Solving Random Differential Equations by Means of Differential Transform Methods

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Abstract

In this paper the random Differential Transform Method (DTM) is used to solve a time-dependent random linear differential equation. The mean square convergence of the random DTM is proven. Approximations to the mean and standard deviation functions of the solution stochastic process are shown in several illustrative examples.

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1 Introduction

Differential transform method (DTM) is a semi-analytical method for obtaining mainly, the solution of differential equations in the form of a Taylor series. Although similar to the Taylor expansion which requires the computation of derivatives of the data functions, DTM constructs an analytical solution in the form of polynomials (truncated infinite...
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series) which involves less computational efforts. In the DTM, certain transformation rules are applied and the governing differential equation and initial/boundary conditions are transformed into a set of algebraic equations in terms of the differential transforms of the original functions. The solution of these algebraic equations gives the desired solution of the problem. The basic idea of DTM was introduced by Zhou [9]. The differential transform method (DTM) has been successfully applied to a wide class of deterministic differential equations arising in many areas of sciences and engineering. Since many physical and engineering problems are more faithfully modeled by random differential equations, a random DTM method based on mean fourth stochastic calculus has been developed in [8].

In this paper, we consider the random time-dependent linear differential equation:

\[
\begin{aligned}
X(t) &= P_n(t)X(t) + B(t); \quad P_n(t) := \sum_{i=0}^{n} A_i t^i, \quad 0 \leq t \leq T, \quad n \geq 0, \\
X(0) &= X_0,
\end{aligned}
\]  

(1.1)

where \( n \) is a nonnegative integer, \( B(t) \) is a stochastic process and \( A_i, 0 \leq i \leq n \), and \( X_0 \) are random variables satisfying certain conditions to be specified later.

The aim of the paper is to give sufficient conditions on the data in such a way that a mean square solution of (1.1) can be expanded as

\[
X(t) = \sum_{k=0}^{\infty} \hat{X}(k) t^k, \quad 0 \leq t \leq T,
\]

(1.2)

where \( \hat{X}(k) \) are random variables. In order to achieve our goal, a random DTM is applied to (1.1) taking advantage of results presented in paper [8]. Similar equations have been studied by the authors in [4,5], but using different approaches. As we will see later, a major advantage of having a solution of the form (1.2) is that it allows to obtain in a practical way, the main statistical functions, namely, expectation and variance (or equivalently standard deviation), of the solution stochastic process.

The paper is organized as follows. Section 2 deals with some definitions, notations and results that clarify the presentation of the paper. Section 3 is addressed to establish sufficient conditions on the data of initial value problem (1.1) in order to obtain a mean square solution of the form (1.2) by applying the random DTM. Several illustrative examples are included in the last section.

\section{2 Preliminaries}

This section begins by reviewing some important concepts, definitions and results related to the stochastic calculus that will play an important role throughout the paper. Further details can be found in [1,6,7]. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, that is, a triplet consisting of a space \( \Omega \) of points \( \omega \), a \( \sigma \)-field \( \mathcal{F} \) of measurable sets \( A \) in \( \Omega \) and,
a probability measure \( \mathbb{P} \) defined on \( \mathcal{F} \), [6]. Let \( p \geq 1 \) be a real number. A real random variable \( U \) defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) is called of order \( p \) (p-r. v.), if

\[
E[|U|^p] < \infty,
\]

where \( E[\cdot] \) denotes the expectation operator. The space \( L_p \) of all the p-r. v.’s, endowed with the norm

\[
\|U\|_p = \left( E[|U|^p] \right)^{1/p},
\]

is a Banach space, [1, p.9]. Let \( \{U_n : n \geq 0\} \) be a sequence of p-r. v.’s. We say that it is convergent in the \( p \)th mean to the p-r. v. \( U \in L_p \), if

\[
\lim_{n \to \infty} \|U_n - U\|_p = 0.
\]

If \( q > p \geq 1 \) and \( \{U_n : n \geq 0\} \) is a convergent sequence in \( L_q \), that is, \( q \)th mean convergent to \( U \in L_q \), then \( \{U_n : n \geq 0\} \) lies in \( L_p \) and it is \( p \)th mean convergent to \( U \in L_p \), [1, p.13].

In this paper, we are mainly interested in the mean square (m. s.) and mean fourth (m. f.) calculus, corresponding to \( p = 2 \) and \( p = 4 \) respectively, but results related with \( p = 8, 16 \) will also be used. For \( p = 2 \), the space \( (L_2, \|\cdot\|_2) \) is not a Banach algebra, i.e., the inequality \( \|UV\|_2 \leq \|U\|_2 \|V\|_2 \) does not hold, in general. The space \( L_2 \) is a Hilbert space and one satisfies the Schwarz inequality

\[
\langle U, V \rangle = E[|UV|] \leq \left( E[|U^2|] \right)^{1/2} \left( E[|V^2|] \right)^{1/2} = \|U\|_2 \|V\|_2.
\]

Clearly, if the p-r. v.’s \( U \) and \( V \) are independent, one gets: \( \|UV\|_p = \|U\|_p \|V\|_p \). However, independence cannot be assumed in many applications. The following inequalities are particularly useful to overcome such situation:

\[
\|UV\|_p \leq \|U\|_{2p} \|V\|_{2p}, \quad \forall U, V \in L_{2p},
\]

which is a direct application of the Schwarz inequality, and

\[
\left\| \prod_{i=1}^m U_i \right\|_p \leq \prod_{i=1}^m \left( \|U_i\|_p^m \right)^{1/m}, \quad \text{with} \ E\left[ (U_i)^{mp} \right] < \infty, \quad 1 \leq i \leq m,
\]

which is proven in [2]. Let \( U_i \) be p-r. v.’s, the set of \( m \)-dimensional random vectors \( \vec{U} = (U_1, U_2, \ldots, U_m)^T \) together with the norm

\[
\|\vec{U}\|_{p,v} = \max_{1 \leq i \leq m} \|U_i\|_p
\]

will be denoted by \( (L_m^m, \|\cdot\|_{p,v}) \). The set of \( m \times m \)-dimensional random matrices \( \mathbf{A} = (A_{i,j}) \), which entries \( A_{i,j} \in L_p \) with the norm

\[
\|\mathbf{A}\|_{p,m} = \sum_{i=1}^m \sum_{j=1}^m \|A_{i,j}\|_p,
\]
will be denoted by \((L^p_{m \times m}, \| \cdot \|_{p,m})\). By using (2.2), it is easy to prove that
\[
\|A\hat{U}\|_{p,v} \leq \|A\|_{2p,m} \|\hat{U}\|_{2p,v}.
\] (2.6)

Let \(\mathcal{T}\) be an interval of the real line. If \(E[|U(t)|^p] < +\infty\) for all \(t \in \mathcal{T}\), then \(\{U(t) : t \in \mathcal{T}\}\) is called a stochastic process of order \(p\) (p.s. p.). If there exists a stochastic process of order \(p\), denoted by \(\hat{U}(t), U^{(1)}(t)\) or \(dU(t)/dt\), such that
\[
\left\| \frac{U(t + h) - U(t)}{h} - \frac{dU(t)}{dt} \right\|_p \to 0 \quad \text{as} \quad h \to 0,
\]
then \(\{U(t) : t \in \mathcal{T}\}\) is said to be \(p\)th mean differentiable at \(t \in \mathcal{T}\).

Furthermore, a p.s. p. \(\{U(t) : |t| < c\}\) is \(p\)th mean analytic on \(|t| < c\) if it can be expanded in the \(p\)th mean convergent Taylor series
\[
U(t) = \sum_{n=0}^{\infty} \frac{U^{(n)}(0)}{n!} t^n,
\]
where \(U^{(n)}(0)\) denotes the derivative of order \(n\) of the s. p. \(U(t)\) evaluated at the point \(t = 0\), in the \(p\)th mean sense. In connection with the \(p\)th mean analyticity, we state without proof the following result that can be found in [3].

**Proposition 2.1.** Let \(\{U(t) : |t| < c\}\) be a \(p\)th mean analytic s. p. given by
\[
U(t) = \sum_{k=0}^{\infty} U_k t^k, \quad U_k = \frac{U^{(k)}(0)}{k!}, \quad |t| < c,
\]
where the derivatives are considered in the \(p\)th mean sense. Then, there exists \(M > 0\) such that
\[
\|U_k\|_p \leq \frac{M}{\rho^k}, \quad 0 < \rho < c, \quad \forall k \geq 0 \quad \text{integer}.
\] (2.8)

We end this section, with a short presentation of the random differential transform method and some of its important operational results that can be found in [8].

The random differential transform of an m. s. analytic s. p. \(U(t)\) is defined as
\[
\hat{U}(k) = \frac{1}{k!} \left[ \frac{d^k (U(t))}{dt^k} \right]_{t=0}, \quad k \geq 0, \quad \text{integer},
\] (2.9)
where \(\hat{U}\) is the transformed s. p. and \(d^k dt^k\) denotes the m. s. derivative of order \(k\). The inverse transform of \(\hat{U}\) is defined as
\[
U(t) = \sum_{k=0}^{\infty} \hat{U}(k) t^k.
\] (2.10)

The next result provides some operational rules for the random differential transform of stochastic processes that will be required later.
Theorem 2.2. Let \( \{F(t) : |t| < c\}, \{G(t) : |t| < c\} \) be m. f. analytic s. p.’s. Then the following results hold:

(i) If \( U(t) = F(t) \pm G(t) \), then \( \hat{U}(k) = \hat{F}(k) \pm \hat{G}(k) \).

(ii) Let \( A \) be a 4-r. v.. If \( U(t) = AF(t) \), then \( \hat{U}(k) = A\hat{F}(k) \).

(iii) Let \( m \) be a nonnegative integer.

\[
\text{If } U(t) = \frac{d^m}{dt^m} F(t), \text{ then } \hat{U}(k) = (k + 1)\ldots(k + m) \hat{F}(k + m). 
\]

(iv) If \( U(t) = F(t)G(t) \), then \( \hat{U}(k) = \sum_{n=0}^{k} \hat{F}(n)\hat{G}(k - n) \).

3 Application of the Random DTM

By applying the operational properties of random DTM given in Theorem 2.2 to the initial value problem (1.1), one obtains

\[
(k + 1)\hat{X}(k + 1) = \sum_{r=0}^{k} \hat{P}_n(r)\hat{X}(k - r) + \hat{B}(k), \quad \text{ (3.1)}
\]

being

\[
\hat{P}_n(r) = \begin{cases} 
A_r, & 0 \leq r \leq n, \\
0, & r > n.
\end{cases} \quad \text{ (3.2)}
\]

Notice that we have applied the deterministic DTM of \( r^i \), \( 1 \leq i \leq n \), is just 1, [9]. Thus equation (3.1) writes

\[
(k + 1)\hat{X}(k + 1) = \sum_{r=0}^{n} A_r\hat{X}(k - r) + \hat{B}(k), \quad k = n, n + 1, \ldots, \quad \text{ (3.3)}
\]

for which the r. v’s \( \hat{X}(0), \ldots, \hat{X}(n) \) are computed from (3.1). The initial value \( \hat{X}(0) \) is computed directly from the initial condition \( X_0 \). Note that from equations (3.1)–(3.3) one can construct an approximation to the solution process of (1.1). In order to prove the m. s. convergence of the inverse transform of \( \hat{X}(k) \) given in (1.2), it is defined:

\[
\vec{Z}(k) := (Z_1(k), Z_2(k), \ldots, Z_n(k), Z_{n+1}(k))^T = (\hat{X}(k - n), \hat{X}(k - n + 1), \ldots, \hat{X}(k - 1), \hat{X}(k))^T. \quad \text{ (3.4)}
\]

Then, recurrence (3.3) is written as

\[
\vec{Z}(k + 1) = A(k)\vec{Z}(k) + \vec{H}(k), \quad \text{ (3.5)}
\]
where

\[
\mathbf{A}(k) = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
A_n & A_{n-1} & A_{n-2} & \ldots & \ldots & A_0
\end{pmatrix},
\]

\[
\mathbf{H}(k) = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
\hat{\mathbf{B}}(k)
\end{pmatrix}.
\]

(3.6)

Iterations of (3.5) yield

\[
\tilde{Z}(k) = \left( \prod_{i=n}^{k-1} \mathbf{A}(i) \right) \tilde{Z}(n) + \sum_{r=n}^{k-1} \left( \prod_{i=r+1}^{k-1} \mathbf{A}(i) \right) \tilde{H}(r),
\]

(3.7)

for \( k = n, n+1, \ldots \) where

\[
\prod_{i=n_0}^{m} \mathbf{A}(i) = \begin{cases} 
\mathbf{A}(m)\mathbf{A}(m-1)\ldots\mathbf{A}(n_0) & \text{if } m \geq n_0, \\
1 & \text{otherwise}. 
\end{cases}
\]

(3.8)

4 Mean Square Convergence

This section is addressed to prove that the inverse transform of \( \hat{X}(k) \) given in (1.2) is an m. s. solution of problem (1.1). This result is summarized as follows

**Theorem 4.1.** Consider problem (1.1) and let us assume that:

i) The polynomial coefficients \( A_j \) are r. v.'s satisfying the following condition:

\[
\exists M, K > 0 : \ E \left[ |A_j^m| \right] \leq KM^m < \infty, \quad \forall m \geq 0, \quad j = 0, 1, \ldots, n.
\]

ii) The initial condition \( X_0 \) is a 16-r. v.

iii) The forcing term \( \{ B(t) : |t| < c \} \) is a 16th mean analytic stochastic process where \( c > \frac{1}{n+1} \).

Then there exists an m. s. solution of the form

\[
X(t) = \sum_{k=0}^{\infty} \hat{X}(k)t^k, \quad |t| < \frac{1}{n+1},
\]

(4.1)

where \( \hat{X}(k) \) are defined by (3.3).
Proof. Expression (3.7) will be used to show that \( \sum_{k=0}^{\infty} \| \tilde{Z}(k) \|_{4,v} |t|^k \) is convergent for \( |t| < d_1 \), where \( d_1 \) is to be determined later. This fact will imply that \( \sum_{k=0}^{\infty} \tilde{X}(k)t^k \) is m. f. convergent for \( |t| < d_2 \) because \( \| \tilde{Z}(k) \|_{4,v} \geq \| \tilde{X}(k) \|_4 \). Note that the m. f. convergence of (4.1) is required because it was a condition for applying Theorem 2.2 to the equation (1.1). Setting \( \tilde{Z}_1(k) = \left( \prod_{i=n}^{k-1} A(i) \right) \tilde{Z}(n) \) and \( \tilde{Z}_2(k) = \sum_{r=n}^{k} \left( \prod_{i=r+1}^{k-1} A(i) \right) \tilde{H}(r) \) one gets: \( \tilde{Z}(k) = \tilde{Z}_1(k) + \tilde{Z}_2(k) \). Then, we will prove that \( \sum_{k=0}^{\infty} \| \tilde{Z}_i(k) \|_{4,v} |t|^k \) are convergent for \( i = 1, 2 \) to achieve our goal. Following previous notation, let us define

\[
\tilde{A} := \prod_{i=n}^{k-1} A(i) = \left( \tilde{A}_{r,s} \right).
\]

From (3.8) it follows that

\[
\tilde{A}_{r,s} = \sum_{s_1,s_2,\ldots,s_{k-n-1}=1}^{n+1} A_{r,s_1}(k-1)A_{s_1,s_2}(k-2)A_{s_2,s_3}(k-3)\ldots A_{s_{k-n-1},s}(n). \tag{4.2}
\]

Note that \( \tilde{A} \) is an \( n + 1 \times n + 1 \)-dimension random matrix. Now, using the norm for (4.2) defined in (2.5) and inequality (2.3) yield

\[
\left\| \prod_{i=n}^{k-1} A(i) \right\|_{2p,m} \leq \sum_{r,s=1}^{n+1} \sum_{s_1,s_2,\ldots,s_{k-n-1}=1}^{n+1} \left\| A_{r,s_1}(k-1)A_{s_1,s_2}(k-2)\ldots A_{s_{k-n-1},s}(n) \right\|_{2p} \leq \sum_{r,s=1}^{n+1} \left( \left\| (A_{r,s_1}(k-1))^{k-n} \right\|_{2p} \right)^{(1-\frac{1}{2p})} \ldots \left( \left\| (A_{s_{k-n-1},s}(n))^{k-n} \right\|_{2p} \right)^{(1-\frac{1}{2p})}. \tag{4.3}
\]

Note that any component \( A_{r,s}(l) \) of the matrix \( A(l) \) can be either 0, 1 or \( \frac{A_j}{l+1} \) with \( j = 0, \ldots, n \) and \( l = n, n+1, \ldots, k-1 \). Then, one obtains

\[
\left( \left\| (A_{r,s}(l))^{k-n} \right\|_{2p} \right)^{(1-\frac{1}{2p})} = \begin{cases} 0 & \text{or}, \\ 1 & \text{or}, \\ \left( E \left( \left( \frac{A_j}{l+1} \right)^{2p(k-n)} \right) \right)^{(1-\frac{1}{2p})}. \end{cases} \tag{4.4}
\]
By i) and (4.4) it follows that
\[
\left( E \left[ \left( \frac{A_j}{l+1} \right)^{2p(k-n)} \right] \right)^\frac{1}{2p(k-n)} \leq K\frac{\gamma_{p(l-n)} M}{l + 1}.
\]  
(4.5)

On the other hand, by (3.3)–(3.4) and observing that the terms involved in \( \vec{Z}(n) \) have the form \( A_0 \ldots A_{\gamma_{n-1}}X_0 \) or \( A_0 \ldots A_{\gamma_{n-1}}B(k) \) for \( k = 0, \ldots, n - 1, \gamma_i \) nonnegative integers such that \( 0 \leq \gamma_1 + \ldots + \gamma_{n-1} \leq n, \) i) and ii) imply that there exists \( C(n) \) such that \( \| \vec{Z}(n) \|_{2p,v} \leq C(n) < \infty. \)

Now we analyze the bounds of \( \vec{Z}_1(k) \) and \( \vec{Z}_2(k). \) To conduct the study, we distinguish two cases.

Case I: If \( \frac{K\gamma_{p(k-n)} M}{l + 1} \leq 1, \) for all \( l \) and \( k \) then \( \left\| \prod_{i=n}^{k-1} A(i) \right\|_{2p,m} \leq (n + 1)^{k-n+1}. \) By using (2.6) with \( p = 4 \) yields
\[
\left\| \vec{Z}_1(k) \right\|_{4,v} \leq \left\| \prod_{i=n}^{k-1} A(i) \right\|_{8,m} \quad \text{and} \quad \left\| \vec{Z}(n) \right\|_{4,v} \leq C(n)(n + 1)^{k-n+1}.
\]

Hence, in this case for each \( t, \) the series \( \sum_{k=0}^{\infty} \left\| \vec{Z}_1(k) \right\|_{2p,v} |t|^k \) can be majorized by \( \sum_{k=0}^{\infty} \alpha_k, \)

being \( \alpha_k = C(n)(n + 1)^{k-n+1} \) which is convergent for \( |t| < \frac{1}{n + 1} \) as it can be checked by D’Alembert’s test.

Also \( \left\| \prod_{i=r+1}^{k-1} A(i) \right\|_{2p,m} \leq (n + 1)^{k-r} \) and by Proposition 2.1 it follows that
\[
\left\| \vec{H}(r) \right\|_{2p,v} = \left\| \hat{B}(r) \right\|_{2p} \leq \frac{M_1}{(r + 1)^\rho},
\]  
(4.6)

for \( 0 < \rho < c \) and \( M_1 > 0. \) Thus,
\[
\sum_{k=n+1}^{\infty} \left\| \vec{Z}_2(k) \right\|_{p,v} |t|^k \leq M_1 \sum_{k=n+1}^{\infty} \left( \sum_{r=n}^{k-1} \frac{1}{r + 1} \left( \frac{1}{(n + 1)\rho} \right)^r \right) (n + 1)^k |t|^k.
\]  
(4.7)

Using D’Alembert’s test, one proves that the series in the right-hand side of (4.7) is convergent for \( |t| < \frac{1}{n + 1} \) for all \( n \geq 1 \) and \( c > \rho \geq \frac{1}{n + 1}. \)

Case II: If \( \frac{K\gamma_{p(k-n)} M}{l + 1} \geq 1, \) for all \( l \) and \( k, \) then
\[
\left\| \prod_{i=n}^{k-1} A(i) \right\|_{2p,m} \leq n! K\frac{\gamma_{p} M^{k-n}(n + 1)^{k-n+1}}{k!}.
\]
On the other hand, we know that \( \| \tilde{Z}(n) \|_{2p,v} \leq C(n) \) and then \( \sum_{k=n+1}^{\infty} \| \tilde{Z}_i(k) \|_{2p,v} |t|^k \) is majorized by
\[
\sum_{k=n+1}^{\infty} C(n)n!K^{\frac{k}{2p}} M^{k-n}(n+1)^{k-n+1} \frac{|t|^k}{k!},
\]
which is convergent on the whole real line.

Taking into account that \( \frac{k-r-1}{k-n} < 1 \), hence for \( K > 1 \) we have that
\[ K^{\frac{k-r-1}{2p(n-k-n)}} < K^{\frac{1}{2p}}. \] (4.8)

By
\[
\left\| \prod_{i=r+1}^{k-1} A(i) \right\|_{2p,m} \leq \frac{(n+1)^{k-r} (r+1)! K^{\frac{k-r}{2p(n-k-n)}} M^{k-r-1}}{k!}
\]
and (4.6), (4.8), it follows for \( K > 1 \) that
\[
\sum_{k=n+1}^{\infty} \| Z_2(k) \|_{p,v} |t|^k \leq M_1 K^{\frac{k-r}{2p}} \sum_{k=n+1}^{\infty} \frac{(r!)^2}{(n+1)^r \rho^r M^{2r}} \frac{(n+1)^k}{k!} M^{k-1} |t|^k. \] (4.9)

It can be checked that the series in the right side of (4.9) is convergent for \(|t| < \rho\). The same remains true for \( K < 1 \). By doing an analogous study of combining Case I and Case II, it is proved that the radios of convergence of the series \( \sum_{k=0}^{\infty} \| \tilde{Z}_i(k) \|_{2p,v} |t|^k \), \( i = 1, 2 \) are greater or equal to \( \frac{1}{n+1} \).

†

In order to obtain approximations of the mean and standard deviation functions by the DTM, we truncate the infinite sum given by (4.1):
\[ X_N(t) = \sum_{k=0}^{N} \hat{X}(k)t^k. \] (4.10)

Thus,
\[ \mathbb{E}[X_N(t)] = \sum_{k=0}^{N} \mathbb{E}[\hat{X}(k)] t^k \] (4.11)
\[ \mathbb{E}[(X_N(t))^2] = \sum_{k=0}^{N} \mathbb{E}\left[\left(\hat{X}(k)\right)^2 \right] t^{2k} + 2 \sum_{k=1}^{N} \sum_{l=0}^{k-1} \mathbb{E}\left[\hat{X}(k)\hat{X}(l)\right] t^{k+l}. \] (4.12)

This completes the proof.
In this section several illustrative examples in connection with the previous development are provided. The mean and standard deviation of the truncated solution are computed from (4.11)–(4.12).

**Example 5.1.** Let us consider equation (1.1) with $P_1(t) = At$, where $A$ is a standard normal r. v. ($A \sim N(0,1)$), $B(t) = e^{-Bt}$, where $B$ is an exponential r. v. with parameter $\lambda = 1$ ($B \sim \text{Exp}(\lambda = 1)$) and $X_0$ is a Beta r. v. with parameters $\alpha = 2$ and $\beta = 3$ ($X_0 \sim \text{Be}(\alpha = 2; \beta = 3)$). To simplify computations, in this example we will assume that $A$, $B$ and $X_0$ are independent r. v.'s. In order that i), iii) of Theorem 4.1 are satisfied, we truncated the r. v.'s $A$ and $B$. Therefore, for those r. v.'s we select $L_1, L_2 > 0$ sufficiently large so as to get $\mathbb{P}(B \geq L_1), \mathbb{P}(|A| \geq L_2)$ arbitrarily small, see the truncated method in [6]. Let us say that $\bar{A}, \bar{B}$ are the corresponding truncated r. v.'s of $A$ and $B$, respectively. It is easy to show that $\tilde{B}(t) := e^{-\bar{B}t}$ is $16$-th mean analytic on the whole real line and $\bar{A}$ satisfies i). Clearly, $X_0$ is a $16$-r. v. Hence, Theorem 4.1 holds and $X(t)$ given by (4.1) is an m. s. solution of equation (1.1). Table 5.1 contains the numerical results of the mean $\mathbb{E}[X_N(t)]$ and the standard deviation $\sigma[X_N(t)]$ using (4.11)–(4.12) with $N = 20$ as well as the mean $\mu_{MC}^m(X(t))$ and the standard deviation $\sigma_{MC}^m(X(t))$ using Monte Carlo sampling with $m = 100000$ simulations. It was chosen $L_1 = 100$ and $L_2 = 8$ that made $\mathbb{P}(B \geq L_1) = 3.72008 \times 10^{-44}$ and $\mathbb{P}(|A| \geq L_2) = 1.22125 \times 10^{-15}$. Notice that the approximations provided by the random DTM are in agreement with Monte Carlo simulations.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\mu_{MC}^m(X(t))$</th>
<th>$\mathbb{E}[X_N(t)]$</th>
<th>$\sigma_{MC}^m(X(t))$</th>
<th>$\sigma[X_N(t)]$</th>
</tr>
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<td>0.250000</td>
<td>0.193389</td>
<td>0.193649</td>
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<tr>
<td>0.10</td>
<td>0.344950</td>
<td>0.345314</td>
<td>0.193448</td>
<td>0.193771</td>
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<tr>
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<td>0.432082</td>
<td>0.432392</td>
<td>0.194243</td>
<td>0.194511</td>
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<tr>
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<td>0.512766</td>
<td>0.197344</td>
<td>0.197620</td>
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<tr>
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<td>0.205386</td>
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<td>0.659248</td>
<td>0.220580</td>
<td>0.220914</td>
</tr>
</tbody>
</table>

Table 5.1: Comparison of the expectation and standard function of Example 5.1 by using expressions (4.11)–(4.12) with $N = 20$ and Monte Carlo method with $m = 100000$ simulations at some selected points of the interval $0 \leq t \leq 1/2$.

**Example 5.2.** We now consider $P_1(t) = At$ where $A \sim \text{Be}(\alpha_1 = 1, \beta_1 = 2)$, $B(t) = B$ where $B \sim \text{Gamma}(\alpha_2 = 2, \beta_2 = 3)$ and $X_0 \sim \text{Unif}([10, 14])$. Condition i) of Theorem 4.1 is satisfied because $A$ has codomain $[0,1]$. Clearly $B$ is $16$-th mean analytic and $X_0$ is a $16$th r. v. therefore $X(t)$ given by (4.1) is an m. s. solution of equation (1.1).
Using the same notation to the mean and standard deviation of Example 5.1, we show the numerical results in Table 5.2. Note that Monte Carlo simulations are in agreement with random DTM results.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\mu^\text{MC}_m (X(t))$</th>
<th>$m = 150000$</th>
<th>$\text{E}[X_N(t)]$</th>
<th>$\sigma^\text{MC}_m (X(t))$</th>
<th>$\sigma[X_N(t)]$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>12.00000</td>
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<td>1.15469</td>
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<tr>
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<td>1.58004</td>
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<tr>
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<td>15.60148</td>
<td>2.19295</td>
<td>2.19409</td>
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</tr>
</tbody>
</table>

Table 5.2: Comparison of the expectation and standard function of Example 5.2 by using expressions (4.11)–(4.12) with $N = 10$ and Monte Carlo method with $m = 150000$ simulations at some selected points of the interval $0 \leq t \leq 1/2$.

**Example 5.3.** The differential equation $\dot{X}(t) = (\gamma \cos t)X(t)$ is a mathematical model for the population $X(t)$ that undergoes yearly seasonal fluctuations [10]. In the deterministic scenario, $\gamma$ is a positive parameter that represents the intensity of the fluctuations. However, in practice, it depends heavily on uncertain factors such as weather, genetics, prey and predator species and other characteristics of the surrounding medium. This motivates the consideration of parameter $\gamma$ as a r. v. rather than a deterministic constant leading to a random differential equation. To apply Theorem 4.1, we use the approximations $\cos t \approx 1 - \frac{1}{2} t^2$ which leads to equation $\dot{X}(t) = (\gamma - \frac{\gamma}{2} t^2)X(t)$. Thus $P_2(t) = \gamma - \frac{\gamma}{2} t^2$ and so $A_0 = \gamma$, $A_1 = 0$, $A_2 = -\frac{\gamma}{2}$. Assuming that $\gamma \sim \text{Be}(\alpha = 2, \beta = 5)$ and $X_0$ is a r. v. such that $X_0 \sim \text{Unif}([110, 140])$, in Table 5.3 we collect the numerical approximations for the expectation and the standard deviation to the solution s. p. $X(t)$ using both Monte Carlo with $10^6$ simulations and random DTM with truncation order $N = 10$. Again, we observe that results provided by both approaches agree.

We would like to point out that the main contribution of this paper is the rigorous construction of a mean square analytic solution of an important class of a time-dependent random differential equation by the random differential transform method. The approach allowed to provide reliable approximations for the first and second moments of the solution stochastic process.
Table 5.3: Comparison of the expectation and standard function of Example 5.3 by using expressions (4.11)–(4.12) with $N = 10$ and Monte Carlo method with $m = 100000$ simulations at some selected points of the interval $0 \leq t \leq 1/2$.

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References


