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An Asymptotic Estimate for Linear Delay Differential Equations with Power Delayed Arguments

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Abstract

The paper gives an analysis of asymptotic behavior of linear differential equations with several delayed terms. There are considered power function coefficients in the equations. Delayed arguments are in the form of power functions with powers from interval $(0,1)$. Some asymptotic estimates are derived and illustrated by several examples.

AMS Subject Classifications: 34K06, 34K25.
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1 Introduction

The systematic development of the theory of delay differential equations started in forties of the last century. It has been mainly motivated by the consideration of delay in feedback mechanisms in control theory. Nowadays, the applications of these equations cover all conceivable disciplines. The importance of qualitative analysis of these equations consists also in the fact that they are mostly solvable just in a numerical way.

This paper presents asymptotic estimates of solutions of linear differential equations with unbounded delays

$$\dot{y}(t) = at^{\alpha}y(t) + \sum_{i=1}^{m} b_i t^{\beta_i} y(t^{\lambda_i}), \qquad t \in [t_0, \infty), \tag{1.1}$$

where $a \neq 0$, α, b_i, β_i and $0 < \lambda_i < 1$, $i = 1, \ldots, m$ are real scalars and $t_0 \geq 1$.

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There are many papers dealing with asymptotic properties of equations with unbounded lags. To recall some closely related works we start with Čermák [2], where a particular case of (1.1) in the form

$$\dot{y}(t) = ay(t) + by(t^\lambda), \quad 0 < \lambda < 1, \quad t \in [1, \infty)$$

and its discrete counterpart were investigated. There were discussed qualitative properties of both (discrete and continuous) cases on compact domain and also on unbounded domain. Paper [3] formulates asymptotic estimates for differential equations with a more general form of coefficients and delays including a forcing term

$$\dot{y}(t) = a(t)y(t) + \sum_{i=1}^{m} b_i(t)y(\tau_i(t)) + f(t), \qquad t \in [t_0, \infty),$$

where $a(t)$ is a negative continuous function, $\tau_i(t) < t$, $i = 1, \dots, m$ and some additional assumptions. The following analysis of the asymptotics of the equation (1.1) particularly takes advantage of the approach used in that paper. Asymptotic estimate for equation (1.1) with extra forcing term was analyzed in [8]. We amend (in a certain sense) the results formulated in [8] in this paper. Other valuable results connected to the studied problems can also be found in e.g., J. Baštinec, L. Berezansky and J. Diblík [1], Diblík [5], Derfel and Vogl [4], Iserles [7], Lakshmikantham, Wen and Zhang [9].

The structure of the paper is the following: In Section 2 the main results are formulated and their proofs are introduced in Section 3. Section 4 concludes the paper with several remarks and examples.

2 Asymptotic Properties

Theorem 2.1. *Consider equation* (1.1) *such that* $a, b_i \neq 0$ *and* $0 < \lambda_i < 1$, $i = 1, \dots, m$.

If either $a > 0$, $\alpha > -1$ *and* β_i *is arbitrary or* $\alpha < -1$ *and* $\beta_i < -1$ *for* $i = 1, \dots, m$, *then for any solution* y *of* (1.1) *there exists a constant* L *(depending on* y*) such that*

$$\lim_{t \to \infty} \exp\left\{ \frac{-a}{\alpha + 1} t^{\alpha+1} \right\} y(t) = L.$$

If $\alpha = -1$ *and* $a > \dfrac{1 + \beta_i}{1 - \lambda_i}$ *for all* $i = 1, \dots, m$, *then for any solution* y *of* (1.1) *there exists a constant* L *such that*

$$\lim_{t \to \infty} \frac{y(t)}{t^a} = L.$$

Note, that Theorem 2.1 does not cover the case of a negative and $\alpha > -1$. To fill this gap we present the following result

Theorem 2.2. *Let* $a < 0$, $\alpha > -1$, $\alpha \geq \beta_i$, $b_i \neq 0$ *and* $0 < \lambda_i < 1$, $i = 1, \dots, m$. *If* y *is a solution of* (1.1)*, then*

$$y(t) = O(\log^{\gamma} t) \quad \text{as } t \to \infty,$$

where

$$\gamma = \max \left\{ \frac{\log \sum_{i=1}^{m} \frac{|b_i|}{-a}}{\log \lambda_j^{-1}}, j = 1, \dots, m \right\}.$$

3 Proofs

Proof of Theorem 2.1. The asymptotic properties of equation (1.1) summarized in Theorem 2.1 are derived from assertion by Pituk [10]. It deals with functional differential equations in the form

$$\dot{z}(t) = \sum_{j=1}^{m} p_j(t) z(\tau_j(t)), \qquad t \in [t_0, \infty), \tag{3.1}$$

where $p_j, \tau_j \in C([t_0, \infty))$, $\tau_j(t) \leq t$ for all $t \in [t_0, \infty)$ and $\inf\{\tau_j(t), t \in [t_0, \infty)\} > -\infty$, $j = 1, \dots, m$.

Theorem 3.1 (Pituk [10])**.** *Consider equation* (3.1)*. If* $\sum_{j=1}^{m} \int_{t_0}^{\infty} |p_j(t)| \mathrm{d}t$ *converges, then every solution* z *of* (3.1) *tends to a finite limit as* $t \to \infty$*.*

We employ this theorem to show the validity of Theorem 2.1. Let y be a solution of (1.1). If $\alpha \neq -1$, then we introduce $z(t) = \exp\left\{ -\frac{a}{\alpha+1} t^{\alpha+1} \right\} y(t)$. Emphasize that function z solves the equation

$$\dot{z}(t) = \sum_{i=1}^{m} b_i t^{\beta_i} \exp\left\{ \frac{a}{\alpha+1} \left(t^{\lambda_i(\alpha+1)} - t^{\alpha+1} \right) \right\} z(t^{\lambda_i}),$$

which is (3.1) with $p_i(t) = b_i t^{\beta_i} \exp\left\{ \frac{a}{\alpha+1} (t^{\lambda_i(\alpha+1)} - t^{\alpha+1}) \right\}$, $i = 1, \dots, m$. Then the convergence of $\sum_{i=1}^{m} \int_{t_0}^{\infty} |p_i(t)| \mathrm{d}t$ is ensured either for $a > 0$ and $\alpha > -1$ or $\alpha < -1$ and $\beta_i < -1$, $i = 1, \dots, m$.

In the case of $\alpha = -1$ we introduce $z(t) = t^{-a}y(t)$, which solves the equation

$$\dot{z}(t) = \sum_{i=1}^{m} b_i t^{\beta_i + a(\lambda_i - 1)} z(t^{\lambda_i}).$$

This equation can be obtained from (3.1) by $p_i(t) = b_i t^{\beta_i + a(\lambda_i - 1)}$, $i = 1, \dots, m$. Then the convergence of $\sum_{i=1}^{m} \int_{t_0}^{\infty} |p_i(t)| \mathrm{d}t$ is ensured if $a > \dfrac{1+\beta_i}{1-\lambda_i}$ for all $i = 1, \dots, m$. □

Proof of Theorem 2.2. The proof is based on a suitable transformation of equation (1.1) to a differential equation with constant delays and boundedness analysis of its solutions. First we introduce the substitution

$$s = \psi(t) = \log\log(t), \qquad z(s) = (\log t)^{-\gamma} y(t) \tag{3.2}$$

on $I := [t_0, \infty)$, $t_0 > 1$, which transforms equation (1.1) into the form

$$\begin{aligned} &\exp\{\gamma s\}(z'(s) + \gamma z(s)) = a(h(s))^{\alpha} \exp\{\gamma s\} h'(s) z(s) \\ &+ \sum_{i=1}^{m} b_i (h(s))^{\beta_i} \exp\{\gamma s - \gamma \log \lambda_i^{-1}\} h'(s) z(\mu_i(s)), \end{aligned} \tag{3.3}$$

where the differentiation with respect to s is denoted by "$'$", $h(s) = \exp\exp(s)$ is the inverse of function ψ and $\mu_i(s) = s - \log \lambda_i^{-1}$ on $\psi(I)$, $i = 1, \dots, m$ are transformed retarded arguments. Emphasize, that function $\psi(t)$ (utilized in substitution) is the common solution of a system of functional equations

$$\psi(t^{\lambda_i}) = \psi(t) - \log \lambda_i^{-1}, \quad t \in I,\ i = 1, \dots, m.$$

This fact enabled us to obtain the differential equation with constant delays (3.3) after the transformation. Now we rewrite equation (3.3) into the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\left[\exp\left\{\gamma s - a\int_{s_0}^{h(s)} u^{\alpha}\mathrm{d}u\right\} z(s)\right] = \\ &\sum_{i=1}^{m} b_i (h(s))^{\beta_i} \lambda_i^{\gamma} \exp\left\{\gamma s - a\int_{s_0}^{h(s)} u^{\alpha}\mathrm{d}u\right\} h'(s) z(\mu_i(s)), \end{aligned} \tag{3.4}$$

where $s_0 \in \psi(I)$ is such that $\gamma > a(h(s))^{\alpha} h'(s)$ for all $s \geq s_0$. Note, that the existence of such s_0 is ensured by assumptions on a and α.

Denote $s_k := s_0 + k\Lambda^{-1}$, $\Lambda = \max\{\lambda_i, i = 1, \dots, m\}$, $J_k := [s_{k-1}, s_k]$, $k = 1, 2, \dots$. Let $s^* \in J_{k+1}$. Then integration of (3.4) over the interval $[s_k, s^*]$ gives

$$\exp\left\{\gamma s - a \int_{s_0}^{h(s)} u^\alpha \mathrm{d}u\right\} z(s)\Bigg|_{s_k}^{s^*} = \sum_{i=1}^{m} \int_{s_k}^{s^*} b_i (h(s))^{\beta_i} \lambda_i^\gamma h'(s) \exp\left\{\gamma s - a \int_{s_0}^{h(s)} u^\alpha \mathrm{d}u\right\} z(\mu_i(s)) \mathrm{d}s.$$

We express the value of z in the instant $s^* \in J_{k+1}$ as

$$\begin{aligned} z(s^*) &= \exp\left\{\gamma(s_k - s^*) + a \int_{h(s_k)}^{h(s^*)} u^\alpha \mathrm{d}u\right\} z(s_k) + \exp\left\{a \int_{s_0}^{h(s^*)} u^\alpha \mathrm{d}u - \gamma s^*\right\} \\ &\quad \times \sum_{i=1}^{m} \int_{s_k}^{s^*} b_i (h(s))^{\beta_i} \lambda_i^\gamma h'(s) \exp\left\{\gamma s - a \int_{s_0}^{h(s)} u^\alpha \mathrm{d}u\right\} z(\mu_i(s)) \mathrm{d}s. \end{aligned}$$

Now we denote $M_k := \max\left\{|z(s)|, s \in \bigcup_{\ell=1}^{k} J_\ell\right\}$, $k = 1, 2, \dots$. Then the value of $z(s^*)$ can be estimated as

$$\begin{aligned} |z(s^*)| &\le M_k \exp\left\{\gamma(s_k - s^*) + a \int_{h(s_k)}^{h(s^*)} u^\alpha \mathrm{d}u\right\} + M_k \exp\left\{a \int_{s_0}^{h(s^*)} u^\alpha \mathrm{d}u - \gamma s^*\right\} \\ &\quad \times \int_{s_k}^{s^*} \sum_{i=1}^{m} |b_i| (h(s))^{\beta_i} \lambda_i^\gamma h'(s) \exp\left\{\gamma s - a \int_{s_0}^{h(s)} u^\alpha \mathrm{d}u\right\} \mathrm{d}s. \end{aligned} \tag{3.5}$$

Further, considering the assumption $\alpha \ge \beta_i$, $i = 1, \dots, m$ we have

$$\sum_{i=1}^{m} |b_i| (h(s))^{\beta_i} \lambda_i^\gamma \le -a(h(s))^\alpha \sum_{i=1}^{m} \frac{|b_i|}{-a} \lambda_i^\gamma \le -a(h(s))^\alpha.$$

Then relation (3.5) can be expressed as

$$\begin{aligned} |z(s^*)| &\le M_k \exp\left\{\gamma(s_k - s^*) + a \int_{h(s_k)}^{h(s^*)} u^\alpha \mathrm{d}u\right\} + M_k \exp\left\{a \int_{s_0}^{h(s^*)} u^\alpha \mathrm{d}u - \gamma s^*\right\} \\ &\quad \times \int_{s_k}^{s^*} \left(-a(h(s))^\alpha\right) h'(s) \exp\left\{\gamma s - a \int_{s_0}^{h(s)} u^\alpha \mathrm{d}u\right\} \mathrm{d}s. \end{aligned}$$

Since a is negative and $\gamma > a(h(s))^\alpha h'(s)$ for all $s > s_0$ the integral on the right-hand side can be estimated as

$$\begin{aligned}
&\int\limits_{s_k}^{s^*} (-a(h(s))^\alpha)\, h'(s) \exp\left\{ \gamma s - a \int\limits_{s_0}^{h(s)} u^\alpha \mathrm{d}u \right\} \mathrm{d}s \\
&= \int\limits_{s_k}^{s^*} \frac{-a(h(s))^\alpha h'(s)}{\gamma - a(h(s))^\alpha h'(s)} \left(\exp\left\{ \gamma s - a \int\limits_{s_0}^{h(s)} u^\alpha \mathrm{d}u \right\} \right)' \mathrm{d}s \\
&\leq \int\limits_{s_k}^{s^*} \left(1 + \frac{|\gamma| \exp\{-s\}}{\gamma \exp\{-s\} - a(h(s))^{\alpha+1}} \right) \left(\exp\left\{ \gamma s - a \int\limits_{s_0}^{h(s)} u^\alpha \mathrm{d}u \right\} \right)' \mathrm{d}s \\
&\leq \exp\left\{ \gamma s - a \int\limits_{s_0}^{h(s)} u^\alpha \mathrm{d}u \right\} \Bigg|_{s_k}^{s^*} (1 + L \exp\{-s_k\}) \,,
\end{aligned}$$

where $L > 0$. Then we obtain

$$\begin{aligned}
|z(s^*)| \;\; &\leq M_k \exp\left\{ \gamma(s_k - s^*) + a \int\limits_{h(s_k)}^{h(s^*)} u^\alpha \mathrm{d}u \right\} + M_k \exp\left\{ -\gamma s^* + a \int\limits_{s_0}^{h(s^*)} u^\alpha \mathrm{d}u \right\} \\
&\quad \times \exp\left\{ \gamma s - a \int\limits_{s_0}^{h(s)} u^\alpha \mathrm{d}u \right\} \Big|_{s_k}^{s^*} (1 + L \exp\{-s_k\}) \leq M_k (1 + L \exp\{-s_k\}).
\end{aligned}$$

Since $s^* \in J_{k+1}$ was arbitrary, we have

$$M_{k+1} \leq M_k(1 + L \exp\{-s_k\}) \leq M_1 \prod_{\ell=1}^{k} (1 + L \exp\{-s_\ell\}).$$

The boundedness of the sequence $\{M_k\}_{k=1}^\infty$ gives the asymptotic estimate

$$y(t) = O(\log^\gamma t) \quad \text{as } t \to \infty$$

utilizing substitution (3.2). □

4 Examples and Final Remarks

In this section we illustrate the usefulness of the presented results. First we compare the direct application of Theorem 3.1 to equation (1.1) with the result stated in Theorem 2.1.

Corollary 4.1. *Consider equation* (1.1). *If* $\alpha < -1$ *and* $\beta_i < -1, i = 1, \ldots, m$, *then every solution* y *of* (1.1) *tends to a finite limit as* $t \to \infty$.

Proof. The assertion follows from Theorem 3.1 by a simultaneous convergence analysis of integrals $\int_{t_0}^{\infty} |a| t^{\alpha} \mathrm{d}t$ and $\int_{t_0}^{\infty} |b_i| t^{\beta_i} \mathrm{d}t$, $i = 1, \ldots, m$. □

Actually, the above assertion does not make the best account of Theorem 3.1. Theorem 2.1 gives a stronger result in the sense of asymptotic behavior of equation (1.1).

Next, we present some specific cases of equation (1.1) to illustrate the application of the introduced asymptotic estimates. First we introduce equation (1.1) with $m = 1$ (one delayed term) and constant coefficients.

Example 4.2. Consider the equation

$$\dot{y}(t) = ay(t) + by(t^{\lambda}), \qquad t \in [1, \infty), \tag{4.1}$$

where $a < 0$, b are real constants, $0 < \lambda < 1$. Then Theorem 2.2 gives the asymptotic estimate

$$y(t) = O(\log^{\gamma} t) \quad \text{as} \quad t \to \infty, \, \gamma = \frac{\log \frac{|b|}{-a}}{\log \lambda^{-1}}.$$

Notice, that the above asymptotic estimate accords with the well-known result of Heard [6].

Theorem 4.3 (Heard [6])**.** *Let* $a < 0$, $b \neq 0$ *be real constants,* $0 < \lambda < 1$. *Then for every solution of* (4.1) *there exists a continuous periodic function* g *with period* $\log \lambda^{-1}$ *such that*

$$y(t) = (\log t)^{\xi} g(\log \log t) + O(\log^{\xi_r - 1} t) \qquad \text{as} \quad t \to \infty,$$

where ξ *is a root of* $a + b\lambda^{\xi} = 0$ *and* $\xi_r = \Re(\xi)$.

The following example illustrates utilization of Theorem 2.1 and Theorem 2.2 in a nonconstant coefficient case of equation (1.1).

Example 4.4. Consider a three-term case of equation (1.1) with $\alpha = 0$, $\beta_1 = \beta_2 = -1$, i.e.,

$$\dot{y}(t) = ay(t) + \frac{1}{t}\left(b_1 y(t^{\lambda_1}) + b_2 y(t^{\lambda_2})\right), \tag{4.2}$$

where a, b_1, b_2 are real nonzero constants and $\lambda_1, \lambda_2 \in (0, 1)$. If $a > 0$, then for any solution $y(t)$ of (4.2) there exists a constant L such that

$$\lim_{t \to \infty} \exp\{-at\} y(t) = L.$$

If $a < 0$, then any solution $y(t)$ of (4.2) satisfies

$$y(t) = O(\log^{\gamma} t) \quad \text{as} \quad t \to \infty, \quad \gamma = \max\left\{ \frac{\log \frac{|b_1| + |b_2|}{-a}}{\log \lambda_1^{-1}}, \frac{\log \frac{|b_1| + |b_2|}{-a}}{\log \lambda_2^{-1}} \right\}.$$

Remark 4.5. In the paper we discussed the asymptotic behavior of equation (1.1). There exist several approaches for the asymptotic investigation of delay differential equations. As shown, in some cases a suitable transformation can give valuable results as well.

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