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Shape of a Travelling Wave in a Time-Discrete Reaction-Diffusion Equation

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Abstract

The aim of this work is to study travelling wavefronts in a discrete-time integrodifference equation with a particular top-hat kernel. An approximation of the wavefront shape by a difference equation solution is presented.

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1 Introduction

The famous Fisher equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}u + ru(1-u) \tag{1.1}$$

was introduced as a model for the spread of an advantageous allele [5]. Later on, the equation was adopted for the spatial population ecology [6,17] as a model of population invasion. In such a case, u denotes population density. The equation (1.1) admits travelling wave solutions, i.e., solutions of the form $u = u(x-ct)$, of all velocities $c \geq 2\sqrt{rD}$. If the initial function possesses a compact support, then all of the solutions converge to a travelling waves of minimum speed $c^* = 2\sqrt{rD}$ [7]. Besides the question of existence and stability of the travelling wave, there is a problem to determine the shape of it. An approximate solution of the problem was obtained by a perturbation method [4].

Equation (1.1) models a spatial spread of a population with overlapping generations growing continuously in time. To describe a spatially distributed population with

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nonoverlapping generations, Kot and Schaffer [10] proposed the integrodifference equation

$$u(t+1,x) = \int_{-\infty}^{\infty} k(x-y) f\big(u(t,y)\big)\mathrm{d}y, \quad t = 0,1,2,\dots, \tag{1.2}$$

where the function f describes a growth of population size and the kernel k models a spatial spread of the population. An explicit solution of the equation fits well to insect dispersal data [9]. Equation(1.2) poses a natural generalization of the impulsive reaction-diffusion model [11]:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R},\ t \in (i, i+1),\ i = 0,1,2,\dots,$$
$$\lim_{t \to i_+} u(t,x) = f\big(u(i,x)\big), \quad x \in \mathbb{R}.$$

The nonnegative function f is defined on $[0,\infty)$ and it should satisfy the conditions

$$f(0) = 0, \quad f'(0) > 0, \quad f(K) = K \text{ for some } K > 0, \tag{1.3}$$

such as the Beverton–Holt stock-recruitment curve [2, 15]

$$f(u) = \frac{K\lambda u}{K + (1-\lambda)u}, \quad \lambda > 1, \tag{1.4}$$

the logistic difference equation [14, 18], or the Ricker curve [16]

$$f(u) = u\left(1 + r - \frac{r}{K}u\right), \qquad f(u) = u\exp\left\{r\left(1 - \frac{u}{K}\right)\right\}, \quad r > 0.$$

The kernel k may be any probability density function, i.e., it should satisfy

$$k \ge 0, \quad \int_{-\infty}^{\infty} k(s)\mathrm{d}s = 1.$$

If an initial function possesses compact support and the kernel has a moment-generating function for some range about the origin, i.e., if there exists a positive constant μ_0 with the property

$$M(\mu) = \int_{-\infty}^{\infty} \mathrm{e}^{\mu s} k(s)\mathrm{d}s < \infty \tag{1.5}$$

for all $|\mu| \le \mu_0$, then travelling waves exists only for c not less than a minimal speed c^*, [1, 3], and the travelling wave with the speed c^* is asymptotically stable [12]. The widely used kernels satisfying the above conditions are the Gaussian and the Laplace (double exponential) ones

$$k(s) = \frac{1}{2\sqrt{\pi D}}\exp\left\{-\frac{s^2}{4D}\right\}, \qquad k(s) = \frac{\alpha}{2}\mathrm{e}^{-\alpha|s|},$$

or the top-hat (uniform) kernel

$$k(s) = \begin{cases} 1/2\beta, & -\beta < s < \beta \\ 0, & \text{else} \end{cases}, \tag{1.6}$$

see [13].

Mark Kot [8] adopted the Canosa's perturbation method to extract the shape of travelling waves from the equation (1.2) with the exponential and double exponential kernels. But the method fails for the top-hat kernel.

The aim of the paper is to suggest a method of approximation the wavefront shape in the case of the top-hat kernel (1.6). The subsequent section summarized prerequisite results that slightly modify some Kot's [8] ideas. The main result-approximation of the wavefront shape by a solution of a difference equation-is presented in Section 3.

2 The Travelling Wave Solution

Travelling waves are solutions of equation (1.2) that satisfy

$$u(t,x) = u(0, x - ct)$$

for some positive constant c. In effect, each iterate yield a lateral translation c with no other change in the shape of solution, c denotes the wave speed. In particular, such a solution u satisfies

$$u(t, x - c) = u\big(0, x - c(t+1)\big) = u(t+1, c).$$

Let $U = U(x)$ be the shape of travelling wave solution, i.e., $U(x) = u(t,x)$ for some t fixed. Then by (1.2) the function U satisfies

$$U(x - c) = \int_{-\infty}^{\infty} k(x-y) f\big(U(y)\big) \mathrm{d}y. \tag{2.1}$$

Let us search the minimum possible wave speed c. It is determined by the local behavior of (2.1) in the neighborhood of $U^* \equiv 0$. The linearization of the function f in such a neighborhood is just

$$f(U) \approx f(0) + f'(0)U = f'(0)U$$

by (1.3). Hence, the linearization of equation (2.1) reads

$$U(x - c) = f'(0) \int_{-\infty}^{\infty} k(x-y) U(y) \mathrm{d}y. \tag{2.2}$$

For a rightward moving wave, one may attempt a solution of the form

$$U(x) = A\mathrm{e}^{-\mu x} \tag{2.3}$$

with μ positive. Substituting it into (2.2), we obtain

$$\mathrm{e}^{\mu c} = f'(0) \int_{-\infty}^{\infty} k(x-y)\mathrm{e}^{-\mu(x-y)}\mathrm{d}y = f'(0)M(\mu), \tag{2.4}$$

where M is the moment-generating function defined by the integral in (1.5). Let us put $\Phi(\mu) = f'(0)M(\mu)$. The function Φ satisfies

$$\Phi(0) = f'(0) > 1, \quad \Phi(\mu) > 0 \text{ for } \mu > 0$$

according to the properties (1.3). Moreover, assumption (1.5) allows us to evaluate

$$\Phi'(\mu) = f'(0) \int_{-\infty}^{\infty} sk(s)\mathrm{e}^{\mu s}\mathrm{d}s = f'(0) \int_{0}^{\infty} s\left(k(s)\mathrm{e}^{\mu s} - k(-s)\mathrm{e}^{-\mu s}\right)\mathrm{d}s \geq$$
$$\geq f'(0) \int_{0}^{\infty} s\left(k(s) - k(-s)\right)\mathrm{d}s = f'(0) \int_{-\infty}^{\infty} sk(s)\mathrm{d}s \geq 0,$$

$$\Phi''(\mu) = f'(0) \int_{-\infty}^{\infty} s^2 k(s)\mathrm{e}^{\mu s}\mathrm{d}s \geq 0.$$

That is, the function Φ is a positive increasing convex function with $\Phi(0) > 1$. Consequently, there exists a $c^* > 0$ such that equation (2.4) has no real root for $c < c^*$ and it has two real roots for $c > c^*$. Hence, c^* is the searched minimal wave speed. Of course, equation (2.4) has the double root μ^* for $c = c^*$, i.e., $c = c^*$ and $\mu = \mu^*$ satisfy the equation

$$c\mathrm{e}^{\mu c} = \Phi'(\mu). \tag{2.5}$$

The provided considerations lead to the conclusion: *The minimal wave speed c and the parameter μ describing the shape of a "leading edge" of the travelling wave are solutions of the system of equations* (2.4), (2.5) *that can be rearranged to the form*

$$c = \frac{M'(\mu)}{M(\mu)}, \qquad \mathrm{e}^{\mu c} = f'(0)M(\mu). \tag{2.6}$$

3 The Wavefront Shape for the Top-Hat Kernel

Let k be the top-hat kernel (1.6) and define

$$F(x) = \frac{\cosh x}{\sinh x} - \frac{1}{x}, \quad G(x) = \frac{x}{\sinh x} \mathrm{e}^{xF(x)}.$$

The first semester calculus yields, that

$$M(\mu) = \frac{\sinh \beta\mu}{\beta\mu}, \quad M'(\mu) = \frac{1}{\beta\mu} \frac{\beta\mu \cosh \beta\mu - \sinh \beta\mu}{\mu}.$$

Now, the first equality of (2.6) takes the form

$$\frac{c}{\beta} = \frac{\cosh \beta\mu}{\sinh \beta\mu} - \frac{1}{\beta\mu} = F(\beta\mu), \tag{3.1}$$

and, subsequently, the second equation of (2.6) reads

$$f'(0) = \frac{\beta\mu}{\sinh \beta\mu} \mathrm{e}^{\beta\mu F(\beta\mu)} = G(\beta\mu). \tag{3.2}$$

Since

$$\lim_{x \to 0_+} G(x) = 1, \quad \lim_{x \to \infty} G(x) = \infty, \quad G'(x) = \frac{(\cosh x)^2 - (x^2+1)}{(\sinh x)^3} \mathrm{e}^{xF(x)} > 0$$

for $x > 0$, the equation (3.2) possesses unique solution $\mu > 0$ provided, that $f'(0) > 1$. Moreover,

$$\lim_{x \to 0_+} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1, \quad F'(x) = \frac{(\sinh x)^2 - x^2}{(x \sinh x)^2} > 0 \text{ for } x > 0.$$

This observation together with equality (3.1) implies $0 < c/\beta < 1$, that is

$$0 < c < \beta. \tag{3.3}$$

Equation (2.1) for the top-hat kernel (1.6) gains the form

$$U(x-c) = \frac{1}{2\beta} \int_{x-\beta}^{x+\beta} f\big(U(y)\big) \mathrm{d}y.$$

Differentiating it by x, we obtain the equality

$$U'(x-c) = \frac{f\big(U(x+\beta)\big) - f\big(U(x-\beta)\big)}{2\beta}.$$

The substitution

$$z = \frac{x}{\beta}, \quad V(z) = U(\beta z), \quad \gamma = \frac{c}{\beta}$$

simplifies the previous equality to

$$2V'(z-\gamma) = f\big(V(z+1)\big) - f\big(V(z-1)\big).$$

Let us denote $v_n = V(n)$. Inequality (3.3) yields $0 < \gamma < 1$ and then the derivative on the right hand side of the previous equality for $z = n$ can be approximated by the term $v_n - v_{n-1}$. Finally, let us introduce the function

$$g(v) = 2v - f(v). \tag{3.4}$$

This way, we obtain the implicit second order difference equation

$$f(v_{n+1}) - 2v_n + g(v_{n-1}) = 0. \tag{3.5}$$

Now, we can conclude: *Let g be the function defined by* (3.4)*. If the difference equation* (3.5) *possesses the solution $\{v_n\}_{n=-\infty}^{\infty}$ such that*

$$\lim_{n\to-\infty} v_n = K \quad \textit{and} \quad \lim_{n\to\infty} v_n = 0, \tag{3.6}$$

then $v_n \approx U(\beta n)$.

The function g defined by (3.4) with the Beverton–Holt function f (1.4) can be inverted,

$$g^{-1}(x) = \frac{K(\lambda-2) + x(\lambda-1) + \sqrt{\big(K(\lambda-2)+x(\lambda-1)\big)^2 + 4Kx(\lambda-1)}}{2(\lambda-1)},$$

and equation (3.5) becomes "backward" explicit,

$$v_{n-1} = g^{-1}\big(2v_n - f(v_{n+1})\big). \tag{3.7}$$

According to (2.3) we can solve this equation for the initial values

$$v_0 = \varepsilon, \quad v_1 = \varepsilon \mathrm{e}^{-\mu\beta},$$

where ε is a small positive number and μ is the unique solution of equation (3.2).

Let us finish with a numerical experiment checking the proposed procedure. The solution of the difference equation (3.7) with the Beverton–Holt function possessing parameters $\lambda = 1.5$, $K = 1$, together with the numerical solution of the integrodifference equation (1.2) with top-hat kernel with parameter $\beta = 0.5$, is plotted on fig. 3.1.

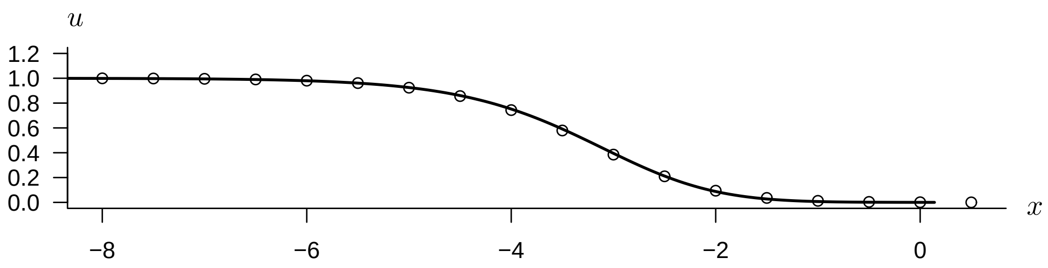

Figure 3.1: The solution of the difference equation (3.5) with the Beverton–Holt function f possessing parameters $\lambda = 15$, $K = 1$ (circles) and a numerical solution of the integrodifference equation (1.2) with the top-hat kernel (1.6) with $\beta = 0.5$ (solid line).

Remark 3.1. The method can be generalized to any symmetric kernel k with compact support, i.e., for k satisfying

$$k(x) = k(-x), \quad k(x) = 0 \text{ for } x \in (-\infty, -\beta) \cup (\beta, \infty), \quad \int_{-\beta}^{\beta} k(x)\mathrm{d}x = 1.$$

In such a case we need more sophisticated calculations to verify inequalities (3.3). The implicit difference equation for a discrete approximation of the wavefront shape takes the form

$$k(\beta)f(v_{n+1}) - v_n + v_{n+1} - k(\beta)f(v_{n-1}) = 0.$$

Of course, the equation can have the solution satisfying the boundary conditions (3.6) only if $k(\beta) \neq 0$.

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