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Spectral Theory of X_1-Laguerre Polynomials

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Abstract

In 2009, Gómez–Ullate, Kamran, and Milson characterized all sequences of polynomials $\{p_n\}_{n=1}^{\infty}$, with $\deg p_n = n \geq 1$, that are eigenfunctions of a second–order differential equation and are orthogonal with respect to a positive Borel measure on the real line having finite moments of all orders. Up to a complex linear change of variable, the only such sequences are the X_1-Laguerre and the X_1-Jacobi polynomials. In this paper, we discuss the self-adjoint operator, generated by the second-order X_1-Laguerre differential expression, that has the X_1-Laguerre polynomials as eigenfunctions.

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1 Introduction

The classical orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ of Jacobi, Laguerre, and Hermite can be characterized, up to a complex linear change of variable, in several ways. Indeed, they are the only positive-definite orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ satisfying any of the following four equivalent conditions:

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(a) the sequence of first derivatives $\{p_n'(x)\}_{n=1}^{\infty}$ is also orthogonal with respect to a positive measure;

(b) the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a Rodrigues formula

$$p_n(x) = K_n^{-1}(w(x))^{-1}\frac{d^n}{dx^n}\left(\rho^n(x)w(x)\right) \quad (n \in \mathbb{N}),$$

where $K_n > 0$, $w > 0$ on some interval $I = (a,b)$, and where ρ is a polynomial of degree ≤ 2;

(c) the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a differential recursion relation of the form

$$\pi(x)p_n'(x) = (\alpha_n x + \beta_n)p_n(x) + \gamma_n p_{n-1}(x) \quad (n \in \mathbb{N},\ p_{-1}(x) = 0),$$

where $\pi(x)$ is some polynomial and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences;

(d) there exists a second-order differential expression

$$m[y](x) = b_2(x)y''(x) + b_1(x)y'(x) + b_0(x)y(x),$$

and a sequence of complex numbers $\{\mu_n\}_{n=0}^{\infty}$ such that $y = p_n(x)$ is a solution of

$$m[y](x) = \mu_n y(x) \quad (n \in \mathbb{N}_0).$$

Three excellent sources for further information on the theory of orthogonal polynomials are the texts of Chihara [3], Ismail [15], and Szegö [26]. The characterization given in (d) above is generally attributed to S. Bochner [2] in 1929 and P. A. Lesky [19] in 1962. However, in recent years, it has become clear that the question was addressed earlier by E. J. Routh [25] in 1885; see [15, p. 509]. Extensions of characterization (d) to a *real* linear change of variable was considered by Kwon and Littlejohn [17] in 1997 and to orthogonality with respect to a Sobolev inner product by Kwon and Littlejohn [18] in 1998.

In 2009, Gómez–Ullate, Kamran, and Milson [8] (see also [7, 9–13]) characterized all polynomial sequences $\{p_n\}_{n=1}^{\infty}$, with $\deg p_n = n \geq 1$, which satisfy the following conditions:

(i) there exists a second-order differential expression

$$\ell[y](x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x),$$

and a sequence of complex numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $y = p_n(x)$ is a solution of

$$\ell[y](x) = \lambda_n y(x) \quad (n \in \mathbb{N});$$

each coefficient $a_i(x)$ is a function of the independent variable x and does not depend on the degree of the polynomial eigenfunctions;

(ii) if c is any nonzero constant, $y(x) \equiv c$ is *not* a solution of $\ell[y](x) = \lambda y(x)$ for any $\lambda \in \mathbb{C}$;

(iii) there exists an interval I and a Lebesgue measurable function $w(x)$ $(x \in I)$ such that

$$\int_I p_n(x)p_m w(x)dx = K_n \delta_{n,m},$$

where $K_n > 0$ for each $n \in \mathbb{N}$;

(iv) all moments $\{\mu_n\}_{n=0}^{\infty}$ of w, defined by

$$\mu_n = \int_I x^n w(x)dx \quad (n = 0, 1, 2, \ldots),$$

exist and are finite.

Up to a complex linear change of variable, the authors in [8] show that the only solutions to this classification problem are the *exceptional* X_1-Laguerre and the X_1-Jacobi polynomials. Their results are spectacular and remarkable; indeed, it was believed, due to the 'Bochner' classification, that among the class of all orthogonal polynomials, only the Hermite, Laguerre, and Jacobi polynomials, satisfy second-order differential equations and are orthogonal with respect to a positive-definite inner product of the form

$$(p, q) = \int_{\mathbb{R}} p\overline{q}d\mu.$$

Even though the authors in [8] introduce the notion of exceptional polynomials via Sturm–Liouville theory, the path that they followed to their discovery was motivated by their interest in quantum mechanics, specifically with their intent to extend exactly solvable and quasi-exactly solvable potentials beyond the Lie algebraic setting. The impetus that led to their work in [8] came from an earlier paper [16] of Kamran, Milson, and Olver on evolution equations reducible to finite–dimensional dynamical systems. It is important to note as well that the work in [8] was not originally motivated by orthogonal polynomials although they set out to construct potentials that would be solvable by polynomials which fall outside the realm of the classical theory of orthogonal polynomials. To further note, their work was inspired by the paper of Post and Turbiner [23] who formulated a *generalized Bochner problem* of classifying the linear differential operators in one variable leaving invariant a given vector space of polynomials.

The X_1-Laguerre and X_1-Jacobi polynomials, as well as subsequent generalizations, are exceptional in the sense that they start at degree ℓ $(\ell \geq 1)$ instead of degree 0, thus avoiding the restrictions of the Bochner classification but still satisfy second-order differential equations of spectral type. Reformulation within the framework of one-dimensional quantum mechanics and shape invariant potentials followed their work by various other authors; for example, see [22, 24]. Furthermore, the two second-order differential equations that they discover in their X_1 classification are important examples illustrating the Stone–von Neumann theory [4, Chapter 12] and the Glazman–Krein–Naimark theory [21, Section 18] of differential operators.

In this paper, we will focus on the spectral theory associated with the X_1-Laguerre polynomials; the main objective will be to fill in the details of the analysis briefly outlined in [5, 8].

2 Summary of Properties of X_1-Laguerre Polynomials

We summarize some of the main properties of the X_1-Laguerre polynomials $\left\{\hat{L}_n^{\alpha}\right\}_{n=1}^{\infty}$, as discussed in [8]. Unless otherwise indicated, we assume that the parameter $\alpha > 0$. The X_1-Laguerre polynomial $y = \hat{L}_n^{\alpha}(x)$ $(n \in \mathbb{N})$ satisfies the second-order differential equation

$$\ell_\alpha[y](x) = \lambda_n y(x) \quad (0 < x < \infty),$$

where

$$\ell_\alpha[y](x) := -xy'' + \frac{(x-\alpha)(x+\alpha+1)}{(x+\alpha)}y' - \frac{(x-\alpha)}{(x+\alpha)}y \tag{2.1}$$

and

$$\lambda_n = n-1 \quad (n \in \mathbb{N}). \tag{2.2}$$

These polynomials are orthogonal on $(0,\infty)$ with respect to the weight function

$$w_\alpha(x) = \frac{x^\alpha e^{-x}}{(x+\alpha)^2} \quad (x \in (0,\infty)). \tag{2.3}$$

The X_1-Laguerre polynomials form a complete orthogonal set in the Hilbert–Lebesgue space $L^2((0,\infty); w_\alpha)$, defined by

$$L^2((0,\infty); w_\alpha) := \left\{ f : (0,\infty) \to \mathbb{C} \mid f \text{ is measurable and } \int_0^\infty |f|^2 w_\alpha < \infty \right\}.$$

More specifically, with

$$\|f\|_\alpha := \left(\int_0^\infty |f(x)|^2 w_\alpha(x)dx \right)^{1/2},$$

it is the case that

$$\left\|\hat{L}_n^{\alpha}\right\|_\alpha^2 = \left(\frac{\alpha+n}{\alpha+n-1}\right)\frac{\Gamma(\alpha+n)}{(n-1)!} \quad (n \in \mathbb{N}).$$

The fact that $\left\{\hat{L}_n^{\alpha}\right\}_{n=1}^{\infty}$ forms a complete set $L^2((0,\infty); w_\alpha)$ is remarkable, and certainly counterintuitive, since $f(x) = 1 \in L^2((0,\infty); w_\alpha)$.

The X_1-Laguerre polynomials can be written in terms of the classical Laguerre polynomials $\{L_n^\alpha(x)\}_{n=0}^\infty$; specifically,

$$\hat{L}_n^\alpha(x) = -(x+\alpha+1)L_{n-1}^\alpha(x) + L_{n-2}^\alpha(x) \quad (n \in \mathbb{N}),$$

where $L_{-1}^\alpha(x) = 0$. Moreover, the X_1-Laguerre polynomials satisfy the (nonstandard) three-term recurrence relation:

$$\begin{aligned} 0 &= (n+1)\left((x+\alpha)^2(n+\alpha)-\alpha\right)\hat{L}_{n+2}^\alpha(x) \\ &\quad + (n+\alpha)\left((x+\alpha)^2(x-2n-\alpha-1)+2\alpha\right)\hat{L}_{n+1}^\alpha(x) \\ &\quad + (n+\alpha-1)\left((x+\alpha)^2(n+\alpha+1)-\alpha)\right)\hat{L}_n^\alpha(x). \end{aligned}$$

However in stark contrast with the classical Laguerre polynomials $\{L_n^\alpha\}_{n=0}^\infty$, whose roots are all contained in the interval $(0,\infty)$ when $\alpha > -1$, the X_1-Laguerre polynomials $\left\{\hat{L}_n^\alpha\right\}_{n=1}^\infty$ have $n-1$ roots in $(0,\infty)$ and one zero in the interval $(-\infty,-\alpha)$ whenever $\alpha > 0$.

3 X_1-Laguerre Spectral Analysis

The background material for this section can be found in the classic texts of Akhiezer and Glazman [1], Hellwig [14], and Naimark [21].

In Lagrangian symmetric form, the X_1-Laguerre differential expression (2.1) is given by

$$\ell_\alpha[y](x) = \frac{1}{w_\alpha(x)}\left(-\left(\frac{x^{\alpha+1}e^{-x}}{(x+\alpha)^2}y'(x)\right)' - \frac{x^\alpha e^{-x}(x-\alpha)}{(x+\alpha)^3}y(x)\right) \quad (x>0), \quad (3.1)$$

where $w_\alpha(x)$ is the X_1-Laguerre weight defined in (2.3).

The maximal domain associated with $\ell_\alpha[\cdot]$ in the Hilbert space $L^2\left((0,\infty);w_\alpha\right)$ is defined to be

$$\Delta := \left\{f:(0,\infty)\to\mathbb{C} \mid f,f' \in AC_{\text{loc}}(0,\infty); f,\ell_\alpha[f] \in L^2((0,\infty);w_\alpha)\right\}.$$

The associated *maximal operator*

$$T_1 : \mathcal{D}(T_1) \subset L^2\left((0,\infty);w_\alpha\right) \to L^2((0,\infty);w_\alpha),$$

is defined by

$$\begin{aligned} T_1 f &= \ell_\alpha[f] \\ f \in \mathcal{D}(T_1) &:= \Delta. \end{aligned} \quad (3.2)$$

For $f, g \in \Delta$, Green's formula can be written as

$$\int_0^\infty \ell_\alpha[f](x)\overline{g}(x)w_\alpha(x)dx = [f,g](x) \mid_{x=0}^{x=\infty} + \int_0^\infty f(x)\ell_\alpha[\overline{g}](x)w_\alpha(x)dx,$$

where $[\cdot,\cdot](\cdot)$ is the sesquilinear form defined by

$$[f,g](x) := \frac{x^{\alpha+1}e^{-x}}{(x+\alpha)^2}\left(f(x)\overline{g}'(x) - f'(x)\overline{g}(x)\right) \quad (0 < x < \infty), \tag{3.3}$$

and where

$$[f,g](x) \mid_{x=0}^{x=\infty} := [f,g](\infty) - [f,g](0).$$

By definition of Δ (and the classical Hölder's inequality), notice that the limits

$$[f,g](0) := \lim_{x\to 0^+}[f,g](x) \text{ and } [f,g](\infty) := \lim_{x\to\infty}[f,g](x)$$

exist and are finite for each $f, g \in \Delta$.

The term 'maximal' comes from the fact that Δ is the largest subspace of functions in $L^2((0,\infty);w_\alpha)$ for which T_1 maps back into $L^2((0,\infty);w_\alpha)$. It is not difficult to see that Δ is dense in $L^2((0,\infty);w_\alpha)$; consequently, the adjoint T_0 of T_1 exists as a densely defined operator in $L^2((0,\infty);w_\alpha)$. For obvious reasons, T_0 is called the *minimal operator* associated with $\ell_\alpha[\cdot]$. From [1] or [21], this minimal operator $T_0 : \mathcal{D}(T_0) \subset L^2((0,\infty);w_\alpha) \to L^2((0,\infty);w_\alpha)$ is defined by

$$\begin{aligned} T_0 f &= \ell_\alpha[f] \\ f \in \mathcal{D}(T_0) &: = \left\{ f \in \Delta \mid [f,g] \mid_{x=0}^{x=\infty} = 0 \text{ for all } g \in \Delta \right\}. \end{aligned} \tag{3.4}$$

The minimal operator T_0 is a closed, symmetric operator in $L^2((0,\infty);w_\alpha)$; furthermore, because the coefficients of $\ell[\cdot]$ are real, T_0 necessarily has equal deficiency indices m, where m is an integer satisfying $0 \leq m \leq 2$. Therefore, from the general Stone–von Neumann [4] theory of self-adjoint extensions of symmetric operators, T_0 has self-adjoint extensions. We seek to find the self-adjoint extension T in $L^2((0,\infty);w_\alpha)$, generated by $\ell[\cdot]$, which has the X_1-Laguerre polynomials $\left\{\hat{L}_n^\alpha\right\}_{n=1}^\infty$ as eigenfunctions. In order to compute the deficiency indices, it is necessary to study the behavior of solutions of $\ell[y] = 0$ near each of the singular endpoints $x = 0$ and $x = \infty$.

3.1 Endpoint Behavior Analysis

The endpoint $x = 0$ is, in the sense of Frobenius, a regular singular endpoint of the X_1-Laguerre equation $\ell_\alpha[y] = \lambda y$, for any value of $\lambda \in \mathbb{C}$. The Frobenius indicial equation at $x = 0$ is

$$r(r+\alpha) = 0.$$

Consequently, two linearly independent solutions of $\ell_\alpha[y] = 0$ will behave asymptotically like

$$z_1(x) := 1 \text{ and } z_2(x) := x^{-\alpha} \quad (0 < x \leq 1).$$

Now, for any $\alpha > 0$,

$$\int_0^1 |z_1(x)|^2 \, w_\alpha(x)dx < \infty;$$

however,

$$\int_0^1 |z_2(x)|^2 \, w_\alpha(x)dx < \infty$$

only when $0 < \alpha < 1$. In the vernacular of the Weyl limit-point/limit-circle analysis (see [14]), this Frobenius analysis shows that the X_1-Laguerre differential expression is in the limit-circle case at $x = 0$ when $0 < \alpha < 1$ and in the limit-point case when $\alpha \geq 1$.

The analysis at the endpoint $x = \infty$ is more complicated since $x = \infty$ is an irregular singular endpoint of the X_1-Laguerre differential expression; consequently, another asymptotic method must be employed. Fortunately, we are able to explicitly solve the differential equation $\ell_\alpha[y](x) = 0$ for a basis $\{y_1(x), y_2(x)\}$ of solutions. Indeed,

$$y_1(x) = x + \alpha + 1 \quad (x > 0)$$

and, for fixed but arbitrary $x_0 > 0$,

$$y_2(x) = (x + \alpha + 1) \int_{x_0}^{x} \frac{e^t \, (t + \alpha)^2}{t^{\alpha+1} \, (t + \alpha + 1)^2} \, dt \quad (x > 0) \tag{3.5}$$

are independent solutions to $\ell_\alpha[y](x) = 0$. In fact, $y_1(x) = \hat{L}_1^\alpha(x)$ is the X_1-Laguerre polynomial of degree 1, while $y_2(x)$ is obtained by the well-known reduction of order method. It is straightforward to see that

$$\int_0^\infty |y_1(x)|^2 \, w_\alpha(x)dx < \infty. \tag{3.6}$$

However, we have the following result

Lemma 3.1. *For any $\alpha > 0$, the solution $y_2(x)$, defined in* (3.5)*, satisfies*

$$\int_1^\infty |y_2(x)|^2 \, w_\alpha(x)dx = \infty; \tag{3.7}$$

that is to say, $y_2 \notin L^2\left((0,\infty); w_\alpha\right)$.

Proof. To begin, we note that, for $x_0 > 0$,

$$\begin{aligned}\int_{x_0}^{x} \frac{e^t (t+\alpha)^2}{t^{\alpha+1}(t+\alpha+1)^2}\,dt &\geq \frac{\alpha^2}{(\alpha+1)^2}\int_{x_0}^{x}\frac{e^t}{t^{\alpha+1}}dt \\ &\geq \frac{\alpha^2}{(\alpha+1)^2}\int_{x_0}^{x} e^{t/2}dt \text{ for large enough } x_0 > 0 \\ &\geq Ae^{x/2} \quad (x \geq x_1 \geq x_0)\end{aligned}$$

for large enough $x_1 \geq x_0$ and where A is some positive constant. It follows that

$$\begin{aligned}|y_2(x)|^2 &= (x+\alpha+1)^2\left(\int_{x_0}^{x}\frac{e^t (t+\alpha)^2}{t^{\alpha+1}(t+\alpha+1)^2}\,dt\right)^2 \\ &\geq A^2(x+\alpha+1)^2 e^x \text{ for } x \geq x_1.\end{aligned}$$

Hence, for any $\alpha > 0$, we see that

$$\begin{aligned}\int_1^\infty |y_2(x)|^2 w_\alpha(x)dx &\geq \int_{x_1}^\infty |y_2(x)|^2 w_\alpha(x)dx \\ &\geq A^2\int_{x_1}^\infty \frac{x^\alpha(x+\alpha+1)^2}{(x+\alpha)^2}dx \\ &\geq A^2\int_{x_1}^\infty x^\alpha dx = \infty.\end{aligned}$$

This concludes the proof. □

To summarize,

Theorem 3.2. *For $\alpha > 0$, let $\ell_\alpha[\cdot]$ be the X_1-Laguerre differential expression, defined in* (2.1) *or* (3.1)*, on the interval* $(0,\infty)$.

(a) $\ell_\alpha[\cdot]$ *is in the limit-point case at $x = 0$ when $\alpha \geq 1$ and is in the limit-circle case at $x = 0$ when $0 < \alpha < 1$;*

(b) $\ell_\alpha[\cdot]$ *is in the limit-point case at $x = \infty$ for any choice of $\alpha > 0$.*

Consequently,

Theorem 3.3. *Let T_0 be the minimal operator in $L^2((0,\infty); w_\alpha)$, defined in* (3.4)*, generated by the X_1-Laguerre differential expression $\ell_\alpha[\cdot]$.*

(a) If $0 < \alpha < 1$, the deficiency index of T_0 is $(1,1)$;

(b) If $\alpha \geq 1$, the deficiency index of T_0 is $(0,0)$.

3.2 Spectral Analysis for $\alpha \geq 1$

From Theorem 3.3(b), the next result follows from the general theory of self-adjoint extensions of symmetric operators [4, Chapter 12, Section 4] and results from Section 2, in particular the completeness of the X_1-Laguerre polynomials in $L^2((0,\infty);w_\alpha)$ and the fact that $\lambda_n \to \infty$ as $n \to \infty$, where λ_n is defined in (2.2).

Theorem 3.4. *For $\alpha \geq 1$, the maximal and minimal operators coincide. In this case, $T_0 = T_1$ is self-adjoint and is the only self-adjoint extension of the minimal operator T_0 in $L^2((0,\infty);w_\alpha)$. Furthermore, in this case, the X_1-Laguerre polynomials $\left\{\hat{L}_n^\alpha\right\}_{n=1}^\infty$ are eigenfunctions of T_0; the spectrum $\sigma(T_0)$ consists only of eigenvalues and is given by*

$$\sigma(T_0) = \mathbb{N}_0.$$

Hence, for $\alpha \geq 1$, no boundary condition restrictions of the maximal domain are needed to generate a self-adjoint extension of the minimal operator T_0.

3.3 Spectral Analysis for $0 < \alpha < 1$

From Theorem 3.3, the general Glazman–Krein–Naimark theory [21, Section 18] implies that one appropriate boundary condition at $x = 0$ is needed in order to obtain a self-adjoint extension of the minimal operator T_0. Necessarily, this boundary condition takes the form

$$[f, g_0](0) = 0 \quad (f \in \Delta),$$

for some appropriately chosen $g_0 \in \Delta \diagdown \mathcal{D}(T_0)$, where $[\cdot,\cdot](\cdot)$ is the sesquilinear form defined in (3.3). In this section, we show that $g_0(x) = 1$ is such an appropriate function which generates the self-adjoint extension T having the X_1-Laguerre polynomials $\left\{\hat{L}_n^\alpha\right\}_{n=1}^\infty$ as eigenfunctions. Notice that, for any $\alpha > 0$, the function $1 \in \Delta$.

The function $y(x) = x^{-\alpha} \in L^2((0,\infty);w_\alpha)$ if and only if $0 < \alpha < 1$. Remarkably, a calculation shows that

$$\ell_\alpha[x^{-\alpha}] = -(\alpha+1)x^{-\alpha};$$

consequently, $x^{-\alpha} \in \Delta$ when $0 < \alpha < 1$. Moreover, from (3.3), we find that

$$[x^{-\alpha}, 1](0) = \alpha \lim_{x \to 0^+} \frac{e^{-x}}{(x+\alpha)^2} = \frac{1}{\alpha} \neq 0.$$

Hence $1 \notin \mathcal{D}(T_0)$. Therefore, from the Glazman–Krein–Naimark Theorem [21, Section 18, Theorem 4], we obtain the following result.

Theorem 3.5. *Suppose $0 < \alpha < 1$. The operator $T : \mathcal{D}(T) \subset L^2((0,\infty);w_\alpha) \to L^2((0,\infty);w_\alpha)$, defined by*

$$Tf = \ell_\alpha[f]$$
$$f \in \mathcal{D}(T) := \{f \in \Delta \mid [f, 1](0) = 0\},$$

is self-adjoint in $L^2((0,\infty);w_\alpha)$ *and has the* X_1*-Laguerre polynomials* $\left\{\hat{L}_n^\alpha\right\}_{n=1}^\infty$ *as eigenfunctions. Moreover, the spectrum of* T *consists only of eigenvalues and is given by*

$$\sigma(T) = \mathbb{N}_0.$$

4 Final Remark: the case $\alpha = 0$

The X_1-Laguerre weight function $w_\alpha(x)$, defined in (2.3), is defined for $\alpha > 0$. When $\alpha = 0$, this weight function reduces to $w_0(x) := x^{-2}e^{-x}$. Consequently, $\alpha = 0$ in the X_1-Laguerre case corresponds to $\alpha = -2$ in the ordinary Laguerre case. Furthermore, in this situation, the X_1-Laguerre equation (2.1) reduces to the ordinary Laguerre equation

$$\ell_0[y](x) := -xy''(x) + (x+1)y'(x) - y(x) = \lambda y(x) \quad (0 < x < \infty). \tag{4.1}$$

For each $n \in \mathbb{N}_0$, the Laguerre polynomial $y(x) = L_n^{-2}(x)$ is a solution of $\ell_0[y](x) = (n-1)y(x)$. The Laguerre polynomials $\left\{L_n^{-2}\right\}_{n=2}^\infty$ (of degree ≥ 2) form a complete orthogonal set in the Hilbert space $L^2((0,\infty);w_0)$; moreover, for $i = 0, 1$, the polynomials $L_i^{-2} \notin L^2((0,\infty);w_0)$. The spectral theory for this Laguerre case, in $L^2((0,\infty);w_0)$, was established in [6]; specifically, the authors in [6] develop the self-adjoint operator in $L^2((0,\infty);w_0)$, generated by the Laguerre expression (4.1), which has the Laguerre polynomials $\left\{L_n^{-2}\right\}_{n=2}^\infty$ as eigenfunctions. These authors also discuss the spectral theory of (4.1) which has the *entire* sequence of Laguerre polynomials $\left\{L_n^{-2}\right\}_{n=0}^\infty$ as eigenfunctions. The analysis, in this case, is in the Hilbert–Sobolev space $(S, (\cdot,\cdot))$, where

$$S = \left\{f : [0,\infty) \to \mathbb{C} \mid f, f' \in AC_{\text{loc}}[0,\infty), f'' \in L^2((0,\infty);e^{-x})\right\},$$

and where $(\cdot,\cdot)$ is the positive-definite inner product defined by

$$(f,g) = 3f(0)\overline{g}(0) - f'(0)\overline{g}(0) - f(0)\overline{g}'(0) + \int_0^\infty f''(x)\overline{g}''(x)e^{-x}dx.$$

The authors find, and discuss, the self-adjoint operator in $(S, (\cdot,\cdot))$ having the Laguerre polynomials $\{L_n^{-2}\}_{n=0}^\infty$ as a complete orthogonal set of eigenfunctions. The key to establishing this analysis of (4.1) in $(S, (\cdot,\cdot))$ is the general left-definite spectral theory developed by Littlejohn and Wellman in [20].

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