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Hybrid Systems by Methods of Time Scales Analysis

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Abstract

The goal of the present paper is to show how to study hybrid systems with complex behavior (including Zeno points) by methods of time scale analysis. We first transform impulsive differential problems with countably many impulses (describing the above hybrid systems) into dynamic problems on a well chosen time scale domain and then apply a version of Peano's theorem for multivalued case on time scales in order to achieve an existence result for the considered problem. We thus offer an alternative proof for [22, Theorem 1], where the tools were coming from the theory of measure driven differential inclusions.

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1 Introduction

Generally, the term "hybrid" means "heterogeneous". When speaking about hybrid systems, we refer to systems containing parts with different attributes, in particular to systems that are combinations of continuous and discrete (in time or values) systems. Hybrid systems arise each time a digital device intercomes into a real world process, but not only: we refer to continuous systems controlled by discrete logic, i.e., the so-called embedded systems (thermostat, aircraft autopilot modes) or to continuous systems with a phased operation (such as, bouncing balls or walking robots). For example, the sampled-data or digital control systems are systems described by differential equations (so, working with continuous-valued variables on a continuous time domain, shortly analog signals) that are controlled by a discrete-time controller, described by difference equations.

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Hybrid systems could be time-driven (studied through differential or difference equations) or event-driven (historically studied via automata or Petri net models). The necessity of an unitary approach is clear when thinking of hybrid systems where the interaction between the continuous and the discrete parts is considerable, such as in automotive engine control. E.g., any accurate model for a four-stroke gasoline engine should take into consideration the fact that the power train and air dynamics are continuous-time processes, while the pistons have four modes of operation corresponding to the stroke they are in (therefore, a discrete event process) and their interconnection is significant: the transition between two phases of the pistons is given by the motion of the power train which, at his turn, depends on the torque produced by each piston (see [3]).

At the two extremities of the spectrum in the current approach of hybrid systems are the extensions of the theories available for continuous systems in order to allow discrete perturbations, respectively the extensions of the methodologies known for discrete systems towards hybrid systems. Our work is situated closer to the former situation: we will focus on hybrid systems whose continuous behavior is subjected to discrete perturbations. In the literature on impulsive systems on bounded intervals, usually only a finite number of impulses intervene (see [21]) or, when their number is infinite, it is imposed that they accumulate only once, at the right (e.g., [17], see also [22] and the references therein). We will allow a countably infinite number of perturbations in a finite interval of time, that could accumulate in a finite number of points (that are known as Zeno points, see [23, p. 78]). Otherwise said, we provide a way to study hybrid systems with complex Zeno behavior.

The chosen method is that of analysis on time scale domains, that are arbitrary closed sets of real numbers (not only real intervals or discrete sets, as in the classical continuous, respectively discrete analysis), introduced in 1988 by S. Hilger in his PhD thesis (see also [19]). At two decades distance from the beginning of this theory, the measure and integral were defined and studied on such domains: the Δ-measure was introduced by Guseinov [18], the Riemann and Lebesgue Δ-integrals were studied by Bohner and Guseinov [7] in the 1-dimensional case and then generalized to the n-dimensional Euclidean space, the integration on curves in the time scales plane [6] and the Green formula [8] were obtained by Bohner and Guseinov and even the Cauchy Δ-integral [20] and the weak Riemann Δ-integral [14] were discussed and many applications were given. We will make use of the notion of Lebesgue Δ-integral, already involved in a considerable number of existence results (see [4, 7, 16]).

After the preliminary section, we will prove a Peano theorem analog for multivalued dynamic problems on time scales and afterwards, by a procedure similar to that described in [15], we provide an existence result for impulsive inclusions describing hybrid systems with an arbitrary number of Zeno points, by embedding this class of problems into the class of dynamic problems on time scales. Thus, we offer an alternative proof of [22, Theorem 1], where the approach was that of measure driven differential inclusions.

2 Notations and Preliminary Facts

In order to make the paper self-contained, we start by introducing some preliminary notions from the time scale analysis (we refer the reader to [1,9,10] and the references therein).

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$ (such as, the real intervals, the subsets of $\mathbb{N}$ or combinations between these two). By $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$ (respectively $[a, b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t < b\}$) we denote the time scales intervals with endpoints $a, b \in \mathbb{T}$.

Definition 2.1. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively.

We say that $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M (so, $\inf \emptyset = \sup \mathbb{T}$) and $\rho(m) = m$ if $\mathbb{T}$ has a minimum m (otherwise said, $\sup \emptyset = \inf \mathbb{T}$).

The jump operators σ and ρ allow the classification of points in time scale in the following way: t is called *right dense*, *right scattered*, *left dense*, *left scattered*, *dense* and *isolated* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ and $\rho(t) < t < \sigma(t)$, respectively.

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}$. Then the Δ-derivative of f at the point t, $f^{\Delta}(t)$, is the element of $\mathbb{R}^n$ satisfying that for any $\varepsilon > 0$ there exists a neighborhood of t on which

$$\big\| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \big\| \leq \varepsilon |\sigma(t) - s|.$$

Remark 2.3. In particular, when $\mathbb{T} = \mathbb{R}$, $f^{\Delta} = f'$ is the usual derivative, while $f^{\Delta} = \Delta f$ is the well-known forward difference operator when $\mathbb{T} = \mathbb{Z}$. Consequently, the time scale analysis gives us the possibility to unify the treatment of differential and difference equations.

Similarly one can define the ∇-derivative:

Definition 2.4. Let $f : \mathbb{T} \to \mathbb{R}^n$ and fix $t \in \mathbb{T}$. Then the ∇–derivative $f^{\nabla}(t)$ is the element of X such that for any $\varepsilon > 0$ there exists a neighborhood of t on which

$$\big\| f(\rho(t)) - f(s) - f^{\nabla}(t)[\rho(t) - s] \big\| \leq \varepsilon |\rho(t) - s|.$$

We will focus on the Δ-part of time scales theory, but if necessary almost the same could be done with the ∇-theory (although a specific care must be taken when switching between the two, as seen in [4]).

As for the measure theory on time scales, for a precise definition and basic properties of the Lebesgue measure on $\mathbb{T}$, that will be denoted by μ_{Δ}, we refer the reader to [12]. For properties of Lebesgue integral on time scales see [9,10].

Let us now recall the notion of absolute continuity on time scales [11] and its characterization, similar to that well-known in the analysis on real intervals. Note that in [11] the authors are concerned with the behavior of real-valued functions, but the generalization to the case of finite-dimensional spaces is immediate.

Definition 2.5. A function $f : \mathbb{T} \to \mathbb{R}^n$ is called absolutely continuous if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\sum_{i=1}^{k} \|f(b_i) - f(a_i)\| < \varepsilon$ whenever $\{[a_i, b_i)_{\mathbb{T}}; 1 \le i \le k\}$ is finite pairwise disjoint family of subintervals of $\mathbb{T}$ satisfying $\sum_{i=1}^{k}(b_i - a_i) < \delta_\varepsilon$.

Suppose in the sequel that the time scale domain $\mathbb{T}$ is bounded and 0 is its minimum.

Theorem 2.6 (See [11, Theorem 4.1])**.** *A function $f : \mathbb{T} \to \mathbb{R}^n$ is absolutely continuous if and only if it is Δ-differentiable μ_Δ-a.e., f^Δ is Lebesgue Δ-integrable and $\int_{[0,t)_{\mathbb{T}}} f^\Delta(s)\Delta s = f(t) - f(0)$, for every $t \in \mathbb{T}$.*

Also,

Proposition 2.7 (See [16, Proposition 2.19])**.** *If $g : \mathbb{T} \to \mathbb{R}^n$ is Lebesgue-Δ-integrable, then the primitive $t \mapsto \int_{[0,t)_{\mathbb{T}}} g(s)\Delta s$ is absolutely continuous and its derivative equals to g, μ_Δ-a.e.*

3 Main Results

We are concerned with the study of hybrid systems with an arbitrary (finite) number of Zeno points (i.e., for which the discrete perturbations affecting the continuous dynamics accumulate in a finite amount of time). More precisely, we focus on systems described by an initial value impulsive differential problem:

$$\dot{y}(t) \in F\left(y(t)\right), \quad a.e.\ \ t \in [0,1] \setminus \{t_1, \ldots, t_m, \ldots\}, \tag{3.1}$$

$$\Delta y(t_i) \in \alpha_i S(y(t_i), \alpha_i), \quad \forall i \in \{1, \ldots, m, \ldots\}, \tag{3.2}$$

$$y(0) = y_0, \tag{3.3}$$

with countably infinitely many impulsive moments $(t_m)_{m\in\mathbb{N}}$ having a finite number of accumulation points.

One may suppose, without restricting the generality, that the sequence $(t_m)_{m\in\mathbb{N}}$ has only one accumulation point and that $t_m \uparrow \tilde{t}$ (if $t_m \downarrow \tilde{t}$, then the same could be done, but using the ∇-derivative instead of Δ-derivative); obviously, if the set of accumulation points would contain more than one element, then the procedure should be repeated for each element.

By an idea that was described in [15] in the case of a finite number of impulses, we choose a sequence of positive numbers $(\gamma_i)_{i\in\mathbb{N}}$ such that $\sum_{i=1}^{\infty}\gamma_i = \gamma$ and define the set (contained in $[0, 1+\gamma]$)

$$\mathbb{T} = [0,t_1] \cup [t_1+\gamma_1, t_2+\gamma_1] \cup [t_2+\gamma_1+\gamma_2, t_3+\gamma_1+\gamma_2] \cup \ldots \cup [\tilde{t}+\gamma, 1+\gamma]$$

which is a time scale domain. Let $x : \mathbb{T} \to \mathbb{R}^n$ be the function defined by

$$x(t) = \begin{cases} y(\beta(t))+, \text{ if } t \in \{t_1+\gamma_1, t_2+\gamma_1+\gamma_2, \ldots\}, \\ y(\beta(t)), \text{ otherwise,} \end{cases}$$

where $\beta : \mathbb{T} \to [0,1]$ is given by

$$\beta(t) = \begin{cases} t, \text{ if } t \in [0,t_1], \\ t-\gamma_1, \text{ if } t \in [t_1+\gamma_1, t_2+\gamma_1], \\ t-(\gamma_1+\gamma_2), \text{ if } t \in [t_2+\gamma_1+\gamma_2, t_3+\gamma_1+\gamma_2], \\ \vdots \\ t-\gamma, \text{ if } t \in [\tilde{t}+\gamma, 1+\gamma]. \end{cases}$$

If N is the null-measure subset of $[0,1]$ where y is not differentiable, then our impulsive differential problem can be stated as a dynamic problem on the described time scale, as follows:

$$\begin{aligned} &x^{\Delta}(t) \in \begin{cases} F\left(x(t)\right), \text{ if } t \notin \beta^{-1}(N), \\ \dfrac{\alpha_j S(x(t)), \alpha_j)}{\gamma_j}, \text{ if } t = t_j+\gamma_1+\ldots+\gamma_{j-1},\ j \in \{1,2,\ldots\}, \end{cases} \\ &x(0) = y_0, \end{aligned}$$

where $\gamma_0 = 0$. Remark that $\beta^{-1}(N)$ contains $\{t_1, t_1+\gamma_1, t_2+\gamma_1, t_2+\gamma_1+\gamma_2, \ldots, \tilde{t}+\gamma\}$, due to the fact that all the points $\tilde{t}, t_m \in N$.

This writing is a consequence of the properties of Δ-derivative (see [10, Theorem 1.3]) which, at a right-scattered point, such as $t = t_j+\gamma_1+\ldots+\gamma_{j-1}$, equals to

$$x^{\Delta}(t) = \frac{x(\sigma(t))-x(t)}{\sigma(t)-t} = \frac{x(\sigma(t))-x(t)}{\gamma_j} \in \frac{\alpha_j S(x(t)), \alpha_j)}{\gamma_j}.$$

Let us note that the function x is Δ-differentiable except on $\beta^{-1}(N) \setminus \{t_1, t_2+\gamma_1, \ldots\}$, that is a μ_Δ-null set.

In this way, it will be possible to study impulsive differential problems with a countable number of impulses on a real bounded interval by methods of analysis on time scales.

Next, without major changes (in the method) comparing to the case of similar problems on real intervals, we state an analog of Peano's theorem, i.e., an existence result for the following set-valued nonautonomous dynamic problem on a bounded time scale:

$$x^{\Delta}(t) \in \widetilde{F}(t, x(t)),\ \mu_{\Delta} - a.e.\ t \in \mathbb{T}, \tag{3.4}$$

$$x(0) = x_0. \tag{3.5}$$

Notice that Peano's theorem on time scales was investigated (in the single-valued case) in [13], an example being also included for infinite dimensional space-valued functions.

Theorem 3.1. *Let* $\widetilde{F} : \Omega \to ck(\mathbb{R}^n)$ *be an upper semicontinuous multifunction, where* $\Omega \subset \mathbb{T} \times \mathbb{R}^n$ *is an open set. Then the dynamic problem* (3.4)–(3.5) *admits Lipschitz continuous solutions on a time-scale interval* $[0, T]_{\mathbb{T}}$.

Proof. As F is upper semicontinuous with compact values, there exist two positive constants M, T such that $[0, T]_{\mathbb{T}} \times (x_0 + TMB_0) \subset \Omega$ and $\|F(t, x)\| \leq M, \forall (t, x) \in [0, T]_{\mathbb{T}} \times (x_0 + TMB_0)$ (here B_0 is the unit ball in $\mathbb{R}^n$). In a classical way (see e.g., [5, Theorem 1, p. 129]), we define the set

$$\mathcal{K} = \{x : [0, T]_{\mathbb{T}} \to \mathbb{R}^n; x \text{ is } M\text{-Lipschitz and } x(0) = x_0\}$$

which is nonempty, convex and compact, by Arzela–Ascoli theorem on time scales ([2]). We also consider the set-valued operator

$$\Xi : \mathcal{K} \to \mathcal{K},\ \Xi(x) = \{z \in \mathcal{K}; z^{\Delta}(t) \in F(t, x(t))\ \mu_{\Delta} - a.e.\ \}$$

with convex and nonempty values (since, by Proposition 2.7, it contains the primitives of all measurable selections of $F(\cdot, x(\cdot))$). An application of Alaoglu's theorem in the dual of the Banach space of Lebesgue-Δ-integrable functions (see [16, Theorem 2.11]) implies that the values are also closed, therefore compact, and that Ξ is upper semicontinuous. In conclusion, it satisfies the hypothesis of Kakutani's fixed point theorem and it possesses fixed points. Each such fixed point is a solution of the announced problem. □

In what follows, we apply Theorem 3.1 in order to obtain an existence result for the impulsive differential problem (3.1)–(3.3), that was proved in [22] by methods of the theory of measure driven differential inclusions (here, the Lipschitz continuity is understood with respect to the Hausdorff–Pompeiu distance).

Let us first clarify the notion of solution that will be obtained in this setting.

Definition 3.2. A function $y : [0, 1] \to \mathbb{R}^n$ is a solution of the problem (3.1)–(3.3) if there is an integrable function $\Phi : [0, 1] \to \mathbb{R}^n$ with $\Phi(t) \in F(y(t))$ a.e. and $\beta_i \in \mathbb{R}^n$ with $\beta_i \in S(y(t_i), \alpha_i)$ such that

$$y(t) = \int_0^t \Phi(s) ds + \sum_{i \in J(t)} \alpha_i \beta_i,$$

where $J(t) = \{i \in \mathbb{N}; t_i < t\}$.

Theorem 3.3 (See [22, Theorem 1])**.** *Suppose that* $F : \mathbb{R}^n \to ck(\mathbb{R}^n)$ *and* $S : \mathbb{R}^n \times [0,1] \to k(\mathbb{R}^n)$ *satisfy the following conditions:*

i) F *is Lipschitz continuous;*

ii) S *is bounded and Lipschitz continuous in the first variable, uniformly with respect to the second one;*

iii) $\sum_{i=1}^{\infty} \alpha_i < \infty$.

Then the impulsive problem (3.1)–(3.3) *admits solutions.*

Proof. Let M_1, resp. M_2 be the Lipschitz constants of F, resp. $S(\cdot\ ,\alpha)$. For each $j \in \mathbb{N}$, let us choose γ_j such that $\dfrac{\alpha_j}{\gamma_j} = \dfrac{M_1}{M_2}$. Obviously, by hypothesis $iii)$, both series of terms α_j and γ_j converge and

$$\sum_{j=1}^{\infty} \gamma_j = \frac{M_2}{M_1} \sum_{j=1}^{\infty} \alpha_j = \gamma.$$

To find solutions of impulsive differential problem (3.1)–(3.3) means to find solutions of the dynamic initial value problem

$$\begin{aligned} &x^{\Delta}(t) \in \widetilde{F}(t, x(t)),\ \mu_{\Delta} - a.e. \\ &x(0) = y_0, \end{aligned}$$

where $\widetilde{F} : \mathbb{T} \times \mathbb{R}^n \to ck(\mathbb{R}^n)$ is given by

$$\widetilde{F}(t,x) = \begin{cases} F\,(x)\,, \text{ if } t \notin \{t = t_j + \gamma_1 + \ldots + \gamma_{j-1},\ j = 1, 2, \ldots\}, \\ \dfrac{\alpha_j S(x, \alpha_j)}{\gamma_j}, \text{ if } t = t_j + \gamma_1 + \ldots + \gamma_{j-1},\ j \in \{1, 2, \ldots\}. \end{cases}$$

We need to show that the conditions of Theorem 3.1 are satisfied for this dynamic problem. The upper semicontinuity of $\widetilde{F}$ comes immediately from hypothesis $i)$ at every point (t,x) where $t \notin \{t = t_j + \gamma_1 + \ldots + \gamma_{j-1},\ j = 1, 2, \ldots\} \cup \{\tilde{t} + \gamma\}$. The upper semicontinuity at an arbitrary point (t,x) with t of the form $t = t_j + \gamma_1 + \ldots + \gamma_{j-1},\ j = 1, 2, \ldots$, follows by the remark that we may suppose, without restricting the generality, that

$$F(x) \subset \frac{M_1}{M_2} S(x, \alpha_j), \forall j \in \mathbb{N}$$

by replacing, if necessary, $F(x)$ by $F(x) \cap \cap_{j=1}^{\infty} \dfrac{M_1}{M_2} S(x, \alpha_j)$. Finally, in order to prove the upper semicontinuity at the points where $t = \tilde{t} + \gamma$, notice that the physical meaning of the set-valued function S naturally implies that we may consider, for every $x \in \mathbb{R}^n$, that

$$\lim_{\alpha \to 0} S(x, \alpha) \subset \frac{M_2}{M_1} F(x).$$

This suffices to our purpose since, when a point of the form $t = t_j + \gamma_1 + \ldots + \gamma_{j-1},\ j = 1, 2, \ldots$ tends to $\tilde{t} + \gamma$, in fact $\lim\limits_{j \to \infty} \alpha_j = 0$. □

Finally, we present an example of hybrid system for which Theorem 3.3 asserts the existence of executions:

Example 3.4 (Example "Ball and Paddle" in [22]). Consider a player trying to keep in the air a table tennis ball by hitting it with a paddle faster and faster (at the same time, lighter and lighter) until it rests on the table and then "reversing" the trajectory (i.e., by hitting it up again). The moment when the ball comes to rest on the table is a Zeno point and if the variable $(x_1(t), x_2(t))$ represents the pair (height,vertical velocity), $F(x_1, x_2) = (x_2, -1)$ and $S(x, \alpha) = (0, \alpha)$, then the preceding result guarantees the existence of executions for this hybrid system.

Remark 3.5. By the same method, namely embedding the impulsive differential problems (with a countable set of impulses occurring in a finite amount of time) into dynamic problems on time scale domains, more existence results could be obtained starting from existence results known on time scales (such as, those proved in [4, 14, 15]), under more general assumptions comparing to those presented above.

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