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Multiple Solutions for Parametric Neumann Problems with Indefinite and Unbounded Potential

Nikolaos S. Papageorgiou and **George Smyrlis**
National Technical University of Athens
Department of Mathematics
Zografou Campus, Athens 157 80, Greece
npapg@math.ntua.gr
gsmyrlis@math.ntua.gr

Abstract

We consider a parametric Neumann problem with an indefinite and unbounded potential. Using a combination of critical point theory with truncation techniques and with Morse theory, we produce four nontrivial smooth weak solutions: one positive, one negative and two nodal (sign changing).

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following Neumann problem

$$-\Delta u(z) + \beta(z)u(z) = \lambda u(z) - f(z, u(z)) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega. \qquad (\mathcal{P})_\lambda$$

Here $\beta \in L^q(\Omega)$ $(q > N)$ and is in general indefinite (i.e., it is sign-changing) and unbounded from below. Also $\lambda \in \mathbb{R}$ is a parameter, $f(z, x)$ is a Carathéodory perturbation and $n(\cdot)$ is the outward unit normal on $\partial\Omega$. Moreover, we assume that for a.a. $z \in \Omega$ $f(z, \cdot)$ is superlinear near $\pm\infty$.

The problem $(\mathcal{P})_\lambda$ was studied in the past under Dirichlet boundary conditions and only in the special case where $\beta \equiv 0$, $f(z, x) = f(x)$, $z \in \Omega$, $x \in \mathbb{R}$. We refer to

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the works [3, 4, 11], where it is proved that for $\lambda > \lambda_2 =$ (the second eigenvalue of the Dirichlet $-\Delta$), the problem $(\mathcal{P})_\lambda$ has at least three nontrivial smooth weak solutions. No sign information is provided for the third solution.

In this paper, we extend the aforementioned works to Neumann boundary value problems with an indefinite and unbounded potential $\beta(\cdot)$ and we produce four nontrivial smooth weak solutions: one positive, one negative and two nodal (sign-changing). Our approach uses variational methods based on the critical point theory, truncation techniques and Morse theory (critical groups).

2 Preliminaries

Let X be a Banach space, X^* its topological dual and let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Palais–Smale condition* (the *PS-condition* for short), if the following holds:

"Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded and $\varphi'(x_n) \to 0$ in X^*, admits a strongly convergent subsequence".

This compactness-type condition leads to a *deformation theorem* from which we derive minimax characterizations of certain critical values of φ, for example the *mountain pass theorem*.

Next, let us recall some basic definitions and facts from *Morse theory*. Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\}, \; K_\varphi = \{x \in X : \varphi'(x) = 0\}.$$

Let Y_1, Y_2 be two topological spaces such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the kth-*relative singular homology group* for the topological pair (Y_1, Y_2). Recall that for $k < 0$, $H_k(Y_1, Y_2) \equiv 0$. The *critical groups* of φ at an isolated $x \in K_\varphi$ are defined by

$$C_k(\varphi, x) = H_k\left(\varphi^c \cap U, \varphi^c \cap U \setminus \{x\}\right) \quad \text{for all} \quad k \geq 0,$$

where $c = \varphi(x)$ and U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The *excision property* of singular homology theory implies that the above definition of critical groups, is independent of the particular neighborhood U. If x is a local minimizer of φ, then $C_k(\varphi, x) = \delta_{k,0}\mathbb{Z}$, for all $k \geq 0$.

Suppose that φ satisfies the PS-condition and $\inf \varphi(K_\varphi) > -\infty$. Let also $c < \inf \varphi(K_\varphi)$. The critical groups of φ at *infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all} \quad k \geq 0.$$

The *Second Deformation Theorem* implies that the above definition of critical groups at infinity, is independent of the particular level $c < \inf \varphi(K_\varphi)$ used. If φ is bounded from below, then $C_k(\varphi, \infty) = \delta_{k,0}\mathbb{Z}$, for all $k \geq 0$.

Suppose that K_φ is finite. We set

$$M(x) = \sum_{k\geq 0}(-1)^k \text{rank} C_k(\varphi, x),\ x \in K_\varphi.$$

A particular case of the *Morse relation* says that

$$\sum_{x\in K_\varphi} M(x) = \sum_{k\geq 0}(-1)^k \text{rank} C_k(\varphi, \infty). \tag{2.1}$$

Suppose $X = H$ is a Hilbert space, $x \in H$, U is a neighborhood of x and $\varphi \in C^2(U)$. If $x \in K_\varphi$, then the *Morse index* of x, denoted by $\mu(x)$, is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(x)$ is negative definite. We say that $x \in K_\varphi$ is *nondegenerate*, if $\varphi''(x)$ is invertible. If $x \in K_\varphi$ is nondegenerate and has Morse index $\mu = \mu(x)$, then

$$C_k(\varphi, x) = \delta_{k,\mu}\mathbb{Z}, \quad \text{for all} \quad k \geq 0. \tag{2.2}$$

Here $\delta_{k,\mu}$ is the Kronecker symbol.

Suppose that $X = Y \oplus V$. We say that $\varphi \in C^1(X)$ has a *local linking at 0*, if there exists $r > 0$ such that

$$\varphi(x) \leq 0 \ \text{(resp.} \ \geq 0) \quad \text{for all} \ \ x \in Y \ \ \text{(resp. for all} \ \ x \in V) \ \text{ with} \ \ ||x|| \leq r.$$

From [12, Proposition 2.3], we have the following.

Proposition 2.1. *If H is a Hilbert space, $\varphi \in C^2(H)$, 0 is an isolated critical point of φ, $\varphi''(0)$ is a Fredholm operator and φ has a local linking at 0 with respect to the direct sum decomposition $H = Y \oplus V$ with dim $Y = \mu =$ the Morse index of 0, then*

$$C_k(\varphi, 0) = \delta_{k,\mu}\mathbb{Z}, \ \ \textit{for all} \ \ k \geq 0.$$

In the study of problem $(\mathcal{P})_\lambda$, we will use the Sobolev space $(H^1(\Omega), ||\cdot||)$ and the ordered Banach space $C^1(\overline{\Omega})$. The order cone of $C^1(\overline{\Omega})$ and its interior are given by

$$C_+ = \left\{u \in C^1(\overline{\Omega}) : u\,|_{\overline{\Omega}}\ \geq 0\,\right\}, \ \ \text{int}C_+ = \left\{u \in C_+ : u\,|_{\overline{\Omega}}\ > 0\,\right\},$$

respectively. We know that int$C_+ \neq \emptyset$.

If $h_1, h_2 \in H^1(\Omega)$ such that $h_1(z) \leq h_2(z)$, a.e. on Ω, then we write $h_1 \preceq h_2$ and we define the order interval $[h_1 \preceq h_2] = \{u \in H^1(\Omega) : h_1 \preceq u \preceq h_2\}$.

If $h_1, h_2 \in C^1(\overline{\Omega})$ with $h_2 - h_1 \in \text{int}C_+$, then we write $h_1 \prec h_2$ and we define the order interval $(h_1 \prec h_2) = \left\{u \in C^1(\overline{\Omega}) : h_1 \prec u \prec h_2\right\}$. The latter is an open subset of the Banach space $(C^1(\overline{\Omega}), ||\cdot||_{C^1(\overline{\Omega})})$.

Finally, let $g(z,x), z \in \Omega$, $x \in \mathbb{R}$ be a Carathéodory function and $u_1, u_2 : \Omega \to \mathbb{R}$ be two measurable functions such that $u_1(z) \le u_2(z)$. a.e. in Ω. The *truncation of* g *at the pair* (u_1, u_2) is defined by

$$\widehat{g}(z,x) = \begin{cases} g(z,u_1(z)) & \text{if} \quad x < u_1(z) \\ \\ g(z,x) & \text{if} \quad u_1(z) \le x \le u_2(z) \\ \\ g(z,u_2(z)) & \text{if} \quad u_2(z) < x. \end{cases}$$

Then $\widehat{g}(z,x)$, $z \in \Omega$, $x \in \mathbb{R}$ is also a Carathéodory function.

3 Regularity and Sign of the Solutions

We impose the following hypotheses on $\beta(\cdot)$, $f(\cdot,\cdot)$.

$\underline{H_\beta}$**:** $\beta \in L^q(\Omega)$ for some $q > N$, $\quad \beta^+ \in L^\infty(\Omega)$.

$\underline{H_f}$**:** $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ a.e. in Ω and

(i) $|f(z,x)| \le \alpha(z) + c|x|^{r-1}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)_+$, $c > 0$, $2 < r < 2^* = 2N/(N-2)$.

(ii) $\displaystyle\lim_{x \to 0} \frac{f(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$.

Proposition 3.1. *Under H_β, H_f(i), (ii), let $u \in H^1(\Omega)$ be a weak solution of problem $(\mathcal{P})_\lambda$. Then $u \in C^1(\overline{\Omega})$. If in addition $u \neq 0$ and $u \ge 0$ (resp. $u \le 0$) on Ω, then $u \in$ intC_+ (resp. $u \in -$ intC_+).*

Proof. Hypotheses H_f(i), (ii) imply that

$$|f(z,x)| \le c_1(\,|x| + |x|^{r-1}\,) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } c_1 > 0. \tag{3.1}$$

We set

$$h(z) = \begin{cases} f(z,u(z))/u(z) & \text{if} \quad u(z) \neq 0 \\ \\ 0 & \text{if} \quad u(z) = 0. \end{cases}$$

Due to H_f(i), (ii), h is measurable. Moreover (3.1) yields

$$|h(z)| \le c_1(1 + |u(z)|^{r-2}), \quad \text{for a.a. } z \in \Omega. \tag{3.2}$$

Since $2 < r < 2^*$, we also have $0 < (r-2)N/2 < 2^*$. By the Sobolev embedding theorem, $u \in H^1(\Omega) \hookrightarrow L^{2^*}(\Omega) \subseteq L^{(r-2)N/2}(\Omega)$, so $h \in L^{N/2}(\Omega)$ (see (3.2)). It follows that $\gamma(\cdot) = \lambda - h(\cdot) - \beta(\cdot) \in L^{N/2}(\Omega)$. But now we have

$$-\Delta u(z) = \gamma(z)u(z) \quad \text{a.e. in } \Omega, \quad \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega. \tag{3.3}$$

Invoking the regularity theory of Wang [15, Lemma 5.1], we get that $u \in L^t(\Omega)$ for all $t \geq 1$. This fact combined with (3.2) and with H_β gives that $\gamma \in L^q(\Omega)$.

To proceed, choose $s \in (N, q)$ and set $p = q/s > 1$. Applying Hölder's inequality for the exponents p and $p' = p/(p-1)$ we infer that $\gamma(\cdot)u(\cdot) \in L^s(\Omega)$ (recall that $u \in L^t(\Omega)$ for all $t \geq 1$). Then $-\Delta u \in L^s(\Omega)$ (see (3.3)). Now the regularity theory of Wang [15, Lemma 5.2] implies that $u \in W^{2,s}(\Omega)$. But the Sobolev embedding theorem says that for $\tau = 1 - N/s \in (0,1)$, $W^{2,s}(\Omega) \hookrightarrow C^{1,\tau}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$. Thus, $u \in C^1(\overline{\Omega})$.

Next, suppose that $u \neq 0$ and $u \geq 0$ on Ω. Since $u \in C^1(\overline{\Omega})$, we have that $h \in L^\infty(\Omega)$ (see (3.2)) and

$$\Delta u(z) = (\beta(z) - \lambda + h(z))u(z) \leq (||\beta^+||_\infty + |\lambda| + ||h||_\infty)u(z), \quad \text{for a.a. } z \in \Omega.$$

It follows that $u \in \mathrm{int}C_+$ (see Vazquez [14]).

We argue in a similar way in the case where $u \not\equiv 0$, $u\,|_\Omega \leq 0$. □

4 The Spectrum of $-\Delta u + \beta u$

In this section we study the spectral properties of the differential operator $u \mapsto -\Delta u + \beta u$. So, we consider the following linear eigenvalue problem:

$$-\Delta u(z) + \beta(z)u(z) = \lambda u(z) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Omega. \tag{4.1}$$

We assume that H_β holds (see Section 3). By using standard arguments mainly based on the continuous embedding $H^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega)$, we may prove the following.

Proposition 4.1. *Let $\sigma : H^1(\Omega) \to \mathbb{R}$ be defined by*

$$\sigma(u) = ||Du||_2^2 + \int_\Omega \beta(z)u(z)^2 dz \quad \text{for all } u \in H^1(\Omega) \ (Du \text{ is the gradient of } u).$$

Then

(i) the smallest eigenvalue $\widehat{\lambda}_1 \in \mathbb{R}$ of (4.1) *is given by*

$$-\infty < \widehat{\lambda}_1 = \inf\left[\frac{\sigma(u)}{||u||_2^2} : u \in H^1(\Omega),\ u \neq 0\right]; \tag{4.2}$$

(ii) there exist $\xi > 0$, $\xi > -\widehat{\lambda}_1$ and $c > 0$ such that

$$\sigma(u) + \xi\,||u||_2^2 \geq c\,||u||^2 \quad \text{for all } u \in H^1(\Omega). \tag{4.3}$$

By using Proposition 4.1 and working in a similar way as in the case of $-\Delta$ we obtain a sequence of eigenvalues $-\infty < \widehat{\lambda}_1 < \widehat{\lambda}_2 < \ldots < \widehat{\lambda}_m < \ldots,\quad \widehat{\lambda}_m \to +\infty$ as $m \to \infty$. Also there is a corresponding sequence $\{\widehat{u}_m\}_{m\geq 1}$ of eigenfunctions of

(4.1) such that $\widehat{u}_m \in C^1(\overline{\Omega})$, for all $m \geq 1$ (see Proposition 3.1) and $\{\widehat{u}_m\}_{m\geq 1}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1(\Omega)$.

For each $m \geq 2$, we have the following *variational characterization*:

$$\widehat{\lambda}_m = \sup\left[\frac{\sigma(u)}{||u||_2^2} : u \in H_m,\ u \neq 0\right] = \inf\left[\frac{\sigma(u)}{||u||_2^2} : u \in H_{m-1}^{\perp},\ u \neq 0\right], \quad (4.4)$$

where $H_m = \bigoplus_{i=1}^{m} E(\widehat{\lambda}_i)$, $H_{m-1}^{\perp} = \overline{\bigoplus_{i\geq m} E(\widehat{\lambda}_i)}$ and $H^1(\Omega) = H_{m-1} \oplus H_{m-1}^{\perp}$.

Here $E(\widehat{\lambda}_i)$ denotes the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_i$. The infimum in (4.2) is realized on $E(\widehat{\lambda}_1)$ and both the supremum and infimum in (4.4) are realized on $E(\widehat{\lambda}_m)$. We know that $\widehat{\lambda}_1$ is simple (i.e., dim $E(\widehat{\lambda}_1) = 1$) and is the only eigenvalue with eigenfunctions of constant sign. All the higher eigenvalues have nodal eigenfunctions. In the sequel by $\widehat{u}_1 \in C_+ \setminus \{0\}$ we denote the L^2-normalized (i.e., $||\widehat{u}_1||_2 = 1$), positive eigenfunction. Then Proposition 3.1 implies that $\widehat{u}_1 \in \text{int}\, C_+$. The eigenspaces $E(\widehat{\lambda}_i)$, $i \geq 1$ all have the so-called *Unique Continuation Property* (*UCP*), namely if $u \in E(\widehat{\lambda}_i)$ and u vanishes on a set of positive measure, then $u \equiv 0$ (see Garofalo–Lin [9] and de Figueiredo–Gossez [8]). An easy application of the UCP and of (4.4), reveals that if $m \geq 1$, then there exists $\widetilde{\xi} > 0$ such that

$$\sigma(u) - \widehat{\lambda}_{m+1}||u||_2^2 \leq -\widetilde{\xi}\,||u||^2, \quad \text{for all } u \in H_m. \quad (4.5)$$

5 Solutions of Constant Sign

In addition to hypotheses H_f(i), (ii), we also assume that

$\underline{H_f}$: (iii) $\lim_{|x|\to\infty} \frac{f(z,x)}{x} = +\infty$ uniformly for a.a. $z \in \Omega$ (superlinearity).

Hypothesis H_f(iii) enables us to construct an upper solution $\overline{u} \in \text{int}C_+$ and a lower solution $\underline{u} \in -\,\text{int}C_+$ of the problem $(\mathcal{P})_\lambda$. In fact, $\overline{u}$ (resp. $\underline{u}$) is an appropriate positive (resp. negative) multiple of $\widehat{u}_1$ (see Section 4).

Proposition 5.1. *If hypotheses $H(\beta)$ and H_f(i)–(iii) hold and $\lambda > \widehat{\lambda}_1$, then problem $(\mathcal{P})_\lambda$ has at least two nontrivial constant sign smooth solutions*

$$u_0 \in [0 \preceq \overline{u}] \cap \textit{int}\, C_+ \quad \textit{and} \quad v_0 \in [\underline{u} \preceq 0] \cap (-\textit{int}\, C_+).$$

Sketch of the proof. First we construct u_0 (the construction of v_0 is similar).

Let $g_+^\lambda(z,x)$ be the truncation of the function $(\lambda + \xi)x - f(z,x)$ at the pair $(0,\overline{u})$ (see Section 2), where ξ appears in (4.3).

We set $G_+^\lambda(z,x) = \int_0^x g_+^\lambda(z,s)ds$ and consider $\varphi_+^\lambda \in C^1(H^1(\Omega))$ defined by

$$\varphi_+^\lambda(u) = \frac{1}{2}||Du||_2^2 + \frac{1}{2}\int_\Omega (\xi + \beta(z))u(z)^2 dz - \int_\Omega G_+^\lambda(z, u(z))dz, \text{ for all } u \in H^1(\Omega).$$

The functional φ_+^λ is coercive (this may be shown with the aid of (4.3)) so, it attains its minimum at some point $u_0 \in H^1(\Omega)$. In particular, u_0 is a critical point of φ_+^λ. By using suitable test functions and taking into account the definition of g_+^λ, we may show that u_0 is a weak solution of the problem $(\mathcal{P})_\lambda$ such that $u_0 \in [0 \preceq \overline{u}]$.

Moreover, employing H_f(ii) together with the hypothesis that $\lambda > \widehat{\lambda}_1$, we may choose $t > 0$ sufficiently small such that $\varphi_+^\lambda(t\widehat{u}_1) < 0 = \varphi_+^\lambda(0)$. This fact implies that $u_0 \neq 0$ (recall that u_0 is a global minimizer of φ_+^λ). Now we apply Proposition 3.1. □

Proposition 5.2. *Under hypotheses of Proposition 5.1, the problem $(\mathcal{P})_\lambda$ has a smallest positive smooth solution u_* and a biggest negative smooth solution v_*.*

Sketch of the proof. We establish the existence of the smallest positive solution (the existence of the biggest negative solution is proved in a similar way). Let

$$S_+^\lambda = \left\{ u \in H^1(\Omega) : u \in [0 \preceq \overline{u}],\ u \neq 0,\ u \text{ is a solution of } (\mathcal{P})_\lambda \right\}.$$

From Propositions 5.1 and 3.1 we know that $S_+^\lambda \neq \emptyset$ and $S_+^\lambda \subseteq \text{int}\, C_+$.

We show that S_+^λ has a minimal element with respect to the pointwise order ($\preceq$). To this end, let $C \subseteq S_+^\lambda$ be a chain. We can find a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq C$ such that $\inf C = \inf_{n\geq 1} u_n$. Each u_n is a weak solution of $(\mathcal{P})_\lambda$ and $\sup_{n\geq 1} ||u_n||_\infty \leq ||\overline{u}||_\infty < \infty$. By using u_n as a test function, we may show that $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ is bounded. By passing to subsequences we may assume that $\{u_n\}_{n\geq 1}$ weakly converges to some $u \in H^1(\Omega)$. Exploiting the strong monotonicity of the operator $u \to -\Delta u$ combined with the Sobolev embedding theorem, we deduce that $u_n \to u$ in $H^1(\Omega)$. Then passing to the limit as $n \to \infty$, we infer that u solves $(\mathcal{P})_\lambda$ and $u \in [0 \preceq \overline{u}]$.

It remains to prove that $u \neq 0$. Arguing indirectly, suppose that $u = 0$. We have $u_n \to 0$ in $H^1(\Omega)$. We set $y_n = \frac{u_n}{||u_n||}$, $n \geq 1$. As before, by passing to subsequences, we may assume that

$$y_n \to y \text{ in } H^1(\Omega) \quad \text{for some } y \in H^1(\Omega) \text{ with } ||y|| = 1,\ y \geq 0 \tag{5.1}$$

and also

$$u_n(z) \to 0, \quad y_n(z) \to y(z) \quad \text{a.e. in } \Omega. \tag{5.2}$$

We set

$$g_n(z) = \frac{f(z, u_n(z))}{||u_n||} = \frac{f(z, u_n(z))}{u_n(z)} y_n(z), \quad n \geq 1,\ z \in \Omega.$$

By using (3.1), (5.1), (5.2), combined with hypothesis H_f(ii) and Lebesgue's theorem, we may check that

$$g_n \to 0 \quad \text{in} \quad L^2(\Omega). \tag{5.3}$$

Now we observe that each y_n is a weak solution of the problem $(\mathcal{P})_\lambda$, where the nonlinearity $f(\cdot,\cdot)$ is replaced by the function $g_n(\cdot)$. Then by passing to the limit as $n \to \infty$ and taking into account (5.1), (5.3) we infer that y is a λ-eigenfunction of the operator $u \mapsto -\Delta u + \beta u$. Since $\lambda > \widehat{\lambda}_1$, y must be nodal (see Section 4) which contradicts (5.1).

Therefore $u = \inf C \in S_+^\lambda$. Since $C \subseteq S_+^\lambda$ is an arbitrary chain, Zorn's lemma guarantees that S_+^λ has a minimal element $u_* \in S_+^\lambda$. But as in [2, Lemma 1], we may show that $(S_+^\lambda, \preceq)$ is *downward directed*, i.e., if $u_1, u_2 \in S_+^\lambda$, we can find $u \in S_+^\lambda$ such that $u \preceq u_1$, $u \preceq u_2$. Hence, $u_* \in \mathrm{int}C_+$ is the smallest positive solution of $(\mathcal{P})_\lambda$. □

6 Nodal Solutions

Under more restrictive hypotheses on the data of the problem $(\mathcal{P})_\lambda$, we are going to construct two nodal smooth solutions. For $\lambda > \widehat{\lambda}_1$, we consider the corresponding *energy functional* φ_λ defined by

$$\varphi_\lambda(u) = \frac{1}{2}\sigma(u) - \frac{\lambda}{2}||u||_2^2 + \int_\Omega F(z, u(z))dz, \quad u \in H^1(\Omega),$$

where $\sigma(u)$ is defined in Proposition 4.1 and

$$F(z,x) = \int_0^x f(z,s)ds, \ z \in \Omega, \ x \in \mathbb{R}.$$

Under hypotheses H_β, H_f(i),(iii), $\varphi_\lambda \in C^1(H^1(\Omega))$ and it is coercive, thus it satisfies the Palais–Smale condition. It is not clear whether if φ_λ has the *Geometry of the mountain pass theorem (GMPT)* or not. Nevertheless, we have the following.

Proposition 6.1. *Under hypotheses H_β, H_f(i)–(iii) and for $\lambda > \widehat{\lambda}_1$, there exists a coercive functional $\psi_\lambda \in C^1(H^1(\Omega))$, such that*

(i) φ_λ, ψ_λ coincide on $[v_* \preceq u_*]$ and $K_{\psi_\lambda} \subseteq K_{\varphi_\lambda} \cap [v_* \preceq u_*]$.

(ii) u_*, v_* are both local minimizers of ψ_λ.

(iii) ψ_λ has the GMPT and there exists $y_0 \in K_{\psi_\lambda} \setminus \{u_*, v_*\}$ such that $C_1(\psi_\lambda, y_0) \neq 0$.

Sketch of the proof. (i), (ii). Let h_+^λ, h_-^λ, h^λ be the truncations of the function $(\lambda+\xi)x - f(z,x)$ at the pairs $(0, u_*)$, $(v_*, 0)$, (v_*, u_*) respectively (see Section 2), where ξ appears in (4.3). We define the functionals $\psi_\pm^\lambda$, $\psi^\lambda \in C^1(H^1(\Omega))$ exactly as the functional φ_+^λ

in the proof of Proposition 5.1, replacing "g_+^λ" by $h_\pm^\lambda$, h^λ respectively. The functionals $\psi_\pm^\lambda$, ψ^λ are all coercive and

$$K_{\psi_+^\lambda} \subseteq K_{\varphi_\lambda} \cap [0 \preceq u_*], \quad K_{\psi_-^\lambda} \subseteq K_{\varphi_\lambda} \cap [v_* \preceq 0], \quad K_{\psi^\lambda} \subseteq K_{\varphi_\lambda} \cap [v_* \preceq u_*].$$

The extremality of u_*, v_* yields $K_{\psi_+^\lambda} = \{0, u_*\}$ and $K_{\psi_-^\lambda} = \{v_*, 0\}$. Now minimizing $\psi_\pm^\lambda$ we obtain that u_*, v_* are both local minimizers of ψ_λ with respect to the norm $||\cdot||_{C^1(\overline{\Omega})}$ (note that ψ^λ coincides with $\psi_\pm^\lambda$ on $\pm\text{int}C_+$). It follows from a result of Brezis–Nirenberg [6] that u_*, v_* are also local minimizers of ψ_λ in $H^1(\Omega)$.

(iii) We may assume that K_{ψ^λ} is finite (or otherwise we already have an infinity of nodal solutions). Then u_*, v_* are isolated critical points and local minimizers of ψ^λ. We may also assume that $\psi^\lambda(v_*) \leq \psi^\lambda(u_*)$. Arguing as in [1, proof of Prop. 29], we may find $\rho \in (0,1)$ such that $||v_* - u_*|| > \rho$ and $\psi^\lambda(v_*) \leq \psi^\lambda(u_*) < \inf[\psi^\lambda(u) : ||u - u_*|| = \rho]$. Hence, ψ_λ has the GMPT. Now the conclusion follows from an improved version of the mountain pass theorem (see for example, [7, p.89]). □

To proceed, we *strengthen* the hypotheses on the data of $(\mathcal{P})_\lambda$. Namely we assume

$\underline{H_f'}$**:** $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f_x'(z,x)| \leq \alpha(z) + c|x|^{r-2}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\alpha \in L^\infty(\Omega)_+$, $c > 0$, $2 < r < 2^*$.

(ii) $f_x'(z,0) = \lim_{x \to 0} \dfrac{f(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$.

(iii) $\lim_{|x| \to \infty} \dfrac{f(z,x)}{x} = +\infty$ uniformly for a.a. $z \in \Omega$.

(iv) $f(z,x)x \geq 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$.

Moreover, we assume that $\lambda > \widehat{\lambda}_2$ (see Section 4). Hypotheses H_β, H_f'(i) imply that $\varphi_\lambda \in C^2(H^1(\Omega))$. This will be crucial in computing the critical groups of φ_λ.

Let $m \geq 2$ be the unique positive integer such that $\lambda \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$. We have.

Proposition 6.2. *If hypotheses H_β, H_f'(i)–(iv) hold, then $C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z}$, for all $k \geq 0$, where $d_m = \dim H_m \geq 2$, $H_m = \bigoplus_{i=1}^m E(\widehat{\lambda}_i)$.*

Sketch of the proof. For all $x \in H^1(\Omega)$,

$$\langle \varphi_\lambda''(0)(x), x \rangle = ||Dx||_2^2 + \int_\Omega [\beta(z) - \lambda] x(z)^2 dz = \sigma(x) - \lambda ||x||_2^2 \ \text{(see } H_f' \text{ (ii))}.$$

Since $\lambda \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$, the variational expressions (4.4) imply that

$$\langle \varphi_\lambda''(0)(x), x\rangle \leq 0 \text{ (resp. } \geq 0 \text{), for all } x \in H_m \text{ (resp. for all } x \in H_m^\perp \text{).}$$

Hence, the Morse index of 0 equals to $\mu = d_m \geq 2$.

If $\lambda \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, then 0 is a nondegenerate critical point of φ_λ and the conclusion follows immediately from (2.2).

Suppose that $\lambda = \widehat{\lambda}_{m+1}$. Then by using H_f'(ii) in conjunction with (4.5), we may find $\delta > 0$ such that for $u \in H_m$ with $||u||_{C^1(\overline{\Omega})} \leq \delta$, we have $\varphi_\lambda(u) \leq 0$. But all the norms on H_m are equivalent, so we may also find $\widehat{\delta} > 0$ such that for $u \in H_m$ with $||u||_{H^1(\Omega)} \leq \widehat{\delta}$, inequality $\varphi_\lambda(u) \leq 0$ still holds. On the other hand, employing (4.4) together with H_f'(iv), we may show that $\varphi_\lambda(u) \geq 0$ for all $u \in H_m^\perp$.

It turns out that φ_λ has a *local linking* at 0 (see Section 2) and the conclusion follows from Proposition 2.1. □

Now we are ready to establish the existence of the first nodal solution.

Proposition 6.3. *Suppose that H_β, H_f'(i)–(iv) hold, $\lambda > \widehat{\lambda}_2$ and let y_0 be given by Proposition 6.1. Then $y_0 \in (v_* \prec u_*) \setminus \{0\}$ and thus, it is a nodal smooth weak solution of $(\mathcal{P})_\lambda$.*

Proof. First we show that $y_0 \neq 0$. Indeed, ψ_λ, φ_λ coincide on $(v_* \prec u_*)$ which is an open neighborhood of 0 with respect to the norm $||\cdot||_{C^1(\overline{\Omega})}$. Meanwhile, the linear space $C^1(\overline{\Omega})$ is $||\cdot||_{H^1(\Omega)}$-dense in the Hilbert space $H^1(\Omega)$. By a result of Palais [10], we infer that for all $k \geq 0$,

$$C_k(\psi_\lambda, 0) = C_k(\psi_\lambda \,|_{C^1(\overline{\Omega})}, 0) = C_k(\varphi_\lambda \,|_{C^1(\overline{\Omega})}, 0) = C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \tag{6.1}$$

(see Proposition 6.2). In particular, $C_1(\psi_\lambda, 0) = 0$ (recall that $d_m \geq 2$), whereas $C_1(\psi_\lambda, y_0) \neq 0$ (see Prop. 6.1(iii)). Thus, $y_0 \neq 0$.

Next, we set $R = \max\{||u_*||, ||v_*||\}$. By using H_f'(i) combined with the mean value theorem we may find $\eta > 0$ such that for a.a. $z \in \Omega$, the function $x \to \eta x - f(z,x)$ is nondecreasing on $[-R, R]$. This fact implies that for $w \in \{y_0 - v_*, u_* - y_0\}$, $\Delta w(z) \leq (\beta(z) + \eta)w(z)$, for a.a. $z \in \Omega$. Now H_β coupled with the Strong Maximum principle of Vazquez [14] yields that $y_0 - v_*, u_* - y_0 \in \text{int}C_+$. □

In order to derive a second nodal solution, we also need to compute the critical groups of ψ_λ at y_0. Namely, we have the following:

Proposition 6.4. $C_k(\psi_\lambda, y_0) = \delta_{k,1}\mathbb{Z}$, *for all $k \geq 0$.*

Proof. Arguing exactly as in the first part of the proof of Proposition 6.3 with 0 being replaced by y_0, we deduce that $C_k(\psi_\lambda, y_0) = C_k(\varphi_\lambda, y_0)$, for all $k \geq 0$. In particular, $C_1(\varphi_\lambda, y_0) = C_1(\psi_\lambda, y_0) \neq 0$ (see Proposition 6.1(iii)). It suffices to prove that $C_k(\varphi_\lambda, y_0) = \delta_{k,1}\mathbb{Z}$, for all $k \geq 0$. To this end, let $\sigma(\varphi_\lambda''(y_0))$ denote the spectrum of

$\varphi''_\lambda(y_0)$ and assume that $\sigma(\varphi''_\lambda(y_0)) \subseteq [0, +\infty)$. This means that the Morse index of y_0 is zero and we have

$$||Du||_2^2 \geq \int_\Omega \eta u^2 dz \quad \text{for all} \;\; u \in H^1(\Omega), \tag{6.2}$$

where $\eta(\cdot) = \lambda - \beta(\cdot) - f'_x(\cdot, y_0(\cdot)) \in L^q(\Omega)$ $(q > N)$ (see H_β, H'_f(i)). If $u \in \ker\varphi''_\lambda(y_0)$, then

$$-\Delta u(z) = \eta(z)u(z) \;\; \text{a.e. in} \;\; \Omega, \quad \frac{\partial u}{\partial n} = 0 \;\; \text{on} \;\; \partial\Omega. \tag{6.3}$$

If $\eta^+ = 0$, then from (6.3) it follows that $u = 0$. If $\eta^+ \neq 0$, then from (6.2) we have $\widetilde{\lambda}_1(\eta) = \inf \left[\, ||Du||_2^2 : \int_\Omega \eta u^2 dz = 1, \; u \in H^1(\Omega) \right] \geq 1$, which is the principal eigenvalue of $-\Delta$ with Neumann boundary conditions and weight η. This eigenvalue is simple (see [13, Theorem 2.5]) and so from (6.3) it follows that dim $\ker\varphi''_\lambda(y_0) \leq 1$. Then invoking a result of Bartcsh [5, Proposition 2.5] we infer that $C_k(\varphi_\lambda, y_0) = \delta_{k,1}\mathbb{Z}$, for all $k \geq 0$, which finishes the proof of Proposition 6.4. □

Now we are ready to establish the existence of a second nodal solution.

Proposition 6.5. *Assume that H_β and H'_f(i)–(iv) hold. Then for each $\lambda > \widehat{\lambda}_2$, the problem $(\mathcal{P})_\lambda$ has a second nodal solution $\widehat{y} \neq y_0$.*

Proof. Regarding the critical groups of ψ_λ at its critical points $0, u_*, v_*, y_0$, we have for all $k \geq 0$,

$$C_k(\psi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z}, \quad C_k(\psi_\lambda, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{(see (6.1) and Proposition 6.4)}, \tag{6.4}$$

$$C_k(\psi_\lambda, u_*) = C_k(\psi_\lambda, v_*) = \delta_{k,0}\mathbb{Z}, \quad \text{(see Proposition 6.1(ii) and Section 2)} \tag{6.5}$$

and

$$C_k(\psi_\lambda, \infty) = \delta_{k,0}\mathbb{Z}, \quad \text{since } \psi_\lambda \text{ is coercive (see Section 2).} \tag{6.6}$$

Assume that $K_{\psi_\lambda} = \{0, u_*, v_*, y_0\}$. Then exploiting the *Morse relation* (2.1) together with (6.4), (6.5) and (6.6) we obtain $(-1)^{d_m} + 1 + 1 + (-1) = 1$ (false!). Hence, we may find $\widehat{y} \in K_{\psi_\lambda} \setminus \{0, u_*, v_*, y_0\}$. Arguing as in the second part of the proof of Proposition 6.3, we may check that $\widehat{y} \in (v_* \prec u_*)$ and thus, $\widehat{y}$ is nodal. □

Summarizing the above results we obtain our main existence result stated below:

Theorem 6.6. *Assume that H_β and H'_f(i)–(iv) hold. Then for each $\lambda > \widehat{\lambda}_2$, the problem $(\mathcal{P})_\lambda$ has at least four nontrivial smooth weak solutions: one positive, one negative and two nodal.*

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