Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 8, Number 1, pp. 95–106 (2013)
http://campus.mst.edu/adsa

Existence of Three Solutions for a Biharmonic System with Weight

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Abstract

Existence and multiplicity of weak solutions for an elliptic system is studied. By using Ekeland's variational principle and the mountain pass theorem, we prove existence of at least three weak solutions.

AMS Subject Classifications: 35J40, 35J67.
Keywords: (p, q)-biharmonic system, weight function, Navier condition, Ekeland's variational principle, mountain pass theorem.

1 Introduction and Main Result

We are concerned with the following elliptic system of (p, q)-biharmonic type:

$$\begin{gathered}\Delta(|\Delta u|^{p-2}\Delta u) = \lambda h_1(x)|u|^{p-2}u + F_u(x, u, v) \quad \text{in } \Omega, \\ \Delta(|\Delta v|^{q-2}\Delta v) = \mu h_2(x)|v|^{q-2}v + F_v(x, u, v) \quad \text{in } \Omega, \\ u = v = \Delta u = \Delta v = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded smooth domain, $p, q > 1$, $F \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$, F_u denotes the partial derivative of F with respect to u, and $h_i \in C(\overline{\Omega})$, $i = 1, 2$, are nonnegative weight functions.

The investigation of existence and multiplicity of solutions for problems with p-biharmonic operators has drawn the attention of many authors, see [3, 4, 6–9]. In [3],

Received May 14, 2012; Accepted November 14, 2012
Communicated by Delfim F. M. Torres

Li and Tang considered the following Navier boundary value problem involving the p-biharmonic operator:

$$\begin{gathered} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $p > \max\left\{1, \dfrac{N}{2}\right\}$ and $\lambda, \mu \geq 0$. Under suitable assumptions, the existence of at least three weak solutions is established. In [4], the authors studied the system

$$\begin{gathered} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda F_u(x,u,v) + \mu G_u(x,u,v) \quad \text{in } \Omega, \\ \Delta(|\Delta v|^{q-2}\Delta v) = \lambda F_v(x,u,v) + \mu G_v(x,u,v) \quad \text{in } \Omega, \\ u = v = \Delta u = \Delta v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $p, q > \max\left\{1, \dfrac{N}{2}\right\}$ and $\lambda, \mu \geq 0$. By a technical approach based on the three critical points theorem of Ricceri, they obtained existence and multiplicity of solutions. In [6], Shen and Zhang studied the following system:

$$\begin{gathered} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = \lambda |u|^{q-2}u + \frac{1}{p^{**}} F_u(x,u,v) \quad \text{in } \Omega, \\ \Delta\left(|\Delta v|^{p-2}\Delta v\right) = \mu |v|^{q-2}v + \frac{1}{p^{**}} F_v(x,u,v) \quad \text{in } \Omega, \\ u = v = \Delta u = \Delta v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $1 < q < p < \dfrac{N}{2}$, $p^{**} = \dfrac{Np}{N-2p}$, $\lambda, \mu > 0$, and $F \in C^1(\overline{\Omega} \times (\mathbb{R}^+)^2), \mathbb{R}^+)$ is positively homogeneous of degree p^{**}. The authors proved the existence of at least two positive solutions when the pair parameters satisfy a certain inequality.

The purpose of this paper is to extend some of the results obtained in the paper [1], for the case of (p,q)-Laplacian to the case of a fourth-order quasilinear system with weight. We prove the existence of at least three weak solutions for system (1.1). Our technical approach is based on Ekeland's variational principle and the mountain pass theorem. We assume that $F(x,u,v)$ satisfies the following condition:

(F1) $\displaystyle\lim_{|s|+|t|\to\infty} \frac{F_s(x,s,t)}{h_1(x)|s|^{p-1}} = 0$, $\displaystyle\lim_{|s|+|t|\to\infty} \frac{F_t(x,s,t)}{h_2(x)|t|^{q-1}} = 0$, uniformly in $x \in \overline{\Omega}$.

We introduce the space

$$X := \left(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\right) \times \left(W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)\right),$$

which is a reflexive Banach space endowed with the norm

$$||(u,v)|| = ||u||_p + ||v||_q,$$

where $||u||_p = \left(\int_\Omega |\Delta u|^p dx\right)^{1/p}$ and $||v||_q = \left(\int_\Omega |\Delta v|^q dx\right)^{1/q}$.

Consider the following problems:

$$\begin{gathered} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda h_1(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.5}$$

and

$$\begin{gathered} \Delta(|\Delta v|^{q-2}\Delta v) = \mu h_2(x)|v|^{q-2}v \quad \text{in } \Omega, \\ v = \Delta v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.6}$$

Let λ_1, μ_1 denote the first eigenvalues of problems (1.5) and (1.6), respectively. According to the work of Talbi and Tsouli [8], since $h_i \in C(\overline{\Omega})$ and $h_i \geq 0$, $i = 1, 2$, λ_1 and μ_1 are positive, simple, isolated and are given by

$$\begin{aligned} \lambda_1 &= \inf\left\{||u||_p^p : u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \int_\Omega h_1(x)|u|^p dx = 1\right\}, \\ \mu_1 &= \inf\left\{||v||_q^q : v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \int_\Omega h_2(x)|v|^q dx = 1\right\}. \end{aligned} \tag{1.7}$$

Therefore,

$$\begin{aligned} &\int_\Omega |\Delta u|^p dx \geq \lambda_1 \int_\Omega h_1(x)|u|^p dx \text{ for all } u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ &\int_\Omega |\Delta v|^q dx \geq \mu_1 \int_\Omega h_2(x)|v|^q dx \text{ for all } v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega). \end{aligned} \tag{1.8}$$

Let φ_1 and ψ_1 be the corresponding normalized eigenfunctions to λ_1 and μ_1, respectively. Moreover, let

$$\begin{aligned} \lambda_2 &= \inf\left\{\lambda : \lambda \text{ is an eigenvalue of (1.5) with } \lambda > \lambda_1\right\}, \\ \mu_2 &= \inf\left\{\mu : \mu \text{ is an eigenvalue of (1.6) with } \mu > \mu_1\right\}. \end{aligned} \tag{1.9}$$

The fact that λ_1 and μ_1 are isolated implies that $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$. It can also be shown (see Lemma 2.1) that there exist $\overline{\lambda} \in (\lambda_1, \lambda_2]$ and $\overline{\mu} \in (\mu_1, \mu_2]$ such that

$$\int_\Omega |\Delta u|^p dx \geq \overline{\lambda} \int_\Omega h_1(x)|u|^p dx \tag{1.10}$$

for all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $\int_\Omega h_1(x)|\varphi_1|^{p-2}\varphi_1 u dx = 0$,

$$\int_\Omega |\Delta v|^q dx \geq \overline{\mu} \int_\Omega h_2(x)|v|^q dx \tag{1.11}$$

for all $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ with $\int_\Omega h_2(x)|\psi_1|^{q-2}\psi_1 v dx = 0$. Now we are ready to state our main result.

Theorem 1.1. *Assume that* $(F1)$ *holds and*

$$\lim_{|s|,|t|\to\infty} F(x, s\varphi_1, t\psi_1) = +\infty, \quad \textit{uniformly in } x \in \Omega. \tag{1.12}$$

Then, for $\lambda < \lambda_1$ *and* $\mu < \mu_1$ *sufficiently close to* λ_1 *and* μ_1*, problem* (1.1) *has at least three solutions.*

Remark 1.2. An example of a nonlinear F that satisfies the assumption (F_1) is:

$$F(x, s, t) = h_1(x)h_2(x)\ln\left(|s|^p + |t|^q + 1\right) \text{ for all } (x, s, t) \in \overline{\Omega} \times \mathbb{R}^2,$$

where $p, q > 1$ and $h_1, h_2 \in C(\overline{\Omega})$ are considered as in problem (1.1).

2 Preliminaries and Proof of Theorem 1.1

Let us denote by $\langle\varphi_1\rangle$ and $\langle\psi_1\rangle$ the linear spans of φ_1 and ψ_1, respectively. Define

$$V = \langle\varphi_1\rangle \times \langle\psi_1\rangle, \tag{2.1}$$

$$W = \left\{(u, v) \in X : \int_\Omega h_1(x)|\varphi_1|^{p-2}\varphi_1 u dx = 0, \int_\Omega h_2(x)|\psi_1|^{q-2}\psi_1 v dx = 0\right\}. \tag{2.2}$$

Then we can decompose X as a direct sum of V and W. In fact, let $(u, v) \in X$. Writing

$$u = \alpha\varphi_1 + w \quad \text{and} \quad v = \beta\psi_1 + z,$$

where $(w, z) \in X$,

$$\alpha = \lambda_1 \int_\Omega h_1(x)|\varphi_1|^{p-2}\varphi_1 u dx \quad \text{and} \quad \beta = \mu_1 \int_\Omega h_2(x)|\psi_1|^{q-2}\psi_1 v dx. \tag{2.3}$$

One has

$$\int_\Omega |\Delta\varphi_1|^p dx = 1 \quad \text{and} \quad \int_\Omega |\Delta\psi_1|^q dx = 1,$$

$$\int_\Omega h_1(x)|\varphi_1|^{p-2}\varphi_1 w dx = 0 \quad \text{and} \quad \int_\Omega h_2(x)|\psi_1|^{q-2}\psi_1 z dx = 0.$$

Therefore, $(w, z) \in W$ and

$$X = V \oplus W.$$

We begin by establishing the existence of $\overline{\lambda}$ and $\overline{\mu}$ for which (1.10) and (1.11) hold.

Lemma 2.1. *There exist* $\overline{\lambda} \in (\lambda_1, \lambda_2]$ *and* $\overline{\mu} \in (\mu_1, \mu_2]$ *such that*

$$\int_\Omega |\Delta u|^p dx \geq \overline{\lambda} \int_\Omega h_1(x)|u|^p dx, \tag{2.4}$$

$$\int_\Omega |\Delta v|^q dx \geq \overline{\mu} \int_\Omega h_2(x)|v|^q dx, \tag{2.5}$$

for all $(u, v) \in W$.

Proof. For simplicity, we set

$$\begin{aligned}\varphi_1^{\perp} &= \left\{ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \int_\Omega h_1(x)|\varphi_1|^{p-2}\varphi_1 u dx = 0 \right\}, \\ \psi_1^{\perp} &= \left\{ v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) : \int_\Omega h_2(x)|\psi_1|^{q-2}\psi_1 v dx = 0 \right\}.\end{aligned} \tag{2.6}$$

Let

$$\lambda = \inf \left\{ ||u||_p^p : u \in \varphi_1^{\perp}, \int_\Omega h_1(x)|u|^p dx = 1 \right\}.$$

This value is attained in $\varphi_1^{\perp}$. To see why this is so, let u_n be a sequence in $\varphi_1^{\perp}$ satisfying $\int_\Omega h_1(x)|u_n|^p dx = 1$ for all n, and $\int_\Omega |\Delta u_n|^p dx \to \lambda$. It follows that u_n is bounded in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and therefore, up to a subsequence, we may assume that

$$u_n \rightharpoonup u \text{ weakly in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ strongly in } L^p(\Omega).$$

From the strong convergence of the sequence in $L^p(\Omega)$, we obtain

$$\int_\Omega h_1(x)|u|^p dx = \lim_{n\to\infty} \int_\Omega h_1(x)|u_n|^p dx = 1$$

and

$$\int_\Omega h_1(x)|\varphi_1|^{p-2}\varphi_1 u dx = \lim_{n\to\infty} \int_\Omega h_1(x)|\varphi_1|^{p-2}\varphi_1 u_n dx = 0,$$

so that $u \in \varphi_1^{\perp}$. By the weakly lower semicontinuity of the norm $|| \cdot ||_p$, we get

$$\lambda \leq \int_\Omega |\Delta u|^p dx \leq \liminf_{n\to\infty} \int_\Omega |\Delta u_n|^p dx = \lambda,$$

and hence λ is attained at u.

Now we claim that $\lambda > \lambda_1$. It follows from (1.7) that $\lambda \geq \lambda_1$. If $\lambda = \lambda_1$, then by simplicity of λ_1 there is $\alpha \in \mathbb{R}$ such that $u = \alpha\varphi_1$. Since $u \in \varphi_1^{\perp}$,

$$\alpha \int_\Omega h_1(x)|\varphi_1|^p dx = 0,$$

which implies $\alpha = 0$. This contradicts the fact that $\int_\Omega h_1(x)|u|^p dx = 1$. So, choose $\overline{\lambda} = \min\{\lambda, \lambda_2\}$. It is clear that $\overline{\lambda}$ satisfies (2.4).

In the same way, we prove the existence of $\overline{\mu} \in (\mu_1, \mu_2]$ such that (2.5) holds, and the proof of the lemma is complete. □

Definition 2.2. We say that $(u,v) \in X$ is a weak solution of problem (1.1) if

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi dx + \int_\Omega |\Delta v|^{q-2}\Delta v \Delta\psi dx - \int_\Omega h_1(x)|u|^{p-2}u\varphi dx \\ - \int_\Omega h_2(x)|v|^{q-2}v\psi dx - \int_\Omega F_u(x,u,v)\varphi dx - \int_\Omega F_v(x,u,v)\psi dx = 0$$

for all $(\varphi,\psi) \in X$.

The corresponding energy functional of problem (1.1) is given by

$$I(u,v) = \frac{1}{p}\int_\Omega |\Delta u|^p dx + \frac{1}{q}\int_\Omega |\Delta v|^q dx - \frac{\lambda}{p}\int_\Omega h_1(x)|u|^p dx - \frac{\mu}{q}\int_\Omega h_2(x)|v|^q dx \\ - \int_\Omega F(x,u,v)dx. \tag{2.7}$$

Let us consider the functional $T(u,v) = \displaystyle\int_\Omega F(x,u,v)dx$.

Lemma 2.3. *Assume that* $(F1)$ *holds. Then* $T \in C^1(X,\mathbb{R})$ *and*

$$\langle T'(u,v),(a,b)\rangle = \int_\Omega F_u(x,u,v)a + F_v(x,u,v)bdx$$

for all $(u,v),(a,b) \in X$.

Proof. It suffices to observe that by $(F1)$, and using the fact that $F_s, F_t \in C(\overline{\Omega}\times\mathbb{R}^2,\mathbb{R})$ for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} |F_s(x,s,t)| &\le \varepsilon h_1(x)|s|^{p-1} + C_\varepsilon, \\ |F_t(x,s,t)| &\le \varepsilon h_2(x)|t|^{q-1} + C_\varepsilon, \end{aligned} \tag{2.8}$$

for all $(x,s,t) \in \overline{\Omega}\times\mathbb{R}^2$. □

In view of Lemma 2.3, we have $I \in C^1(X,\mathbb{R})$.

Lemma 2.4. *Assume that* $(F1)$ *holds. Then, for* $\lambda < \lambda_1$ *and* $\mu < \mu_1$*, the functional* I *is coercive in* X *and bounded from below on* W*. Moreover, there exists a constant* m*, independent of* λ *and* μ*, such that* $\inf_W I(u,v) \ge m$*.*

Proof. By Hölder's inequality, from (2.8) we have

$$\begin{aligned} |F(x,u,v)| &\le \left|\int_0^u |F_s(x,s,v)|ds + \int_0^v |F_t(x,0,t)|dt + F(x,0,0)\right| \\ &\le \left|\int_0^u (\varepsilon h_1(x)|s|^{p-1} + C_\varepsilon)ds + \int_0^v (\varepsilon h_2(x)|t|^{q-1} + C_\varepsilon)dt\right| + M \\ &\le \frac{\varepsilon}{p}h_1(x)|u|^p + C_\varepsilon|u| + \frac{\varepsilon}{q}h_2(x)|v|^q + C_\varepsilon|v| + M, \end{aligned}$$

where $M = \max_{x\in\overline{\Omega}} |F(x,0,0)|$. It follows from (1.8) that

$$\begin{aligned}
&\int_\Omega |F(x,u,v)|dx \\
&\quad \le \varepsilon \left(\frac{1}{p} \int_\Omega h_1(x)|u|^p dx + \frac{1}{q} \int_\Omega h_2(x)|v|^q dx \right) \\
&\qquad + C_\varepsilon \left(\int_\Omega |u|dx + \int_\Omega |v|dx \right) + M|\Omega| \\
&\quad \le \frac{\varepsilon}{p\lambda_1} \int_\Omega |\Delta u|^p dx + \frac{\varepsilon}{q\mu_1} \int_\Omega |\Delta v|^q dx \\
&\qquad + C_\varepsilon |\Omega|^{\frac{p-1}{p}} S_1 \left(\int_\Omega |\Delta u|^p dx \right)^{\frac{1}{p}} + C_\varepsilon |\Omega|^{\frac{q-1}{q}} S_2 \left(\int_\Omega |\Delta v|^q dx \right)^{\frac{1}{q}} + M|\Omega| \\
&\quad \le \frac{\varepsilon}{p\lambda_1} \int_\Omega |\Delta u|^p dx + \frac{\varepsilon}{q\mu_1} \int_\Omega |\Delta v|^q dx \\
&\qquad + C'_\varepsilon \left[\left(\int_\Omega |\Delta u|^p dx \right)^{\frac{1}{p}} + \left(\int_\Omega |\Delta v|^q dx \right)^{\frac{1}{q}} \right] + M|\Omega| \\
&\quad \le \frac{\varepsilon}{p\lambda_1} \int_\Omega |\Delta u|^p dx + \frac{\varepsilon}{q\mu_1} \int_\Omega |\Delta v|^q dx + C'_\varepsilon ||(u,v)|| + M|\Omega|, \qquad (2.9)
\end{aligned}$$

where S_1 and S_2 are the embedding constants of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$, respectively, and

$$C'_\varepsilon = C_\varepsilon \max \left\{ |\Omega|^{\frac{p-1}{p}} S_1, |\Omega|^{\frac{q-1}{q}} S_2 \right\}.$$

For $\lambda < \lambda_1$ and $\mu < \mu_1$, from (1.8), (2.7) and (2.9), we get

$$\begin{aligned}
I(u,v) \ge \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_1} - \frac{\varepsilon}{\lambda_1} \right) \int_\Omega |\Delta u|^p dx + \frac{1}{q} \left(1 - \frac{\mu}{\mu_1} - \frac{\varepsilon}{\mu_1} \right) \int_\Omega |\Delta v|^q dx \\
- C'_\varepsilon ||(u,v)|| - M|\Omega|. \qquad (2.10)
\end{aligned}$$

Choose $\varepsilon = \frac{1}{2} \min \left\{ \lambda_1 \left(1 - \frac{\lambda}{\lambda_1} \right), \mu_1 \left(1 - \frac{\mu}{\mu_1} \right) \right\}$. Thus

$$\begin{aligned}
I(u,v) &\ge \frac{\lambda_1 - \lambda}{2p\lambda_1} \int_\Omega |\Delta u|^p dx + \frac{1}{2q} \left(1 - \frac{\mu}{\mu_1} \right) \int_\Omega |\Delta v|^q dx - C'_\varepsilon ||(u,v)|| - M|\Omega| \\
&\ge \frac{1}{2} \min \left\{ \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_1} \right), \frac{1}{q} \left(1 - \frac{\mu}{\mu_1} \right) \right\} \left(\int_\Omega |\Delta u|^p dx + \int_\Omega |\Delta v|^q dx \right) \\
&\quad - C'_\varepsilon ||(u,v)|| - M|\Omega|. \qquad (2.11)
\end{aligned}$$

Let us make the following remark.

Remark 2.5. For all $s, t \geq 0$ we have

$$\left(t^{\frac{1}{p}} + s^{\frac{1}{q}}\right)^{\min(p,q)} \leq 2^{\min(p,q)}(t + s + 1).$$

Then,

$$\int_\Omega |\Delta u|^p dx + \int_\Omega |\Delta v|^q dx \geq \frac{1}{2^{\min(p,q)}} \left[\left(\int_\Omega |\Delta u|^p dx\right)^{\frac{1}{p}} + \left(\int_\Omega |\Delta v|^q dx\right)^{\frac{1}{q}}\right]^{\min(p,q)} - 1.$$

It follows from (2.11) that

$$\begin{aligned} I(u,v) \geq{} & \frac{1}{2^{1+\min(p,q)}} \min\left\{\frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right), \frac{1}{q}\left(1 - \frac{\mu}{\mu_1}\right)\right\} \left(||(u,v)||^{\min(p,q)} - 2^{\min(p,q)}\right) \\ & - C'_\varepsilon ||(u,v)|| - M|\Omega|. \end{aligned} \tag{2.12}$$

Since $p, q > 1$, I is coercive in X. Similarly, let $(u, v) \in W$. By Lemma 2.1 we get

$$\begin{aligned} I(u,v) &\geq \frac{1}{p}\left(1 - \frac{\lambda}{\overline{\lambda}} - \frac{\varepsilon}{\lambda_1}\right) \int_\Omega |\Delta u|^p dx + \frac{1}{q}\left(1 - \frac{\mu}{\overline{\mu}} - \frac{\varepsilon}{\mu_1}\right) \int_\Omega |\Delta v|^q dx \\ &\quad - C'_\varepsilon ||(u,v)|| - M|\Omega| \\ &\geq \frac{1}{p}\left(1 - \frac{\lambda_1}{\overline{\lambda}} - \frac{\varepsilon}{\lambda_1}\right) \int_\Omega |\Delta u|^p dx + \frac{1}{q}\left(1 - \frac{\mu_1}{\overline{\mu}} - \frac{\varepsilon}{\mu_1}\right) \int_\Omega |\Delta v|^q dx \\ &\quad - C'_\varepsilon ||(u,v)|| - M|\Omega|. \end{aligned}$$

Choose $\varepsilon = \frac{1}{2} \min\left\{\lambda_1\left(1 - \frac{\lambda_1}{\overline{\lambda}}\right), \mu_1\left(1 - \frac{\mu_1}{\overline{\mu}}\right)\right\}$. Thus

$$\begin{aligned} I(u,v) &\geq \frac{1}{2p}\left(1 - \frac{\lambda_1}{\overline{\lambda}}\right) \int_\Omega |\Delta u|^p dx + \frac{1}{2q}\left(1 - \frac{\mu_1}{\overline{\mu}}\right) \int_\Omega |\Delta v|^q dx \\ &\quad - C'_\varepsilon ||(u,v)|| - M|\Omega| \\ &\geq \frac{1}{2^{1+\min(p,q)}} \min\left\{\frac{1}{p}\left(1 - \frac{\lambda_1}{\overline{\lambda}}\right), \frac{1}{q}\left(1 - \frac{\mu_1}{\overline{\mu}}\right)\right\} \left(||(u,v)||^{\min(p,q)} - 2^{\min(p,q)}\right) \\ &\quad - C'_\varepsilon ||(u,v)|| - M|\Omega|. \end{aligned} \tag{2.13}$$

Hence I is bounded from below on W. Moreover, we can find a constant m independent of λ and μ such that $\inf\limits_W I(u,v) \geq m$, and the proof of the lemma is complete. □

Lemma 2.6. *Assume that $(F1)$ and (1.12) hold. Then, for $\lambda < \lambda_1$ and $\mu < \mu_1$ sufficiently close to λ_1 and μ_1, respectively, there exist $s^- < 0 < s^+$ and $t^- < 0 < t^+$ such that $I(s^+\varphi_1, t^+\psi_1) < m$ and $I(s^-\varphi_1, t^-\psi_1) < m$, where m is given by Lemma 2.4.*

Proof. By definition of λ_1 and μ_1, we have

$$\begin{aligned}
I(s\varphi_1, t\psi_1) &= \frac{|s|^p}{p}\int_\Omega |\Delta\varphi_1|^p dx + \frac{|t|^q}{q}\int_\Omega |\Delta\psi_1|^q dx \\
&\quad - \lambda\frac{|s|^p}{p}\int_\Omega h_1(x)|\varphi_1|^p dx - \mu\frac{|t|^q}{q}\int_\Omega h_2(x)|\psi_1|^q dx - \int_\Omega F(x, s\varphi_1, t\psi_1)dx \\
&= \frac{|s|^p}{p}\int_\Omega |\Delta\varphi_1|^p dx - \frac{\lambda|s|^p}{\lambda_1 p}\int_\Omega |\Delta\varphi_1|^p dx \\
&\quad + \frac{|t|^q}{q}\int_\Omega |\Delta\psi_1|^q dx - \frac{\mu|t|^q}{\mu_1 q}\int_\Omega |\Delta\psi_1|^q dx - \int_\Omega F(x, s\varphi_1, t\psi_1)dx \\
&= \frac{|s|^p}{p}\left(1 - \frac{\lambda}{\lambda_1}\right) + \frac{|t|^q}{q}\left(1 - \frac{\mu}{\mu_1}\right) - \int_\Omega F(x, s\varphi_1, t\psi_1)dx. \qquad (2.14)
\end{aligned}$$

By Fatou's lemma and from (1.12), there exist $s^+, t^+ > 0$ such that

$$\int_\Omega F(x, s^+\varphi_1, t^+\psi_1)dx > -m + 1 \qquad (2.15)$$

for $\lambda_1 - \dfrac{p\lambda_1}{2(s^+)^p} < \lambda < \lambda_1$ and $\mu_1 - \dfrac{q\mu_1}{2(t^+)^q} < \mu < \mu_1$. Relations (2.14) and (2.15) imply that

$$I(s^+\varphi_1, t^+\psi_1) < m.$$

Similarly, we get $I(s^-\varphi_1, t^-\psi_1) < m$, for some $s^-, t^- < 0$, and the proof of the lemma is complete. □

Proof of Theorem 1.1. First we show that I satisfies the (PS) condition in X. Let $\{z_n = (u_n, v_n)\} \subset X$ be a (PS) sequence. Since I is coercive, z_n is bounded in X, so up to subsequence, we may assume that $z_n \rightharpoonup z = (u, v)$ weakly in X. Therefore,

$$\langle I'(u_n, v_n), (u_n - u, 0)\rangle = o_n(1). \qquad (2.16)$$

By Hölder's inequality, we have

$$\int_\Omega |h_1(x)|u_n|^{p-2}u_n(u_n - u)|dx \le |h_1|_\infty \left(\int_\Omega |u_n|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |u_n - u|^p dx\right)^{\frac{1}{p}}. \qquad (2.17)$$

Since $u_n \to u$ in $L^p(\Omega)$,

$$\lim_{n\to\infty}\int_\Omega h_1(x)|u_n|^{p-2}u_n(u_n - u)dx = 0. \qquad (2.18)$$

By (2.8), it is easy to see that

$$\lim_{n\to\infty}\int_\Omega F_u(x, u_n, v_n)(u_n - u)dx = 0. \qquad (2.19)$$

Combining (2.16), (2.18) and (2.19), we obtain

$$\lim_{n\to\infty}\int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta(u_n-u)dx = 0.$$

In the same way, we have

$$\lim_{n\to\infty}\int_\Omega |\Delta u|^{p-2}\Delta u \Delta(u_n-u)dx = 0.$$

Therefore,

$$\begin{aligned} 0 &= \lim_{n\to\infty}\int_\Omega (|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u)\Delta(u_n-u)dx \\ &\geq \lim_{n\to\infty}(||u_n||_p^{p-1} - ||u||_p^{p-1})(||u_n||_p - ||u||_p), \end{aligned}$$

and $||u_n||_p \to ||u||_p$. By the uniform convexity of $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$, it follows that $u_n \to u$ strongly in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Similar arguments yield that $v_n \to v$ strongly in $W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)$, so that $z_n \to z$ strongly in X, and I satisfies the (PS) condition. Let

$$\Lambda^\pm = \{z \in X : z = \pm(s\varphi_1, t\psi_1) + w,\ s,t > 0,\ w \in W\} \tag{2.20}$$

and $\{z_n\} \subset \Lambda^+$ be such that $I(z_n) \to c < m$ and $I'(z_n) \to 0$ as $n \to \infty$. Then $z_n \to z$ strongly in X. Note that $\partial\Lambda^+ = W$. So, if $z \in \partial\Lambda^+$, then it follows from $\inf\limits_W I \geq m$ that $I(z_n) \to c = I(z) \geq m$, which is impossible. Therefore, $z \in \Lambda^+$, and hence I, satisfies the $(PS)_{c,\Lambda^+}$ for all $c < m$. Similarly, I satisfies the $(PS)_{c,\Lambda^-}$ for all $c < m$. In view of Lemma 2.6, for $\lambda < \lambda_1$ and $\mu < \mu_1$ sufficiently close to λ_1 and μ_1, respectively, we have

$$-\infty < \inf_{\Lambda^+} I < m. \tag{2.21}$$

By Ekeland's variational principle in $\overline{\Lambda^+}$, there exists a sequence $\{z_n\} \subset \Lambda^+$ such that

$$I(z_n) \to \inf_{\Lambda^+} I \quad \text{and} \quad I'(z_n) \to 0.$$

Since I satisfies the $(PS)_{c,\Lambda^+}$ for all $c < m$, there exists $z^+ \in \Lambda^+$ such that

$$I(z^+) = \inf_{\Lambda^+} I.$$

Similarly, we find $z^- \in \Lambda^-$ such that $I(z^-) = \inf\limits_{\Lambda^-} I$. Hence I has two distinct critical points z^+ and z^-. Now, we prove the existence of the third solution. To fix ideas, suppose that $I(z^+) \leq I(z^-)$ and put

$$J(z) := I(z + z^-) - I(z^-), \quad e = z^+ - z^-.$$

So, $J(0) = 0,\ J(e) \leq 0$. We can find $r > 0$ such that $\overline{B(z^-, r)} \subset \Lambda^-$, thus

$$\inf_{||z-z^-||=r} I(z) \geq I(z^-)$$

and hence $\inf\limits_{||z||=r} J(z) \geq 0$. Let

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \tag{2.22}$$

where

$$\Gamma = \left\{\gamma \in C([0,1], X) : \gamma(0) = z^-,\ \gamma(1) = z^+\right\}.$$

Since J also satisfies the (PS) condition and $J' = I'$, it follows from the mountain pass theorem (see [5], see also Theorem 6.2 in [2]) that c is a critical value of I. Note that all paths joining z^- to z^+ pass through W, $c \geq m$. Therefore, the third solution is obtained, and the proof of our theorem is complete. $\square$

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