiltonian systems, we refer to the fundamental paper [15] and the book [19] summarizing the results of the Russian school in this area. Our "eventual" aim is to establish a similar theory for symplectic difference systems (1.1) and this paper can be regarded as a first step in this direction.

The first observation immediately reveals that if a matrix $\mathcal{S}$ is symplectic then the matrix $\lambda\mathcal{S}$ is generally no longer symplectic. This suggests the problem of a "natural" dependence of $\mathcal{S}$ on λ which preserves symplectic structure of the system, similarly as the Hamiltonian structure is preserved when $\mathcal{H}(t,\lambda) = \lambda\mathcal{H}(t)$. In a part of our paper we are going to discuss this problem using the time scale calculus which unifies discrete and continuous theories, see [3].

The paper is organized as follows. In the next section we consider a special case of the λ-dependence in (1.1) and in Section 3 we show, using the time scales approach, that this dependence is natural, in a certain sense, and corresponds to the simple multiplication in case of (1.5). In the last section we present a result on the existence of a central stability zone in the case of λ-dependence treated in Section 2.

2 Exponential Dependence

In this section we suppose that

$$S_k(\lambda) = I + \sum_{j=1}^{\infty} R_k^j \frac{\lambda^j}{j!} = \exp\{\lambda R_k\}, \tag{2.1}$$

where

$$R_k^* \mathcal{J} + \mathcal{J} R_k = 0, \quad k \in \mathbb{Z}. \tag{2.2}$$

When $\mathcal{S}$ is a symplectic matrix (in particular, $\mathcal{S}$ is nonsingular), then we have

$$\mathcal{S}^\lambda = \exp\{\lambda \log \mathcal{S}\} = \sum_{j=0}^{\infty} (\log \mathcal{S})^j \frac{\lambda^j}{j!}$$

and it is known (see [7, p. 8]) that the logarithm of a symplectic matrix is the Hamiltonian matrix, i.e.,

$$(\log S)^* \mathcal{J} + \mathcal{J} \log S = 0$$

(here * denotes the conjugate transpose of the matrix indicated). This justifies the assumption (2.2). Next we show that (1.1) with $\mathcal{S}_k(\lambda)$ given by (2.1) is really a symplectic difference system for every $\lambda \in \mathbb{R}$.

For any $\lambda \in \mathbb{C}$ (we consider here also complex λ since these values of λ are needed in general stability theory of periodic systems)

$$\begin{aligned}
S^*(\lambda)\mathcal{J}S(\lambda) &= \mathcal{J} + \bar{\lambda} R^* \mathcal{J} + \lambda \mathcal{J} R + \frac{1}{2!}\left[(\bar{\lambda})^2 (R^*)^2 \mathcal{J} + 2\bar{\lambda}\lambda R^* \mathcal{J} R + \lambda^2 \mathcal{J} R^2\right] + \dots \\
&\quad + \frac{1}{n!}\left[(\bar{\lambda})^n (R^*)^n \mathcal{J} + n\lambda(\bar{\lambda})^{n-1}(R^*)^{n-1}\mathcal{J}R + \dots + n\lambda^{n-1}\bar{\lambda} R^* \mathcal{J} R^{n-1} + \lambda^n \mathcal{J} R^n\right] + \dots \\
&= \mathcal{J} + (\bar{\lambda} - \lambda)(-\mathcal{J}R) - \frac{(\bar{\lambda}-\lambda)^2}{2!} R^* \mathcal{J} R - \frac{(\bar{\lambda}-\lambda)^3}{3!} R^*(-\mathcal{J}R)R \\
&\quad + \frac{(\bar{\lambda}-\lambda)^4}{4!}(R^*)^2 \mathcal{J} R^2 + \dots + (-1)^n \frac{(\bar{\lambda}-\lambda)^{2n}}{(2n)!}(R^*)^n \mathcal{J} R^n \\
&\quad + (-1)^n \frac{(\bar{\lambda}-\lambda)^{2n+1}}{(2n+1)!}(R^*)^n(-\mathcal{J}R)R^n + \dots \\
&= \mathcal{J} + (\bar{\lambda} - \lambda)\Bigg[-\mathcal{J}R - \frac{\bar{\lambda}-\lambda}{2!} R^* \mathcal{J} R + \dots + (-1)^n \frac{(\bar{\lambda}-\lambda)^{2n-1}}{(2n)!}(R^*)^n \mathcal{J} R^n \\
&\quad + (-1)^n \frac{(\bar{\lambda}-\lambda)^{2n}}{(2n+1)!}(R^*)^n(-\mathcal{J}R)R^n + \dots\Bigg] \\
&= \mathcal{J} + (\bar{\lambda} - \lambda)\Bigg[\sum_{j=0}^{\infty}(-1)^j \frac{(\bar{\lambda}-\lambda)^{2j}}{(2j+1)!}(R^*)^j(-\mathcal{J}R)R^j \\
&\quad + \sum_{j=1}^{\infty}(-1)^j \frac{(\bar{\lambda}-\lambda)^{2j-1}}{(2j)!}(R^*)^j \mathcal{J} R^j\Bigg].
\end{aligned}$$

Hence, $\mathcal{S}_k(\lambda)$ are symplectic for $\lambda \in \mathbb{R}$ and for every $k \in \mathbb{Z}$.

3 Time Scales Approach

In this section we briefly discuss the problem of λ dependence in (1.1) from time scales point of view. First we recall some essentials of the time scale calculus. A *time scale* $\mathbb{T}$ is any closed subset of the real numbers $\mathbb{R}$ with the inherited topology. The *forward jump* operator σ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : \ s \in \mathbb{T}\}$ and the difference $\mu(t) := \sigma(t) - t$ is called the *graininess* of a time scale. For a sufficiently smooth function $f : \mathbb{T} \to \mathbb{R}$ we define

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

Of course, we have $f^{\Delta}(t) = f'(t)$ in the continuous case $\mathbb{T} = \mathbb{R}$ and $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ in the discrete case $\mathbb{T} = \mathbb{Z}$.

An important role in our consideration is played by the so-called *regressive matrix group*, see [4] and also [11]. Let $\mu \geq 0$ (later it will play the role of the graininess) and define for $X, Y \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$, and $\lambda \in \mathbb{R}$

$$X \oplus_{\mu} Y := X + Y + \mu\, XY$$

and for $I + \mu X$ nonsingular

$$\lambda \odot_{\mu} X := \begin{cases} \dfrac{1}{\mu}\left[(I + \mu X)^{\lambda} - I\right] & \text{if } \mu > 0 \\ \lambda X & \text{if } \mu = 0 \end{cases},$$

where the power λ of the nonsingular matrix $I + \mu X$ we define by

$$(I + \mu X)^{\lambda} = \exp\{\lambda \log(I + \mu X)\}.$$

It is not difficult to show that for $\lambda = n \in \mathbb{N}$ we have $n \odot_{\mu} X = X + \ldots + X$ where the sum on the right-hand side has n summands. The regressive matrix group is closely connected with the time scale *symplectic dynamic system* (see [1,6])

$$z^{\Delta} = S(t) z, \quad t \in \mathbb{T}, \tag{3.1}$$

which is the first order system whose matrix satisfies

$$S^T(t)\mathcal{J} + \mathcal{J}S(t) + \mu(t) S^T(t)\mathcal{J}S(t) = 0. \tag{3.2}$$

Under this assumption, the fundamental matrix of (3.1) is a symplectic matrix for every $t \in \mathbb{T}$ (whenever it is symplectic at an initial condition). This is also the reason why systems (3.1) with the matrix S satisfying (3.2) are called *symplectic* dynamic systems. If $\mathbb{T} = \mathbb{R}$ (i.e., $\mu(t) \equiv 0$), then (3.1) reduces to linear Hamiltonian differential system (1.4) and in the discrete case $\mathbb{T} = \mathbb{Z}$ ($\mu(t) \equiv 1$) we have the system

$$z_{k+1} = (I + S_k) z_k \tag{3.3}$$

and (3.2) implies that (3.3) is a symplectic difference system (1.2) since then (3.2) is equivalent to

$$S^T \mathcal{J} + \mathcal{J} S + S^T \mathcal{J} S + \mathcal{J} = (I+S)^T \mathcal{J} (I+S) = \mathcal{J},$$

which means that (3.3) is really a symplectic difference system.

The main statement of this section reads as follows.

Theorem 3.1. *Suppose that the matrix* $S : \mathbb{T} \to \mathbb{R}^{2n\times 2n}$ *satisfies* (3.2) *and* $\lambda \in \mathbb{R}$. *Then the system*

$$z^\Delta = \big[\lambda \odot_{\mu(t)} S(t)\big] z \tag{3.4}$$

is again a symplectic dynamic system, i.e., its matrix $\lambda \odot_\mu S$ *satisfies identity* (3.2).

Proof. First consider $t \in \mathbb{T}$ for which $\mu(t) = 0$ (the so-called *right dense* points in the time scales terminology). Then $\lambda \odot_\mu S = \lambda S$ and form (3.2) with $\mu = 0$ we have obviously $(\lambda S)^T \mathcal{J} + \lambda \mathcal{J} S = 0$, i.e., λS satisfies (3.2) as well.

Hence, consider the case $\mu(t) > 0$ (the so-called *right scattered* points). Then we have

$$\begin{aligned}
[\lambda \odot_\mu S]^T \mathcal{J} + &\mathcal{J}[\lambda \odot_\mu S] + \mu[\lambda \odot_\mu S]^T \mathcal{J} [\lambda \odot_\mu S] \\
&= \frac{1}{\mu}\left[(I+\mu S^T)^\lambda - I\right] \mathcal{J} + \frac{1}{\mu} \mathcal{J}\left[(I+\mu S)^\lambda - I\right] \\
&\quad + \frac{1}{\mu}\left[(I+\mu S^T)^\lambda - I\right] \mathcal{J} \left[(I+\mu S)^\lambda - I\right] \\
&= \frac{1}{\mu}\left[-\mathcal{J} + (I+\mu S^T)^\lambda \mathcal{J} (I+\mu S)^\lambda\right].
\end{aligned}$$

Hence, it suffices to prove that $(I+\mu S)^\lambda$ is a symplectic matrix. Using the matrix logarithm we have

$$\big(I+\mu S\big)^\lambda = \exp\{\lambda \log(I+\mu S)\} = \sum_{j=0}^\infty \frac{1}{j!}[\log(I+\mu S)]^j.$$

Since (3.2) with $\mu > 0$ implies that the matrix $I + \mu S$ is symplectic, its logarithm $R = \log(I+\mu S)$ is a matrix satisfying (2.2) (see [7, p. 8]) and by the computation from the previous section (where the matrix R plays the role of $\log(I+\mu S)$) the matrix $(I+\mu S)^\lambda$ is really symplectic. □

Remark 3.2. If $\mathbb{T} = \mathbb{Z}$, then $\lambda \odot_{\mu=1} S = (I+S)^\lambda$ and hence (3.4) takes the form

$$z_{k+1} = (I+S_k)^\lambda z_k.$$

which is system (2.1) with $R_k = \log(I+S_k)$ satisfying (2.2). From this point of view, the dependence of $\mathcal{S}_k$ on λ studied in the previous section is a natural discrete counterpart of the multiplicative dependence in the continuous case in (1.5).

4 Central Stability Zone

In this concluding section we present a result concerning the existence of a central stability zone for symplectic system (1.1) with the λ-dependence given by the formula (2.1). We suppose that the matrix sequence R_k is N-periodic, i.e.,

$$R_{k+N} = R_k \tag{4.1}$$

for some integer $N \in \mathbb{N}$.

Recall that a symplectic difference system (1.1) is said to be *stable*, if all its solutions are bounded for $k \in \mathbb{Z}$. A point $\lambda = \lambda_0$ is called the *strong stability point* if there exists $\varepsilon > 0$ such that (1.1) is stable for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Hence the set of strong stability points is an open set, i.e., a union of open intervals, and the interval (if any) containing inside $\lambda_0 = 0$ is called the *central stability zone*.

Let us write the matrix $S_k(\lambda)$ given by (2.1) in the form

$$\mathcal{S}_k(\lambda) = I + \lambda R_k + o(\lambda) \quad \text{as } \lambda \to 0.$$

Then the monodromy matrix of (1.1) is

$$Z_N(\lambda) = \mathcal{S}_{N-1}(\lambda) \cdots \mathcal{S}_0(\lambda) = I + \lambda \sum_{k=0}^{N-1} R_k + o(\lambda).$$

Theorem 4.1. *Suppose that* (2.2) *and* (4.1) *hold. Denote*

$$\mathcal{S}^{[1]} = \sum_{k=0}^{N-1} R_k.$$

If the matrix $\mathcal{J}\mathcal{S}^{[1]}$ is negative definite and the eigenvalues s_j of $\mathcal{S}^{[1]}$ are distinct, then there exists $l > 0$ such that solutions of (1.2) *are bounded for $|\lambda| < l$, i.e., the interval $(-l, l)$ is contained in the central stability zone of* (1.1) *and hence this zone is nonempty.*

Proof. The proof is based on the statement that if A is a symmetric positive definite matrix, then the equation $\det(A + i\mu\mathcal{J}) = 0$ has only real roots, see [9]. So, let $A = -\mathcal{J}\mathcal{S}^{[1]}$ be positive definite and look for eigenvalues of $\mathcal{S}^{[1]}$ in the form $i\sigma$. Then

$$0 = \det(\mathcal{S}^{[1]} - i\sigma I) = \det \mathcal{J}(-\mathcal{J}\mathcal{S}^{[1]} + i\sigma\mathcal{J}) = \det(A + i\sigma\mathcal{J}),$$

hence $\sigma \in \mathbb{R}$ and $s_j = i\sigma_j$. Now, for $\lambda \in \mathbb{R}$, $|\lambda|$ small, the numbers $\gamma_j(\lambda) = i\sigma\lambda + o(\lambda)$ are still different. However, at the same time, for $\lambda \in \mathbb{R}$, the matrix $\mathcal{U}_N(\lambda)$ is $\mathcal{J}$-unitary, i.e., $\mathcal{U}_N^*(\lambda)\mathcal{J}\mathcal{U}_N(\lambda) = \mathcal{J}$, hence its eigenvalues $\rho_j(\lambda) = \exp\{\gamma_j(\lambda)\}$ are symmetric with respect to the unit circle and hence $\gamma_j(\lambda)$ are symmetric with respect to the imaginary axis. But this implies that $\gamma_j(\lambda)$ itself are on the imaginary axis, i.e., $|\rho(\lambda)| = 1$. This implies the stability of the system in view of the statement that a periodic difference system is stable provided its monodromy matrix has all eigenvalues simple and situated on the unit circle in the complex plane. □

Remark 4.2. As we have mentioned in Introduction, this paper is just a first step towards general stability theory of symplectic difference systems (1.1) similar to that for linear Hamiltonian differential system presented in the book [19]. Symplectic difference systems cover as a special case second order matrix difference systems and linear Hamiltonian difference systems. Elements of the stability theory of these periodic systems are established in the papers [10, 16, 17] and we are going to extend them to the general symplectic systems (1.1).

Remark 4.3. Throughout the paper we consider a rather special dependence on the parameter λ given by (2.1), even if it is natural in a certain sense. If we ask the dependence on λ to be analytic in a neighborhood of $\lambda_0 = 0$, the general formula for $\mathcal{S}_k(\lambda)$ is

$$\mathcal{S}_k(\lambda) = \sum_{j=0}^{\infty} S_k^{[j]} \lambda^j, \quad k \in \mathbb{Z}, \tag{4.2}$$

with $S_k^{[j]}$ satisfying identities which imply that the system is really symplectic and with $S_k^{[0]} = I$ for every $k \in \mathbb{Z}$ (since we ask $\lambda_0 = 0$ to be a stability point which is the case when all solutions are constant, similarly as for $\lambda = 0$ in the continuous case (1.5)). We hope to develop the stability theory of general periodic symplectic systems with the matrices $\mathcal{S}_k$ given by (4.2) in a subsequent paper.

Acknowledgements

The research is supported by the grant 201/10/1032 of the Czech Grant Agency (GAČR).

References

[1] C. D. Ahlbrandt, M. Bohner, and J. Ridenhour. Hamiltonian systems on time scales. *J. Math. Anal. Appl.*, 250(2):561–578, 2000.

[2] M. Bohner, O. Došlý, and W. Kratz. Sturmian and spectral theory for discrete symplectic systems. *Trans. Amer. Math. Soc.*, 361(6):3109–3123, 2009.

[3] M. Bohner and A. Peterson. *Dynamic equations on time scales. An introduction with applications*. Birkhäuser Boston, Inc., Boston, MA, 2001.

[4] M. Bohner and A. Peterson. *Advances in dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[5] W. A. Coppel. *Disconjugacy*, volume 220 of *Lecture Notes in Mathematics*. Springer–Verlag, Berlin–New York, 1971.

[6] O. Došlý and R. Hilscher. Disconjugacy, transformations and quadratic functionals for symplectic dynamic systems on time scales. *J. Differ. Equations Appl.*, 7(2):265–295, 2001.

[7] I. Ekeland. *Convexity methods in Hamiltonian mechanics*. Springer–Verlag, Berlin, 1990.

[8] J. V. Elyseeva. Transformations and the number of focal points for conjoined bases of symplectic difference systems. *J. Difference Equ. Appl.*, 15(11–12), 2009.

[9] F. R. Gantmacher. *The theory of matrices*, volume 1. AMS Chelsea Publishing, Providence, RI, 1998.

[10] A. Halanay and V. Rasvan. Stability and boundary value problems for discrete-time linear Hamiltonian systems. *Dynam. Systems Appl.*, 8(3–4):439–459, 1999.

[11] S. Hilger. Matrix Lie theory and measure chains. dynamic equations on time scales. *J. Comput. Appl. Math.*, 141(1–2):197–217, 2002.

[12] R. Hilscher and V. Zeidan. Multiplicities of focal points for discrete symplectic systems: revisited. *J. Difference Equ. Appl.*, 15(10), 2009.

[13] W. Kratz. *Quadratic functionals in variational analysis and control theory*, volume 6 of *Mathematical Topics*. Akademie–Verlag, Berlin, 1995.

[14] W. Kratz. Discrete oscillation. In honour of professor Allan Peterson on the occasion of his 60th birthday, Part II. *J. Difference Equ. Appl.*, 9(1):135–147, 2003.

[15] M. G. Krein. The basic propositions of the theory of λ-zones of stability of a canonical system of linear differential equations with periodic coefficients (Russian). In memory of Aleksandr Aleksandrovi Andronov. *Izdat. Akad. Nauk SSSR, Moscow*, pages 413–498, 1955. English translation: American Mathematical Society Translations, Ser. 2, Vol. 120. Four papers on ordinary differential equations. Edited by Lev J. Leifman. American Mathematical Society Translations, Series 2, **120**, 1–70. American Mathematical Society, Providence, R.I., 1983.

[16] V. Rasvan. On stability zones for discrete-time periodic linear Hamiltonian systems. *Adv. Difference Equ.*, 2006:1–13. Art. 80757.

[17] V. Rasvan. Stability zones for discrete time Hamiltonian systems. *Arch. Math. (Brno)*, 36(suppl.):563–573, 2000.

[18] W. T. Reid. *Sturmian theory for ordinary differential equations*, volume 31 of *Applied Mathematical Sciences*. Springer–Verlag, New York-Berlin, 1980.

[19] V. A. Yakubovich and V. M. Starzhinskii. *Linear differential equations with periodic coefficients*, volume 1, 2 of *Israel Program for Scientific Translations*. Jerusalem–London, 1975.