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Gauge Symmetries in Fractional Variational Problems

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Abstract

We prove a second Noether theorem for Lagrangian densities with fractional derivatives defined in the Riesz–Caputo sense. An application to the fractional electromagnetic field is given.

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1 Introduction

Emmy Noether proved, in 1918, two theorems and their converses which revealed the general connection between symmetries and conservation laws in physics. They led to understanding of laws such as conservation of energy, angular momentum, etc., and also were instrumental in discoveries of gauge field symmetries. The first theorem applies to symmetries associated with finite dimensional Lie groups (global symmetries); the second to symmetries associated with infinite dimensional Lie groups (local symmetries).

Fractional calculus is a discipline that studies integrals and derivatives of noninteger (real or complex) order ([11, 12, 18]). The field was born more than three centuries ago and became an ongoing topic with many well-known mathematicians contributing to its theory (see [22] for a review). The subject is nowadays very active due to its many applications in mechanics, chemistry, biology, economics, and control theory (see [23] for a review).

The study of fractional variational problems was introduced by Riewe [20]. The original motivation was to show that a Lagrangian involving fractional time derivatives

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leads to an equation of motion with nonconservative forces such as friction. It is a remarkable result since frictional and nonconservative forces are beyond the usual macroscopic variational treatment, and consequently, beyond the most advanced methods of classical mechanics. Riewe generalized the usual variational calculus, by considering Lagrangians that depend on fractional derivatives, in order to deal with nonconservative forces. The fractional variational calculus has recently attracted the attention of several researchers (see, e.g., [1,2,4,16,17,19,21] and references therein). For the state of the art we refer the reader to the recent book [15].

Noether's first theorems have been extended to fractional variational problems using several approaches ([3,6–9]). In this paper we generalize the second Noether theorem to fractional setting using Riesz–Caputo calculus. The paper is organized as follows. At first, in Section 2, we fix notation by recalling the basic definitions and facts from the fractional calculus. In Section 3 we prove the second Noether-type theorem for Lagrangian densities with fractional derivatives defined in the Riesz–Caputo sense. We end with Section 4, of application our results to a fractional electromagnetic action.

2 Preliminaries

In this section we fix notation by recalling the necessary definitions and facts from the fractional calculus.

Let $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$, $f \in L_1([a,b],\mathbb{R})$. By the left Riemann–Liouville fractional integral of f on the interval $[a,b]$ we mean a function ${}_aI_x^\alpha f$ defined by

$${}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x \in [a,b] \text{ a.e.},$$

by the right Riemann–Liouville fractional integral of f on the interval $[a,b]$ we mean a function ${}_xI_b^\alpha f$ defined by

$${}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x \in [a,b] \text{ a.e.},$$

where $\Gamma(\cdot)$ represents the Gamma function. For $\alpha = 0$, we set ${}_aI_x^0 f = {}_xI_b^0 f := If$, the identity operator. If the function ${}_aI_x^\alpha f$ is absolutely continuous on the interval $[a,b]$, then the left Riemann–Liouville fractional derivative is given by

$${}_aD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha} f(t)dt.$$

If the function ${}_xI_b^{1-\alpha} f$ is absolutely continuous on the interval $[a,b]$, then the right Riemann–Liouville fractional derivative is given by

$${}_xD_b^\alpha f(x) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b (t-x)^{-\alpha} f(t)dt.$$

Let $f \in AC([a,b])$. By the left Caputo fractional derivative of f on the interval $[a,b]$ we mean a function ${}_a^C D_x^\alpha f$ defined by

$$ {}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} \frac{d}{dt} f(t) dt, $$

and by the right Caputo fractional derivative of f on the interval $[a,b]$ we mean a function ${}_x^C D_b^\alpha f$ defined by

$$ {}_x^C D_b^\alpha f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} \frac{d}{dt} f(t) dt. $$

The Riesz fractional integral ${}_a^R I_b^\alpha f$ of order α is defined by

$$ {}_a^R I_b^\alpha f(x) = \frac{1}{2\Gamma(\alpha)} \int_a^b |x-\theta|^{\alpha-1} f(\theta) d\theta. $$

Observe that, from definitions of the Riemann–Liouville and the Riesz fractional integrals, it follows that

$$ {}_a^R I_b^\alpha f(x) = \frac{1}{2} \left[{}_a I_x^\alpha f(x) +_x I_b^\alpha f(x) \right] . $$

The Riesz fractional derivative ${}_a^R D_b^\alpha f$ and the Riesz–Caputo fractional derivative ${}_a^{RC} D_b^\alpha f$ of order α ($0 < \alpha < 1$) are defined by

$$ {}_a^R D_b^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^b |x-\theta|^{-\alpha} f(\theta) d\theta \, , $$

$$ {}_a^{RC} D_b^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^b |x-\theta|^{-\alpha} \frac{d}{d\theta} f(\theta) d\theta. $$

Therefore,

$$ {}_a^R D_b^\alpha f(x) = \frac{1}{2} \left[{}_a D_x^\alpha f(x) - {}_x D_b^\alpha f(x) \right] $$

and

$$ {}_a^{RC} D_b^\alpha f(x) = \frac{1}{2} \left[{}_a^C D_x^\alpha f(x) - {}_x^C D_b^\alpha f(x) \right] . $$

In the discussion to follow, we need a fractional integration by parts formula. Let $0 < \alpha < 1$. If $f, g \in AC([a,b])$, then

$$ \int_a^b g(x) \, {}_a^C D_x^\alpha f(x) dx = f(x) {}_x I_b^{1-\alpha} g(x) \Big|_{x=a}^{x=b} + \int_a^b f(x) {}_x D_b^\alpha g(x) dx, $$

$$ \int_a^b g(x) \, {}_x^C D_b^\alpha f(x) dx = -f(x) {}_a I_x^{1-\alpha} g(x) \Big|_{x=a}^{x=b} + \int_a^b f(x) {}_a D_x^\alpha g(x) dx. $$

Thus, in the case of the Riesz–Caputo fractional derivative, one has

$$\int_a^b g(x)\, {}^{RC}_{a}D^{\alpha}_{b} f(x)dx = f(x) {}^{R}_{x}I^{1-\alpha}_{b} g(x)\Big|_{x=a}^{x=b} - \int_a^b f(x) {}^{R}_{a}D^{\alpha}_{b} g(x)dx. \tag{2.1}$$

Partial fractional integrals and derivatives are a natural generalization of the corresponding one-dimensional fractional integrals and derivatives, being taken with respect to one or several variables. For $(x_1, \ldots, x_n)$, $(\alpha_1, \ldots, \alpha_n)$, where $0 < \alpha_i < 1$, $i = 1, \ldots, n$ and $[a_1, b_1] \times \cdots \times [a_n, b_n]$, the partial Riemann–Liouville fractional integrals of order α_k with respect to x_k are defined by

$$ {}_{a_k}I^{\alpha_k}_{x_k} f(x_1, \ldots, x_n) = \frac{1}{\Gamma(\alpha_k)} \int_{a_k}^{x_k} (x_k - t_k)^{\alpha_k - 1} f(x_1, \ldots, t_k, \ldots, x_n)dt_k,$$

$$ {}_{x_k}I^{\alpha_k}_{b_k} f(x_1, \ldots, x_n) = \frac{1}{\Gamma(\alpha_k)} \int_{x_k}^{b_k} (t_k - x_k)^{\alpha_k - 1} f(x_1, \ldots, t_k, \ldots, x_n)dt_k.$$

Partial Riemann–Liouville and Caputo derivatives are defined by

$$ {}_{a_k}D^{\alpha_k}_{x_k} f(x_1, \ldots, x_n) = \frac{1}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int_{a_k}^{x_k} (x_k - t_k)^{-\alpha_k} f(x_1, \ldots, t_k, \ldots, x_n)dt_k,$$

$$ {}_{x_k}D^{\alpha_k}_{b_k} f(x_1, \ldots, x_n) = \frac{-1}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int_{x_k}^{b_k} (t_k - x_k)^{-\alpha_k} f(x_1, \ldots, t_k, \ldots, x_n)dt_k,$$

$$ {}^{C}_{a_k}D^{\alpha_k}_{x_k} f(x_1, \ldots, x_n) = \frac{1}{\Gamma(1-\alpha_k)} \int_{a_k}^{x_k} (x_k - t_k)^{-\alpha_k} \frac{\partial}{\partial t_k} f(x_1, \ldots, t_k, \ldots, x_n)dt_k,$$

$$ {}^{C}_{x_k}D^{\alpha_k}_{b_k} f(x_1, \ldots, x_n) = \frac{-1}{\Gamma(1-\alpha_k)} \int_{x_k}^{b_k} (t_k - x_k)^{-\alpha_k} \frac{\partial}{\partial t_k} f(x_1, \ldots, t_k, \ldots, x_n)dt_k.$$

Partial Riesz and Riesz–Caputo derivatives are defined by

$$ {}^{R}_{a_k}D^{\alpha_k}_{b_k} f(x_1, \ldots, x_n) = \frac{1}{2} \left[{}_{a_k}D^{\alpha_k}_{x_k} f(x_1, \ldots, x_n) - {}_{x_k}D^{\alpha_k}_{b_k} f(x_1, \ldots, x_n) \right],$$

and

$$ {}^{RC}_{a}D^{\alpha_k}_{b} f(x_1, \ldots, x_n) = \frac{1}{2} \left[{}^{C}_{a_k}D^{\alpha_k}_{x_k} f(x_1, \ldots, x_n) - {}^{C}_{x_k}D^{\alpha_k}_{b_k} f(x_1, \ldots, x_n) \right].$$

3 Main Results

Consider a system characterized by a set of functions

$$u^j(t, x_1, \ldots, x_m), \quad j = 1, \ldots, n, \tag{3.1}$$

depending on time t and the space coordinates $x_1, \dots, x_m$. We can simplify the notation by interpreting (3.1) as a vector function $u = (u^1, \dots, u^n)$ and writing $t = x_0$, $x = (x_0, x_1, \dots, x_m)$, $dx = dx_0 dx_1 \cdots dx_m$. Then (3.1) becomes simply $u(x)$ and is called a vector field. Define the action functional in the form

$$\mathcal{J}(u) = \int_\Omega \mathcal{L}(x, u, {}^{RC}\nabla^\alpha u) dx, \tag{3.2}$$

where $\Omega = R \times [a_0, b_0]$, $R = [a_1, b_1] \times \cdots \times [a_m, b_m]$, and ${}^{RC}\nabla^\alpha$ is the operator

$$\left({}^{RC}_{a_0}D^{\alpha_0}_{b_0}, {}^{RC}_{a_1}D^{\alpha_1}_{b_1}, \cdots, {}^{RC}_{a_m}D^{\alpha_m}_{b_m} \right),$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_m)$, $0 < \alpha_i \le 1$, $i = 0, \dots, m$. The function $\mathcal{L}(x, u, {}^{RC}\nabla^\alpha u)$ is called the fractional Lagrangian density of the field. We assume that:

(i) $u^j \in C^1(\Omega, \mathbb{R})$, $j = 1, \dots, n$;

(ii) $\mathcal{L} \in C^1(\mathbb{R}^{m+1} \times \mathbb{R}^n \times \mathbb{R}^{n(m+1)}; \mathbb{R})$;

(iii) $x \to \dfrac{\partial \mathcal{L}}{\partial {}^{RC}_{a_i}D^{\alpha_i}_{b_i} u_j}$ are C^1-functions for every $u^j \in C^1(\Omega, \mathbb{R})$, $i = 0, \dots, m$ and $j = 1, \dots, n$.

We define the admissible set of functions $A(\Omega)$ by

$$A(\Omega) := \{u : \Omega \to \mathbb{R}^n : u(x) = \varphi(x) \quad \text{for} \quad x \in \partial\Omega\},$$

where $\varphi : \partial\Omega \to \mathbb{R}^n$ is a given function.

Applying the principle of stationary action to (3.2) we obtain fractional Euler–Lagrange equations for the field (cf. [1, 13]).

Theorem 3.1. *A necessary condition for the function $u \in A(\Omega)$ to provide an extremum for the action functional* (3.2) *is that its components satisfy the following n multidimensional fractional Euler–Lagrange equations:*

$$\frac{\partial \mathcal{L}}{\partial u^j} - \sum_{i=0}^m {}^{R}_{a_i}D^{\alpha_i}_{b_i} \frac{\partial \mathcal{L}}{\partial {}^{RC}_{a_i}D^{\alpha_i}_{b_i} u^j} = 0, \quad j = 1, \dots, n.$$

Proof. The result of the theorem is proved by writing down the variation of the action, performing an appropriate integration by parts, and using the fundamental lemma of the calculus of variations. □

We define

$$E^f_j(\mathcal{L}) := \frac{\partial \mathcal{L}}{\partial u^j} - \sum_{i=0}^m {}^{R}_{a_i}D^{\alpha_i}_{b_i} \frac{\partial \mathcal{L}}{\partial {}^{RC}_{a_i}D^{\alpha_i}_{b_i} u^j}, \quad j = 1, \dots, n,$$

which are called the fractional Lagrange expressions. We shall study infinitesimal transformations that depend upon arbitrary functions of the independent variables and their partial fractional derivatives in the sense of Riesz–Caputo. Let

$$\begin{cases} \bar{x} = x, \\ \bar{u}^j(x) = u^j(x) + T^{j1}(p_1(x)) + \cdots + T^{jr}(p_r(x)), \end{cases} \tag{3.3}$$

$j = 1, \ldots, n$, where p_s, $s = 1, \ldots, r$, are r arbitrary independent C^1-functions defined on Ω and T^{js} is a linear fractional differential operator:

$$T^{js} := c^{js}(x) + \sum_{i=0}^{m} c_i^{js}(x) {}_{a_i}^{RC}D_{b_i}^{\beta_{jsi}}, \quad 0 < \beta_{jsi} \leq 1,$$

with ${}_{a_i}^{RC}D_{b_i}^{\beta_{jsi}} p_s$, c^{js}, c_i^{js} C^1-functions defined on Ω, $s = 1, \ldots, r$, $i = 1, \ldots, m$.

Now we define invariance similarly to the classical case. The functional (3.2) is invariant under transformations (3.3) if, and only if, for all $u \in C^1(\Omega, \mathbb{R}^n)$ we have

$$\int_\Omega \mathcal{L}\left(x, \bar{u}, {}^{RC}\nabla^\alpha \bar{u}\right) dx = \int_\Omega \mathcal{L}\left(x, u, {}^{RC}\nabla^\alpha u\right) dx.$$

Theorem 3.2. *If functional* (3.2) *is invariant under transformations* (3.3)*, then there exist* r *identities of the form*

$$\sum_{j=1}^{n} \tilde{T}^{js}\left(E_j^f(\mathcal{L})\right) = 0, \quad s = 1, \ldots, r,$$

where $\tilde{T}^{js}$ *is the adjoint of* T^{js} *(see Remark 3.3).*

Proof. By the definition of invariance we have

$$\begin{aligned} 0 &= \int_\Omega \mathcal{L}\left(x, \bar{u}, {}^{RC}\nabla^\alpha \bar{u}\right) dx - \int_\Omega \mathcal{L}\left(x, u, {}^{RC}\nabla^\alpha u\right) dx \\ &= \int_\Omega \left(\mathcal{L}\left(x, \bar{u}, {}^{RC}\nabla^\alpha \bar{u}\right) - \mathcal{L}\left(x, u, {}^{RC}\nabla^\alpha u\right)\right) dx. \end{aligned}$$

Then, by the Taylor formula,

$$0 = \sum_{j=1}^{n} \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u^j} T^{js}(p_s) + \sum_{i=0}^{m} \frac{\partial \mathcal{L}}{\partial {}_{a_i}^{RC}D_{b_i}^{\alpha_i} u^j} {}_{a_i}^{RC}D_{b_i}^{\alpha_i} T^{js}(p_s)\right) dx, \tag{3.4}$$

where $T^{js}(p_s) = \sum_{s=1}^{r} T^{js}(p_s)$. The Fubini theorem allows us to rewrite the integrals as iterated integrals so that we can use the integration by parts formula (2.1):

$$\int_\Omega \sum_{i=0}^{m} \frac{\partial \mathcal{L}}{\partial {}_{a_i}^{RC}D_{b_i}^{\alpha_i} u^j} {}_{a_i}^{RC}D_{b_i}^{\alpha_i} T^{js}(p_s)dx$$

$$= -\int_\Omega \sum_{i=0}^{m} {}_{a_i}^{R}D_{b_i}^{\alpha_i} \frac{\partial \mathcal{L}}{\partial {}_{a_i}^{RC}D_{b_i}^{\alpha_i} u^j} T^{js}(p_s)dx + [\cdot]|_{\partial\Omega}, \quad (3.5)$$

$j = 1, \ldots, n$, where $[\cdot]|_{\partial\Omega}$ represent the boundary terms – the $m+1$-volumes integrals. Since p_s are arbitrary, we may choose p_s such that $p_s(x)|_{\partial\Omega} = 0$ and ${}_{a_i}^{RC}D_{b_i}^{\beta_{jsi}} p_s(t)|_{\partial\Omega} = 0$, $s = 1, \ldots, r$, $i = 1, \ldots, l$. Therefore, the boundary term in (3.5) vanishes and substituting (3.5) into (3.4) we get

$$0 = \sum_{j=1}^{n} \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u^j} - \sum_{i=0}^{m} {}_{a_i}^{R}D_{b_i}^{\alpha_i} \frac{\partial \mathcal{L}}{\partial {}_{a_i}^{RC}D_{b_i}^{\alpha_i} u^j} \right) T^{js}(p_s)dx.$$

Now we define the adjoint operator $\tilde{T}^{js}$ of a fractional differential operator T^{js} by

$$\int_\Omega q(x)T^{js}(p_s(x))dx = \int_\Omega p_s(x)\tilde{T}^{js}(q(x))dx + [\cdot]|_{\partial\Omega},$$

where we use the Fubini theorem. Again appealing to the arbitrariness of p_s, we can force the boundary term to vanish (by putting $p_s(x)|_{\partial\Omega} = 0$). Therefore,

$$0 = \sum_{j=1}^{n} \int_\Omega \sum_{s=1}^{r} \tilde{T}^{js} \left(\frac{\partial \mathcal{L}}{\partial u^j} - \sum_{i=0}^{m} {}_{a_i}^{R}D_{b_i}^{\alpha_i} \frac{\partial \mathcal{L}}{\partial {}_{a_i}^{RC}D_{b_i}^{\alpha_i} u^j} \right) p_s dx.$$

Finally, by the fundamental lemma of the calculus of variations, we conclude that

$$\sum_{j=1}^{n} \tilde{T}^{js} \left(E_j^f(\mathcal{L}) \right) = 0, \quad s = 1, \ldots, r.$$

This concludes the proof. □

Remark 3.3. The adjoint of T^{js} is given by the expression

$$\tilde{T}^{js}(q) = c^{js}q - \sum_{i=0}^{m} {}_{a_i}^{R}D_{b_i}^{\beta_{jsi}}(c_i^{js}q), \quad j = 1, \ldots, n.$$

4 An Example

Historically, the first example of gauge symmetry to be discovered was classical electromagnetism. Here, we will propose a fractional electromagnetic action which is invariant under a fractional gauge transformation. To illustrate our result we will use the Lagrangian density for the electromagnetic field (see [10]):

$$\mathcal{L} = \frac{1}{8\pi}(\mathbf{E}^2 - \mathbf{H}^2), \tag{4.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric field vector and the magnetic field vector, respectively. Using the method presented in [5, 14], we generalize (4.1) to the fractional Lagrangian density by changing classical partial derivatives by fractional. Let $x = (x_0, x_1, x_2, x_3) \in \Omega$ and $\mathbf{A}(x) = (A_1(x), A_2(x), A_3(x))$, $A_0(x)$ be a vector potential and a scalar potential, respectively. They are defined by setting

$$\mathbf{E} = {}^{RC}\nabla^{(\alpha_1,\alpha_2,\alpha_3)}A_0 - {}^{RC}_{a_0}D^{\alpha_0}_{a_0}\mathbf{A}, \quad \mathbf{H} = \mathrm{curl}\mathbf{A}, 0 < \alpha_i \leq 1, \quad i = 0, \ldots, 3, \tag{4.2}$$

where

$$\begin{aligned} {}^{RC}\nabla^{(\alpha_1,\alpha_2,\alpha_3)}A_0 &= \mathbf{i}^{RC}_{a_1}D^{\alpha_1}_{b_1}A_0 + \mathbf{j}^{RC}_{a_2}D^{\alpha_2}_{b_2}A_0 + \mathbf{k}^{RC}_{a_3}D^{\alpha_3}_{b_3}A_0, \\ {}^{RC}_{a_0}D^{\alpha_0}_{b_0}\mathbf{A} &= \mathbf{i}^{RC}_{a_0}D^{\alpha_0}_{b_0}A_1 + \mathbf{j}^{RC}_{a_0}D^{\alpha_0}_{b_0}A_2 + \mathbf{k}^{RC}_{a_0}D^{\alpha_0}_{b_0}A_3, \end{aligned}$$

$$\begin{aligned} \mathrm{curl}\mathbf{A} = \mathbf{i}\left({}^{RC}_{a_2}D^{\alpha_2}_{b_2}A_3 - {}^{RC}_{a_3}D^{\alpha_3}_{b_3}A_2\right) + \mathbf{j}\left({}^{RC}_{a_3}D^{\alpha_3}_{b_3}A_1 - {}^{RC}_{a_1}D^{\alpha_1}_{b_1}A_3\right) \\ + \mathbf{k}\left({}^{RC}_{a_1}D^{\alpha_1}_{b_1}A_2 - {}^{RC}_{a_2}D^{\alpha_2}_{b_2}A_1\right). \end{aligned}$$

Replacing $\mathbf{E}$ and $\mathbf{H}$ in (4.1) by their expressions (4.2), we obtain the fractional Lagrangian density

$$\mathcal{L} = \frac{1}{8\pi}\left[\left({}^{RC}\nabla^{(\alpha_1,\alpha_2,\alpha_3)}A_0 - {}^{RC}_{a_0}D^{\alpha_0}_{b_0}\mathbf{A}\right)^2 - (\mathrm{curl}\mathbf{A})^2\right]. \tag{4.3}$$

Observe that, similar to the integer case, the potential $(A_0, \mathbf{A})$ is not uniquely determined by the vectors $\mathbf{E}$ and $\mathbf{H}$. Namely, $\mathbf{E}$ and $\mathbf{H}$ do not change if we make a fractional gauge transformation:

$$\tilde{A}_j(x) = A_j(x) + {}^{RC}_{a_j}D^{\alpha_j}_{b_j}f(x), \quad j = 0, \ldots, 3, \tag{4.4}$$

where $f : \Omega \to \mathbb{R}$ is an arbitrary function of class C^2 in all of its argument. Therefore, the Lagrangian density (4.3), and hence the action functional, is invariant under transformation (4.4). By Theorem (3.2), we conclude that

$$\sum_{j=0}^{3} {}^{R}_{a_j}D^{\alpha_j}_{b_j}\left(E^f_j(\mathcal{L})\right) = 0,$$

where $E^f_j(\mathcal{L})$ are Lagrange expressions corresponding to (4.3). Equations $E^f_j(\mathcal{L}) = 0$ do not uniquely determine the potential $(A_0, \mathbf{A})$ and to avoid this lack of uniqueness, a fractional Lorentz-type condition can be imposed on $(A_0, \mathbf{A})$.

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